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Robust Improvement in Estimation of a Covariance Matrix in an Elliptically Contoured Distribution Respect to Quadratic Loss Function

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Abstract. Let $\mathbf{S}$ be matrix of residual sum of square in linear model $\mathbf{Y} = \mathbf{A}\boldsymbol{\beta} + \mathbf{e}$ where matrix $\mathbf{e}$ is distributed as elliptically contoured with unknown scale matrix $\mathbf{\Sigma}$. In present work, we consider the problem of estimating $\mathbf{\Sigma}$ with respect to squared loss function, $L(\hat{\mathbf{\Sigma}}, \mathbf{\Sigma}) = \mathrm{tr}(\hat{\mathbf{\Sigma}}\mathbf{\Sigma}^{-1} - \mathbf{I})^2$. It is shown that improvement of the estimators were obtained by James, Stein [7], Dey and Srivasan [1] under the normality assumption remains robust under an elliptically contoured distribution respect to squared loss function.

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1. Introduction and Summary

Consider the multivariate linear model

$$\mathbf{Y} = \mathbf{A}\boldsymbol{\beta} + \mathbf{e},$$

where $\mathbf{Y}$ is an $N \times p$ random matrix, $\mathbf{A}$ is a known $N \times m$ matrix with full rank, $\boldsymbol{\beta}$ is an $m \times p$ matrix of unknown parameters and $\mathbf{e}$ is an $N \times p$ matrix of random errors. We assume that the error matrix $\mathbf{e}$ is distributed as an elliptically contoured distribution with its density

$$|\boldsymbol{\Sigma}|^{-\frac{N}{2}} f\big(\mathrm{tr}(\boldsymbol{\Sigma}^{-1}\mathbf{e}'\mathbf{e})\big), \tag{1}$$

where $f(\cdot)$ is a differentiable and nonnegative real-value function on $\mathbf{R}^{+}$ and $\boldsymbol{\Sigma}$ is a positive-definite matrix. In this article $|\mathbf{B}|$, $\mathbf{B}'$ and $\mathrm{tr}(\mathbf{B})$ used for the determinant, the transpose and the trace of a square matrix $\mathbf{B}$, respectively. The model (1) is called elliptically contoured distribution which we refer to as the ECD model in this paper.

There were many studies to robust improvement estimation of a covariance matrix and precision matrix under elliptically contoured distribution. For the estimation of the covariance matrix and precision matrix see [7,5], respectively. Kubokawa and Srivastava [7] Hisayuki Tsukuma [5] showed that improvement of a minimax estimator for a covariance matrix and the precision matrix obtained under normality assumption remains robust under an elliptically contoured distribution respect to Stein's loss function, respectively. In this paper, the problem of estimating the scale matrix of an elliptically contoured distribution with respect to squared loss function is investigated.

The identity for the elliptically contoured distribution which was derived by Kubokawa and Srivastava [7] known in the literature as the "Stein-Haff identity", is applied to compute risk functions. We define the risk functions as the expected value of the loss functions, i.e.

$$R(\hat{\mathbf{\Sigma}}, \mathbf{\Sigma}) = E[L(\hat{\mathbf{\Sigma}}, \mathbf{\Sigma})|\mathbf{\Sigma}].$$

Let $\hat{\mathbf{\Sigma}}$ and $\hat{\mathbf{\Sigma}}_*$ be two estimators of $\mathbf{\Sigma}$, $\hat{\mathbf{\Sigma}}$ dominates $\hat{\mathbf{\Sigma}}_*$ if $R(\hat{\mathbf{\Sigma}}, \mathbf{\Sigma}) \leqslant R(\hat{\mathbf{\Sigma}}_*, \mathbf{\Sigma})(\forall \mathbf{\Sigma})$ ([3, 4]).

