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# $C^*$-Algebra of Cancellative Semigroupoids

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**Abstract.** In this paper the definition and some properties of semigroupoids are considered. Representations, tight representations, and universal representations of a cancellative semigroupoid are discussed. Also, the $C^*$-algebra of a semigroupoid is introduced and it is shown that source elements transfer to zero by tight representations.

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## 1. Introduction

Because of the immensity of the class of all $C^*$-algebras it has become important to identify and study special types of $C^*$-algebras. The theory of $C^*$-crossed products by group actions, specially group $C^*$-algebras, $C^*(G)$, and reduced group $C^*$-algebras, $C_r^*(G)$, are very well developed. In 1982, J. R. Wordingham proved that the left regular representation of $\ell^1(S)$ on $\ell^2(S)$ is faithful ([11]). Following Wordingham, $C^*$-algebras of an inverse semigroup, has been investigated by Duncan and Paterson as a generalization of crossed product of discrete groups ([2, 3, 8]).

The notion of partial crossed product of a $C^*$-algebra by a discrete group is introduced by R. Exel ([4]) and generalized by McClanahan ([7]). Nándor Sieben in his master thesis, at the Arizona State University, under the supervision of J. Quigg defined the $C^*$-crossed product by action of an inverse semigroup and published the results in ([9]).

Partial actions of groups and actions of inverse semigroups have been studied by R. Exel in ([5]), where an inverse semigroup, $S(G)$, is associated to a given group $G$. R. Exel in ([5]) proved that there is a one-to-one correspondence between actions of $S(G)$ and partial actions of $G$. Also, he introduced a "partial" version of the group $C^*$-algebra, that is, partial group $C^*$-algebra, $C^*_P(G)$. Partial inverse semigroup $C^*$-algebra is introduced in ([10]). Now, following ([6]) we will consider the $C^*$-algebra of a cancellative semigroupoid.

The organization of this paper is as follows:

Semigroupoids and its properties are considered in Section 2. Section 3 is devoted to the representations, tight representations, universal representations, the $C^*$-algebra of a cancellative semigroupoid; and it is shown that source elements transfer to zero by tight representation.

## 2. Semigroupoids

In this section the concepts of semigroupoid, cancellative semigroupoid, divisiblity, and source element of a semigroupoid are introduced. An equivalence relation is defined on a special subset of a given semigroupoid. Also, it is shown that the disjoint union of the quotient space of this equivalence relation with the given semigroupoid is a new semigroupoid which has no source.

Let $G$ be a non-empty set and $G^{(2)}$ be a special subset of $G \times G$, that is, $G^{(2)}$ is the set of all ordered pairs on which a kind of multiplication is meaningful. With this in mind we have the following definition.

**Definition 2.1.** *By a semigroupoid $G$ we shall mean a triple $(G, G^{(2)}, .)$ such that*

$$. : G^{(2)} \longrightarrow G$$

*is an associative binary operation in the following sense:*
*For given $x, y, z \in G$ if either*
(i)$(x, y) \in G^{(2)}$ *and* $(y, z) \in G^{(2)}$, *or*
(ii) $(x, y) \in G^{(2)}$ *and* $(xy, z) \in G^{(2)}$, *or*
(iii) $(y, z) \in G^{(2)}$ *and* $(x, yz) \in G^{(2)}$,
*then all of $(x, y), (y, z), (xy, z)$ and $(x, yz)$ are in $G^{(2)}$ and $x(yz) = (xy)z$.*

**Example 2.2.** *Let $E = (E^1, E^0, r, s)$ be a graph. Then the path space of $E, F^+(E)$, consists of all finite paths including the vertices, is a semigroupoid with a product $xy$ if $s(x) = r(y)$. In particular, $x = xs(x) = r(x)x$.*

Before we give the definition of *divisiblity* we need to know that:

If a semigroupoid, say $G$, has not a unit element it is possible to add a unit element to it. That is, to pick some element from the universe of outside of $G$, call it 1, and set $\tilde{G} = G\dot{\cup}\{1\}$. Obviously, $1x = x1 = x$ for every $x$ in $G$.

