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Numerical Solution of Volterra Integral Equations of First Kind by Using a Recursive Scheme

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Abstract. First kind integral equations can be solved numerically with several methods. In this paper we describe a recursive method for solving Volterra integral equation that don't need to solve system of algebraic equation. This method offers several advantages in reducing computational burden. Finally by comparison of numerical results, simplicity and efficiency of this method will be shown.

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1. Introduction

Many phenomena in physics and engineering reduced to an integral equation of the first or second kind. There are several numerical methods for solving these equations. In the most of these methods the integral equation is transformed to a system of linear algebraic equations.
Unfortunately system of algebraic equations corresponding to a first kind integral equation is ill-conditioned, meaning that small changes in the data of the problem cause very large changes in its solution. Hence

it is necessary to use other methods, such as preconditions, to convert the system of equations to a well-conditioned system. In this article we present a method in which the coefficients of the Taylor expansion of the exact solution are calculate by a recursive schemes,without solving any system of equations.
First, we propose method for solving Volterra integral equation of the first kind. Then we apply the proposed method on some examples to show the simplicity and efficiency of the method.

2. Assumptions and Definitions

Consider the following Volterra integral equation of the first kind

$$\int_0^s k(s,t)x(t)dt = y(s), \tag{1}$$

where $k(s,t)$ and $y(s)$ are known functions and $x(s)$ is unknown. Also suppose that $y(s)$ is a polynomial of s and $k(s,t)$ is a polynomial of $(s-t)$. Otherwise, we can substitute the Taylor expansion of $y(s)$ at $s=0$ and the Taylor expansion of $k(s,t)$ at $s=t$ (or $s=-t$ if it is suitable). Hence we assume

$$k(s,t) = \sum_{j=0}^{\infty} c_j (s-t)^j = c_o + c_1(s-t) + c_2(s-t)^2 + \cdots , \tag{2}$$

$$y(s) = \sum_{i=0}^{\infty} b_i s^i = b_o + b_1 s + b_2 s^2 + \cdots . \tag{3}$$

Let us consider the solution of (1) as

$$x(s) = \sum_{i=0}^{\infty} a_i s^i = a_o + a_1 s + a_2 s^2 + \cdots \tag{4}$$

and truncate $x(s)$ by

$$x_k(s) = \sum_{i=0}^{k} a_i s^i = a_o + a_1 s + a_2 s^2 + \cdots + a_k s^k, \tag{5}$$

then equation (1) writes as below

$$\int_0^s \left[c_o + c_1(s-t) + c_2(s-t)^2 + \cdots\right]\left[a_o + a_1 t + a_2 t^2 + \cdots\right] dt$$
$$= \left[b_o + b_1 s + b_2 s^2 + \cdots\right]. \tag{6}$$

Comparing both sides of (6) implies that $b_0 = 0$. The main study of the method is calculating the coefficients in (4).

3. Study of the Method

Theorem 1. *Under assumptions* (2), (3) *and* (4), *and* $k(s,s) \neq 0$ *for all* $s \geqslant 0$, *the coefficients of the expansion of solution of problem* (1) *satisfy the following recursive formula*

$$a_0 = \frac{b_1}{c_0} \tag{7}$$

$$a_n = (n+1)\frac{b_{n+1} - \sum_{i+j=n, j<n} c_i a_j u_{ij}}{c_0}, \qquad n = 1, 2, \cdots \tag{8}$$

where

$$u_{ij} = \frac{1}{s^{i+j+1}} \int_0^s (s-t)^i t^j dt = \sum_{r=0}^{i} \frac{\binom{i}{r}}{r+j+1}. \tag{9}$$

Proof. For obtaining a_0 note that the derivative of (7) is

$$k(s,s)x(s) + \int_0^s k_s(s,t)x(t)dt = y\prime(s)$$

$$\Rightarrow x(s) = \frac{y\prime(s)}{k(s,s)} - \int_0^s \frac{k_s(s,t)}{k(s,s)} x(t)dt$$

therefore

$$a_0 = x(0) = \frac{y\prime(0)}{k(0,0)} = \frac{b_1}{c_0}.$$

Now suppose

$$x(s) \approx x_1(s) = a_0 + a_1 s \tag{10}$$

substituting (10) in (7), we have

$$\int_0^s \left[c_o + c_1(s-t) + c_2(s-t)^2 + \cdots\right][a_o + a_1 t]\,dt = \left[b_1 s + b_2 s^2 + \cdots\right] \tag{11}$$

by neglecting all terms after s^2 or $(s-t)^2$ in both sides, the above equation reduce to