The unbiased estimator $\hat{\mathbf{\Sigma}}_{UB} = n^{-1}\mathbf{S}$ is the best estimator in the estimators of the kind $a\mathbf{S}$ where a is scalar that it has the minimum risk for squared loss defined above. James and Stein [6] obtained a minimax estimator of the form

$$\hat{\mathbf{\Sigma}}_{JS} = \mathbf{TDT}',$$

where $\mathbf{D} = \mathrm{diag}(d_1, \cdots, d_p)$ with $d_i = (n + p + 1 - 2i)^{-1}$, $i = 1, 2, \cdots$ and $\mathbf{T}$ is a $p \times p$ lower triangular matrix with positive diagonal elements such that $\mathbf{S} = \mathbf{TT}'$. Stein [9, 10], Dey and Srinivasan [1] obtained an orthogonally invariant estimator

$$\hat{\mathbf{\Sigma}}_{SDS} = \mathbf{H}\mathrm{diag}(d_1 l_1, \cdots, d_p l_p)\mathbf{H}',$$

where $\mathbf{H}$ is a $p \times p$ orthogonal matrix and $l_1, \cdots, l_p$ are the ordered eigenvalues of the random matrix $\mathbf{S}$ such that $\mathbf{S} = \mathbf{H}\text{diag}(l_1, \cdots, l_p)\mathbf{H}'$ [8]. Kubokawa and Srivastava [7] showed that James-Stein estimator $\hat{\mathbf{\Sigma}}_{JS}$ dominates the unbiased estimator $\hat{\mathbf{\Sigma}}_{UB}$ and Orthogonally invariant estimator $\hat{\mathbf{\Sigma}}_{SDS}$ dominates $\hat{\mathbf{\Sigma}}_{JS}$ under Stein's loss for any possible function of f in (1). Our objective is to establish that the above dominance results hold for every ECD model under squared loss. Based on Haff-stein identity we prove the robustness of the two dominance result, $\hat{\mathbf{\Sigma}}_{JS}$ improving $\hat{\mathbf{\Sigma}}_{UB}$ and $\hat{\mathbf{\Sigma}}_{SDS}$ improving $\hat{\mathbf{\Sigma}}_{JS}$.

2. Expected Values

Let $\mathbf{S}$ be matrix of residual sum of square, i.e.,

$$\mathbf{S} = \mathbf{Y}^{'}(\mathbf{I}_N - \mathbf{A}(\mathbf{A}^{'}\mathbf{A})^{-1}\mathbf{A}^{'})\mathbf{Y}.$$

Under the normality and ECD assumption, the expected values for various function of $\mathbf{S}$ have been derived. see [2,5], respectively. According to the same notation used in [5], let $f^{(0)}(x) = f(x)$,

$$f^{(k+1)}(x) = \frac{1}{2}\int_x^\infty f^{(k)}(t)dt, \quad k = 0, 1,$$

and

$$E^{(k)}_{\mathbf{\Sigma}}[v(\mathbf{S})] = \int v(\mathbf{S})\ |\mathbf{\Sigma}|^{-\frac{N}{2}} f^{(k)}\Big(\text{tr}\big(\mathbf{\Sigma}^{-1}(\mathbf{y} - \mathbf{A}\boldsymbol{\beta})^t(\mathbf{y} - \mathbf{A}\boldsymbol{\beta})\big)\Big)d\mathbf{y},$$

where $v(\mathbf{S})$ is an integrable function of $\mathbf{S}$. Moreover, we use the transformation to polar coordinates to get

$$E_{\mathbf{\Sigma}}^{(k)}[1] = \gamma(k) = \frac{2\pi^{NP/2}}{\Gamma(NP/2)} \int_0^\infty r^{Np-1} f^{(k)}(r^2) dr, \quad k = 0, 1,$$

and assume $\gamma(i) < \infty$, for more details see [5].