It should be noted that $\tilde{G}$ may not be a semigroupoid. Because if it is a semigroupoid, since for given $x, y$ in $\tilde{G}$ the products $x1$ and $1y$ are meaningful we have $(x, 1)$ and $(1, y)$ are elements of $G^{(2)}$. By the Definition 2.1 we conclude that $xy = (x1)y$ is a meaningful product and we know that it is not always the case.

For given $x$ in $\tilde{G}$, we would like to determine the set of all elements of $G$, say $y$, for which $xy$ is meaningful. Therefore we have

$$G^x = \{y \in G : (x, y) \in G^{(2)}\} \text{ and } G^1 = G.$$

**Definition 2.3.** *Let $\tilde{G}$ be a unital semigroupoid and $x, y \in G$. We shall say that $x$ divids $y$ or $y$ is a multiple of $x$, in symbols $x|y$, if there exists $z$ in $\tilde{G}$ such that $(x, z) \in G^{(2)}$ and $y = xz$.*

**Lemma 2.4.** *The divisiblity relation is reflexive, transitive, and invariant under multiplication on the left.*

**Proof.** Let $x \in G$. Since $1x = x1 = x$, we see that the relation is reflexive. To prove the transitivity, if $x, y, z$ are in $G$ such that $x|y$ and $y|z$ we should show that $x|z$. If $x = y$ from $y|z$ we conclude that $x|z$.

Similarly if $y = z$, the relation $x|y$ shows that $x|z$. Otherewise from $x|y$ we have a $u$ in $G$ such that $(x, u) \in G^{(2)}$ and $y = xu$. Also, by $y|z$ we conclude that there exists $v$ in $G$ such that $(y, v) \in G^{(2)}$ and $z = yv$. Since $(x, u), (y, v) \in G^{(2)}$, that is $(x, u), (xu, v) \in G^{(2)}$ we see that $(u, v) \in G^{(2)}$. Consequently, $z = yv = (xu)v = x(uv)$. This shows that $x|z$.

To prove the last part of the lemma, let $x, y, k \in G, x|y, (k, x) \in G^{(2)}$ and $(k, y) \in G^{(2)}$. We should show that $ky$ is a multiple of $kx$. From $x|y$ we conclude that there exists $u$ in $G$ such that $(x, u) \in G^{(2)}$ and $xu = y$. Since $(k, x)$ and $(x, u)$ are elements of $G^{(2)}$ we see that $(kx, u) \in G^{(2)}$ and $(k, y) = (k, xu) \in G^{(2)}$. As a consequence we have $(kx)u = k(xu) = ky$, that is, $kx|ky$. This completes the proof. □

The following important concept is pivotal in our work.

**Definition 2.5.** *We shall say that an element $x \in G$ is cancellative if for every $y, z \in G$ the equation $xy = xz$ implies $y = z$. If every element of $G$ is cancellative, then $G$ is called a cancellative semigroupoid.*

*Some elements of $G$ has special properties, that is, given $x \in G$, there exists $y \in G$ such that $xy$ is not a legal multiplicative. Here, we would like to introduce the set of all such elements.*

**Definition 2.6.** *An element $x$ of $G$ is called source if $G^x = \phi$.*

*If $G^x \neq \phi$, then it is called the multiplicative set of $x$.*

*Here, we make an attempt to introduce a semigroupoid without sources.*

**Theorem 2.7.** *If $G$ is a semigroupoid which has sources, then there exists a semigroupoid which has no source and contains $G$.*

**Proof.** Let $G^0 = \{x \in G : x \text{ is a source }\}$. Also, let

$$\psi : G \longrightarrow G$$

defined by $\psi(x) = e'_x$ be a one-to-one map, and $E' = \psi(G)$. For any source $y$ and any $x$ such that $y \in G^x$, we observe that if $t \in G^y$, that is, $(y,t) \in G^{(2)}$ then $(xy,t) \in G^{(2)}$. This shows that $G^x \subseteq G^{xy}$. On the other hand if $s \in G^{xy}$, that is, $(xy,s) \in G^{(2)}$ then $(y,s) \in G^{(2)}$. So, $s \in G^y$ and $G^{xy} \subseteq G^y$. Consequently $G^y = G^{xy}$, and we conclude that $xy$ is also a source.