$$\int_0^s [c_1 a_0 (s-t) + c_0 a_1 t]\,dt = b_2 s^2$$

$$\Rightarrow c_1 a_0 u_{10} + c_0 a_1 u_{01} = b_2$$

but $u_{01} = \frac{1}{2}$ and then

$$a_1 = 2\frac{b_2 - c_1 a_0 u_{10}}{c_0}.$$

Similarly, for calculating a_n let

$$x(s) \approx x_n(s) = a_0 + a_1 s + \cdots + a_n s^n \tag{12}$$

and consider first $n+1$ terms in all factors in (12) to obtain,

$$\int_0^s \left[c_0 a_n t^n + c_1 a_{n-1}(s-t)t^{n-1} + \cdots + c_n a_0 (s-t)^n\right] dt = b_{n+1} s^{n+1}.$$

$$Thus \quad c_0 a_n u_{0n} + c_1 a_{n-1} u_{1,n-1} + \cdots + c_n a_0 u_{n0} = b_{n+1}.$$

Replacing u_{0n} by $\frac{1}{n+1}$, we have

$$a_n = (n+1)\frac{b_{n+1} - \sum_{i+j=n, j<n} c_i a_j u_{ij}}{c_0},$$

$$u_{ij} = \frac{1}{s^{i+j+1}} \int_0^s (s-t)^i t^j \,dt$$

$$= \frac{1}{s^{i+j+1}} \int_0^s \sum_{r=0}^{i} \binom{i}{r} s^{i-r} t^{r+j} dt = \sum_{r=0}^{i} \frac{\binom{i}{r}}{r+j+1}.$$

Hence the theorem is proved. $\square$

4. Convergence Analysis

In this section we prove that the above recursive method, converges to the solution of (8).

Theorem 2. *Let $k(s,t)$ and $y(s)$ be kernel and function in $C^{\infty}[a,b]$ and $C^{\infty}([a,b]\times[a,b])$, respectively . Then the solution of*

$$\int_0^s k(s,t)x(t)dt = y(s), \qquad a \leqslant s \leqslant b, \tag{13}$$

is also a function in $C^{\infty}[a,b]$.

Proof. See [1] and [2]. □

Theorem 3. *Let $x_n(s)$ be a solution of (8) that produced by recursive relations (7) and (8). Then $x_n(s)$ converges strongly to the solution of Volterra integral equation (1), when $n \to +\infty$.*

Proof. In (2) and (3) we consider the coefficients of Taylor expansion of $k(s,t)$ and $y(s)$. Thus coefficients of solution using (7) and (8) are also the coefficients of Taylor expansion. Therefore $x_n(s)$ converges strongly to $x(s)$, when $n \to +\infty$. On the other hand let

$$e_n(s) = y(s) - \int_0^s k(s,t)x_n(t)dt, \tag{14}$$

then we have

$$e_n(s) = \int_0^s k(s,t)x(t)dt - \int_0^s k(s,t)x_n(t)dt = \int_0^s k(s,t)[x(t)-x_n(t)]dt, \tag{15}$$

therefore

$$\|e_n\| \leqslant \ \|k\| \cdot \|x - x_n\| \tag{16}$$

since $\|k\|$ is bounded. Thus $\|x-x_n\| \to 0$ (*strongly*) implies that $\|e_n\| \to 0$ (*strongly*) and proof is completed. □

5. Numerical Examples

In this section we use the presented method for solving some examples. The computations associated with the examples were performed by Matlab 7.

Example 1. Consider the following first kind integral equation:

$$\int_0^s e^{s+t} x(t) dt = s e^s \tag{17}$$

where the exact solution is $x_T(s) = e^{-s}$ for $0 \leqslant s < 1$. This Example is selected from [3]. The Exact solution, Approximate solution of proposed method and Approximate solution obtained in [3] are denoted by $x_T(s)$, $\bar{x}_1(s)$ and $\bar{x}_2(s)$, respectively. To show accuracy of mentioned method, results of both methods are compared with exact solution by

$$E_i(s) = |x_T(s) - \bar{x}_i(s)|, \qquad i = 1, 2. \tag{18}$$

at 10 points. Results are shown in Table 1.