2.1. The Stein-Haff Identity

Let $T(\mathbf{S}) = (t_{ij}(\mathbf{S}))$ be a $p \times p$ matrix whose elements are the function of $\mathbf{S} = (s_{ij})$. Denote

$$\{D_{\mathbf{S}} T(\mathbf{S})\}_{ij} = \sum_{i=1}^{p} \frac{1}{2}(1 + \delta_{ia}) \frac{\partial t_{aj}(\mathbf{S})}{\partial s_{ia}},$$

where δ_{ia} is Kronecker's delta. From Lemma 1 in [7], for suitable choice of a matrix $T(\mathbf{S})$, the Stein-Haff identity is given by

$$E_{\mathbf{\Sigma}}^{(k)}\Big[\mathrm{tr}\{\mathbf{\Sigma}^{-1} T(\mathbf{S})\}\Big] = E_{\mathbf{\Sigma}}^{(k+1)}\Big[(n-p-1)\ \mathrm{tr}\{\mathbf{S}^{-1} T(\mathbf{S})\} + 2\ \mathrm{tr} D_{\mathbf{S}} T(\mathbf{S})\Big]. \tag{2}$$

3. The Main Results

The following two theorems are the main results of this paper. The proofs are postponed to section 4.

Theorem 3.1. *for any $f(\cdot)$ in (1), the James-Stein estimator $\hat{\mathbf{\Sigma}}_{JS}$ dominates unbiased estimator $\hat{\mathbf{\Sigma}}_{UB}$ under squared loss if $n - p + 1 > 0$ and p=1 .*

Theorem 3.2. *$\hat{\mathbf{\Sigma}}_{SDS}$ is better than $\hat{\mathbf{\Sigma}}_{JS}$ uniformly for every unknown $f(\cdot)$ under squared loss if*

$$n\gamma(2)\mathrm{tr}(\mathbf{\Sigma}^{-1}\mathbf{HDH'\Sigma HDH'}) \leqslant 4E^{(1)}_{\mathbf{\Sigma}}[\sum_{i>j}\frac{d_i-d_j}{l_i-l_j}]. \tag{3}$$

4. Mathematical Details

4.1. Preliminaries

The latter calculations depends on Lemmas below:

Lemma 4.1 [Hisayuki Tsukuma 2005]. *Let $\mathbf{Q}$ be a $p\times p$ matrix of constants. Under the conditions of Lemma* 2.1. *in [5], we have*
i) $E^{(0)}_{\mathbf{\Sigma}}[\mathbf{S}] = n\gamma(1)\mathbf{\Sigma}$,
ii)$E^{(0)}_{\mathbf{\Sigma}}[\mathbf{SQS}] = \gamma(2)\{n^2\mathbf{\Sigma Q\Sigma} + n\mathbf{\Sigma Q}^t\mathbf{\Sigma} + n\ \mathrm{tr}(\mathbf{Q\Sigma})\mathbf{\Sigma}\}$.

Lemma 4.2. *Under the conditions of Lemma* 1. *in [7], we have*
i) $E^{(0)}_{\mathbf{\Sigma}}[\mathrm{tr}(n^{-1}\mathbf{S\Sigma}^{-1})] = p\gamma(1)$,
ii) $E^{(0)}_{\mathbf{\Sigma}}[\mathrm{tr}(n^{-1}\mathbf{S\Sigma}^{-1})^2] = n^{-2}\gamma(2)[(n^2+n)p+np^2]$.