Let "$\sim$" be any equivalence relation on $E'$ such that $e'_{xy} \sim e'_y$ for any source $y$ and any $x$ for which $y \in G^x$. Also, let $e_x = [e'_x] = \{t \in E' : x \sim\ t\}$, and the quotient space, $\frac{E'}{\sim}$, be denoted by $E$. Take $\Gamma = G \dot{\cup} E$,

$$\Gamma^{(2)} = G^{(2)} \cup \{(y, e_y) : y \in G^0\} \cup \{(e_y, e_y) : y \in G^0\}, \text{ and}$$

define the multiplication

$$. : \Gamma^{(2)} \longrightarrow \Gamma$$

which is nothing but the multiplication on $G$ when restricted to $G^{(2)}$, with

$$y.e_y = e_y \ \ , \ \ e_y.e_y = e_y \qquad \forall y \in G^0.$$

Now we can prove that $(\Gamma, \Gamma^{(2)}, .)$ is a semigroupoid which contains $G$ and has no source. To show this, let $r, s, t \in \Gamma$. If $r, s, t \in G$ it is finished, otherwise $r = e_x$, $s = e_y$ and $t = e_z$ for some $x, y, z \in G^0$.
*Case 1.* If $e_x = y$ and $e_z = e_y$, then

$$(r, s) = (e_x, e_y) = (y, e_y) \in \Gamma^{(2)}, \text{ and } (s, t) = (e_y, e_z) = (e_y, e_y) \in \Gamma^{(2)}.$$

That is, $(r, s)$ and $(s, t) \in \Gamma^{(2)}$ and by Definition 2.1 part (i) we conclude that $\Gamma$ is a semigroupoid.
Proofs of other cases are similar to the proof of case 1 and is left to the reader. □

## 3. Representations of Semigroupoids

In this section the notion of representation of a semigroupoid is introduced. Also, a universal $C^*$-algebra is associated to a cancellative semigroupoid. The concept of a *tight representation* and the fact that a source element transfres to the zero operator by a tight representation are discussed.
Throught this section, $G$ is a semigroupoid and A is a unital $C^*$-algebra.

**Definition 3.1.** *Let $x, y \in G$. We shall say that $x$ and $y$ intersect if they have a common multiple, that is, if there exists an element $m$ of $G$ such that $x|m$ and $y|m$. The fact that $x$ and $y$ are intersect is denoted by $x \cap y$. Otherwise we will say that $x$ and $y$ are disjoint and is denoted* *by* $x \perp y$.

*The next concept is crucial in understanding the definition of a tight representation.*

**Definition 3.2.** *If* $X$ *is any subset of* $G$ *and* $Z \subseteq X$, *then* $Z$ *is called a covering of* $X$ *if for every* $x \in X$, *there exists* $h \in Z$ *such that* $x$ *and* $h$ *are intersect.*

*The next definition is the first step in bridging semigroupoids and operator algebras.*

**Definition 3.3.** *By a representation of* $G$ *in* $A$ *we mean a mapping*

$$\Pi : G \to A$$

*such that* $\Pi(x) = \Pi_x$ *is a partial isometry and if* $x, y \in G$ *then*

$$\Pi_x \Pi_y = \begin{cases} \Pi_{xy}, & \text{if } (x,y) \in G^{(2)}, \\ 0, & \text{otherwise.} \end{cases}$$

*Moreover the initial projections* $Q_x = \Pi_x^* \Pi_x$, *and the final projections* $P_y = \Pi_y \Pi_y^*$ *should commute among themselves and satisfy to the following conditions:*

(i)$P_x P_y = 0$, *if* $x \perp y$ ;

(ii) $Q_x P_y = P_y$ , *if* $(x,y) \in G^{(2)}$ ;

(iii) $Q_x P_y = 0$ , *if* $(x,y) \notin G^{(2)}$ .

It should be noted that any representation extends to $\tilde{G}$ by taking $\Pi(1) = \Pi_1 = 1$ and $Q_1 = P_1 = 1$.