Table 1. Solution of example 1.

s	$x_T(s)$	$\bar{x}_1(s)$ n=10	$\bar{x}_2(s)$ m=64	$E_1(s)$	$E_2(s)$
0	1.00000000	1.00000000	0.994792	0	0.005208
0.1	0.90483742	0.90483742	0.905768	1.11×10^{-14}	0.000931
0.2	0.81873037	0.81873075	0.824711	5.55×10^{-14}	0.005980
0.3	0.74081822	0.74081822	0.735426	4.31×10^{-12}	0.005392
0.4	0.67032005	0.67032005	0.669613	1.01×10^{-10}	0.000707
0.5	0.60653066	0.60653066	0.603372	1.17×10^{-9}	0.003159
0.6	0.54881164	0.54881164	0.549376	8.65×10^{-9}	0.000564
0.7	0.49658530	0.49658530	0.500213	4.67×10^{-8}	0.003628
0.8	0.44932896	0.44932897	0.446058	2.01×10^{-7}	0.003271
0.9	0.40656966	0.40656967	0.406141	7.31×10^{-7}	0.000429

Example 2. Consider the following first kind integral equation:

$$\int_0^s \cos(s-t)x(t)dt = s\sin(s) \tag{19}$$

where the exact solution is $x_T(s) = 2\sin(s)$ for $0 \leqslant s < 1$. This Example is selected from [3]. Under notations of example 1, results are shown in Table 2.

Table 2. Solution of example 2.

s	$x_T(s)$	$\bar{x}_1(s)$ n=10	$\bar{x}_2(s)$ m=128	$E_1(s)$	$E_2(s)$
0	0.00000000	0.00000000	0.005208	0	0.005208
0.1	0.19966683	0.19966683	0.192399	2.77×10^{-15}	0.007267
0.2	0.39733866	0.39733866	0.398412	1.05×10^{-13}	0.001073
0.3	0.59104041	0.59104041	0.589930	8.87×10^{-12}	0.001110
0.4	0.77883668	0.77883668	0.785758	2.09×10^{-10}	0.006921
0.5	0.95885108	0.95885108	0.963098	2.44×10^{-9}	0.004246
0.6	1.1292849	1.1292849	1.122812	1.81×10^{-8}	0.006473
0.7	1.2884354	1.2884354	1.289847	9.87×10^{-8}	0.001412
0.8	1.4347122	1.4347122	1.433200	4.28×10^{-7}	0.001512
0.9	1.5666538	1.5666538	1.572171	1.56×10^{-6}	0.005517

Example 3. Consider the following first kind integral equation:

$$\int_0^s \frac{exp(s-t)}{1+s^2}x(t)dt = -\frac{4\pi\cos(4\pi s) + \sin(4\pi s) - 4\pi exp(s)}{(1+s^2)(1+16\pi^2)} \tag{20}$$

where the exact solution is $x_T(s) = \sin(4\pi s)$. This Example is selected from [4]. The coefficients of the expansion of solution are shown in table 3. The results agree with the coefficients of Taylor expansion of $x_T(s)$.

Table 3. Comparing expansion of proposed solution and Taylor series for example 3.

n	$a(n)$	Coefficients of Taylor expansion
0	0.0000	0
1	12.5664	12.5664
2	0.0000	0
3	-330.7336	-330.7336
4	0.0000	0
5	$2.6114 \times 10^{+3}$	$2.6114 \times 10^{+3}$
6	0.0000	0
7	$-9.8184 \times 10^{+3}$	$-9.8184 \times 10^{+3}$
8	0.0000	0
9	$2.1534 \times 10^{+4}$	$2.1534 \times 10^{+4}$
10	0.0000	0

6. Conclusion

In this paper, a recursive method based on calculating coefficients in expansion of solution was developed to approximate the solution of Volterra integral equations of first kind. Numerical examples show that this method have several advantages, as:

Convergence. Above theorems illustrate that the method is convergent.

Coefficient determination. Coefficient determination of the unknown function $x(s)$ using mentioned recursive method, don't need to solve any system of algebraic equations.

Accuracy. In terms of accuracy and error analysis, above examples show that our method is better than the methods discussed in [3] and [4].

References

[1] Kendall E. Atkinson, *The Numerical Solutions of Integral Equations of the Second Kind.* , Cambridge University Press, 1997.

[2] L.M. Delves and J. L. Mohammed, *Computational Methods for Integral Equations* , Cambridge University Press, 1985.

[3] E. Babolian and Z. Masouri, Direct method to solve Volterra integral equation of the first kind using operational matrix with block-pulse functions, *Journal of Computational and Applied Mathematics*, 220 (2008), 51-57.

[4] K. Maleknejad, R. Mollapourasl, and M. Alizadeh, Numerical solution of Volterra type integral equation of the first kind with wawelet basis, *Applied Mathematics and Computation*, 194 (2007), 400-405.

[5] A. Tahmasbi and O. S. Fard, Numerical solution of Linear Volterra integral equations system of the second kind, *Applied Mathematics and Computation*, 201 (2008), 547-552.

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