Here we are able to present the reason why we choose cancellative semigroupoid.

If $G$ is not a cancellative semigroupoid, that is, there exists $x$ in $G$ such that for a distinct pair of elements $y, z \in G$ we have $xy = xz$. For given representation $\Pi$, since $\Pi_x$ is a partial isometry we have

$$\Pi_y = \Pi_y \Pi_y^* \Pi_y = (\Pi_y \Pi_y^*)\Pi_y = P_y \Pi_y = Q_x P_y \Pi_y$$

$$= \Pi_x^* \Pi_x \Pi_y \Pi_y^* \Pi_y = \Pi_x^* \Pi_x \Pi_y = \Pi_x^*(\Pi_x \Pi_y) = \Pi_x^* \Pi_{xy}.$$

And,

$$\Pi_z = \Pi_z \Pi_z^* \Pi_z = (\Pi_z \Pi_z^*)\Pi_z = P_z \Pi_z =$$

$$Q_x P_z \Pi_z = \Pi_x^* \Pi_x \Pi_z \Pi_z^* \Pi_z = \Pi_x^* \Pi_x \Pi_z = \Pi_x^*(\Pi_x \Pi_z) = \Pi_x^* \Pi_{xz}.$$

Since $xz = xy$ we have $\Pi_{xy} = \Pi_{xz}$, that is $\Pi_y = \Pi_z$ whereas $y \neq z$. This shows that if $G$ is not a cancellative semigroupoid, then we may have $\Pi(y) = \Pi(x)$ for some $x, y$ such that $x \neq y$.

Before we present the definition of a *tight* representation we need to know some more about representations.

For given $x \in G$ and $z \in G^x$, since $(x, z) \in G^{(2)}$ we know that the initial projection, $Q_x = \Pi_x^* \Pi_x$, and the final projection, $P_z = \Pi_z \Pi_z^*$, commute and $Q_x P_z = P_z$. Also, we know that $Q_x P_z = P_z$ is equivalent to $P_z \leqslant Q_x$. So, if $z_1, z_2 \in G^x$ we have $P_{z_1} \leqslant Q_x$ and $P_{z_2} \leqslant Q_x$.

Consequently $P_{z_1} \vee P_{z_2} \leqslant Q_x$, and if $H$ is a finite subset of $G^x$ we have

$$\bigvee_{z \in H} P_z \leqslant Q_x.$$

If $y \in \tilde{G}$ and $z \in G - G^y$ then $(y, z) \notin G^{(2)}$, hence $Q_y P_z = P_z Q_y = 0$. Therefore, from $P_z = P_z$ we have $P_z = P_z - P_z Q_y = P_z(1 - Q_y)$ which is equivalent to $P_z \leqslant 1 - Q_y$. Since $z$ is an arbitrary element of $G - G^y$ we conclude that if $H$ is a finite subset of $G - G^y$, then we have

$$\bigvee_{z \in H} P_z \leqslant 1 - Q_y$$

Now for given finite subsets $X, Y$ of $G$, let

$$G^{X,Y} = (\bigcap_{x \in X} G^x) \cap (\bigcap_{y \in Y} G - G^y).$$

If $z \in G^{X,Y}$, then from $z \in \bigcap_{x \in X} G^x$ we conclude that $P_z \leqslant Q_x$ for all $x \in X$ and as a consequence

$$P_z \leqslant \prod_{x \in X} Q_x. \tag{1}$$

Also, from $z \in \bigcap_{y \in Y} G - G^y$ we have $P_z \leqslant 1 - Q_y$ for all $y \in Y$, and consequently

$$P_z \leqslant \prod_{y \in Y} (1 - Q_y). \tag{2}$$

From (1) and (2), for given $z \in G^{X,Y}$ we have

$$P_z \leqslant \prod_{x \in X} Q_x \prod_{y \in Y} (1 - Q_y).$$

Also, for given finite subset $H$ of $G^{X,Y}$, we conclude that

$$\bigvee_{z\in H} P_z \leqslant \prod_{x\in X} Q_x \prod_{y\in Y}(1-Q_y).$$

With this in mind we can present the following important definition.

**Definition 3.4.** *A representation $\Pi$ of $G$ in $A$ is said to be tight if for every subsets $X, Y$ of $\tilde{G}$ and every covering $H$ of $G^{X,Y}$ the following equality holds*

$$\bigvee_{z\in H} P_z = \prod_{x\in X} Q_x \prod_{y\in Y}(1-Q_y).$$

It should be noted that if no such covering exists, then any representation is tight vacuously.

Before we present the definition of the $C^*$-algebra of a semigroupoid we need to introduce the concept of a universal $C^*$-algebra.

In recent years, universal constructions play a crucial role in the theory of operator algebras, specially in the theory of $C^*$-algebras. In other words, many important $C^*$-algebras can be expressed as universal $C^*$-algebras generated by a given set and a set of relations which satisfy in certain conditions. In what follows we will describe that, what do we mean by a universal $C^*$-algebra generated by a set and a set of relations.

Suppose a set $B = \{b_i : i \in \Omega\}$ of generators and a set $R$ of relations are given. It should be noted that the relations can be of a very general

nature. Usually, some algebraic relations between generators and their adjoints exist. The only restriction on the relation is that:
(i) they must be realizable among operators on a Hilbert space.
(ii) each generator should have an upper bound when realized as an operator.

A *representation* of $(B|R)$ is a set $\{T_i : i \in \Omega\}$ of bounded operators on a Hilbert space $H$ which satisfying in the given relations. Each such representation of $(B|R)$ defines a $*$-representation of the free $*$-algebra $\mathcal{A}$ on the set $B$. For given $x \in \mathcal{A}$, let

$$||x|| = \sup\{||\Pi(x)|| : \Pi \text{ is a representation of } (B|R)\}.$$

This supremum defines a $C^*$-seminorm on $\mathcal{A}$ provided that it is finite. If the elements of seminorm 0 are divided out, then the completion of $\mathcal{A}$ is called the *universal $C^*$-algebra generated by* $B, R$, or the *universal $C^*$-algebra on* $(B|R)$, and is denoted by $C^*(B|R)$.

**Example 3.5.** *Let $B = \{x\}$ and $R = \{x = x^*, ||x|| < 1\}$. Then $C^*(B|R)$ is the universal $C^*$-algebra generated by a single self-adjoint element of norm 1.*

Note that there is no universal $C^*$-algebra generated by a single self-adjoint element, because there is no bound on the norm of the element. For more on universal $C^*$-algebras, see Chapter II of [1].

Here, we introduce a universal $C^*$-algebra which contains the $C^*$-algebra of semigroupoid.

**Definition 3.6.** *Let $G$ be a semigroupoid, $B = \{\Pi_x\}_{x\in G}$ be a family of partial isometries and $R$ be the set of all relations such that the correspondence $x \to \pi_x$ is a tight representation of $G$. The unital universal $C^*$-algebra generated by $B, R$, that is, $C^*(B|R)$ denoted by $\tilde{O}(G)$.*
*In order to give the definition of the $C^*$-algebra of a semigroupoid $G$ we need to know that, what do we mean by the universal representation of $G$?*

**Definition 3.7.** *A collection of partial isometries, $\{\Pi_x\}_{x\in G}$, such that the correspondence $x \to \Pi_x$ is a tight representation of $G$ is called the universal representation of $G$.*
*Now, we are ready to present the definition of the $C^*$-algebra of a semigroupoid.*

**Definition 3.8.** *The closed $*$-subalgebra of $\tilde{O}(G)$ which is generated by the range of the universal representation of $G$ denoted by $O(G)$, is the $C^*$-algebra of $G$.*
*We close this section by the following important theorem.*

**Theorem 3.9.** *If $\Pi$ is a tight representation of a semigroupoid, $G$ and $x \in G$ is a source element, then $\Pi_x = 0$.*

**Proof.** Since $x$ is a source element, we have $G^x = \phi$. From the fact that empty set is a covering for $G^x$, we conclude that $Q_x = 0$. Since $Q_x = \Pi_x^* \Pi_x$, we have $\Pi_x = 0$. $\square$

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