Proof. (i): Using Lemma 4.1. we obtain

$$E^{(0)}_{\mathbf{\Sigma}}[\mathrm{tr}(n^{-1}\mathbf{S\Sigma}^{-1})] = E^{(0)}_{\mathbf{I}}[\mathrm{tr}(n^{-1}\mathbf{S})] = p\gamma(1).$$

(ii): Taking $\mathbf{Q}=\mathbf{I}$ in Lemma 4.1. we have

$$\begin{aligned} E^{(0)}_{\mathbf{\Sigma}}[\mathrm{tr}(n^{-1}\mathbf{S\Sigma}^{-1})^2] &= E^{(0)}_{\mathbf{I}}[\mathrm{tr}(n^{-1}\mathbf{S})^2] \\ &= n^{-2}\gamma(2)[(n^2+n)p+np^2]. \end{aligned}$$

Lemma 4.3. *Let* $\mathbf{D} = \mathrm{diag}(d_1, \ldots, d_p)$ *with* $d_i = (n+p+1-2i)^{-1}, i = 1, 2, \ldots, p$ *and* $\mathbf{T}$ *is a* $p \times p$ *lower triangular matrix with positive diagonal elements such that* $\mathbf{S} = \mathbf{TT}'$*. Under the conditions of Lemma 2.1. in [5], we have*

i) $E_{\mathbf{\Sigma}}^{(0)}[\mathrm{tr}\mathbf{TDT}'\mathbf{\Sigma}^{-1}] = pE_{\mathbf{\Sigma}}^{(1)}[1] = p\gamma(1)$,

ii) $E_{\mathbf{\Sigma}}^{(0)}[\mathrm{tr}(\mathbf{TDT}')^2] = \gamma(2)[(n^2+n)\sum_{i=1}^{p} d_i^2 + n(\sum_{i=1}^{p} d_i)^2]$.

Proof. (i): By using the equation $E_{\mathbf{I}}^{(0)}(\mathbf{T}'\mathbf{T}) = \mathbf{D}^{-1}\gamma(1)$ [7], we arrive at

$$\begin{aligned} E_{\mathbf{\Sigma}}^{(0)}[\mathrm{tr}\mathbf{TDT}'\mathbf{\Sigma}^{-1}] &= E_{\mathbf{I}}^{(0)}[\mathrm{tr}\mathbf{TDT}'] \\ &= E_{\mathbf{I}}^{(0)}[\mathrm{tr}\mathbf{DT}'\mathbf{T}] \\ &= p\gamma(1), \end{aligned}$$

which gives the desired result.

(ii): From Lemma 4.1. with $\mathbf{Q} = D$ and $\mathbf{S} = \mathbf{T}'\mathbf{T}$, we can see that

$$\begin{aligned} E_{\mathbf{\Sigma}}^{(0)}[\mathrm{tr}(\mathbf{TDT}'\mathbf{\Sigma}^{-1})^2] &= E_{\mathbf{I}}^{(0)}[\mathrm{tr}(\mathbf{TDT}')^2] \\ &= E_{\mathbf{I}}^{(0)}[\mathrm{tr}(\mathbf{TDT}'\mathbf{TDT}')] \\ &= E_{\mathbf{I}}^{(0)}[\mathrm{tr}(\mathbf{T}'\mathbf{TDT}'\mathbf{TD})] \\ &= E_{\mathbf{I}}^{(0)}[\mathrm{tr}(\mathbf{SDSD})] \\ &= \gamma(2)[n^2\mathrm{tr}\mathbf{D}^2 + n\mathrm{tr}\mathbf{D}^2 + n(\mathrm{tr}\mathbf{D})^2] \\ &= \gamma(2)[(n^2+n)\sum_{i=1}^{p} d_i^2 + n(\sum_{i=1}^{p} d_i)^2]. \ \square \end{aligned}$$

Lemma 4.4 ([7]). *Let* $\mathbf{S} = \mathbf{HLH}'$, $\mathbf{L} = \mathrm{diag}(l_1, \ldots, l_p)$ *and* $l_1 \geqslant \cdots \geqslant l_p$. *Also, consider the estimator* $\hat{\mathbf{\Sigma}}(\mathbf{\Phi}) = \mathbf{H}\ \mathrm{diag}(\Phi_1(L), \ldots, \Phi_p(L))\mathbf{H}'$. *Then, under suitable conditions corresponding to those of Lemma 2.2. in [7], we have*

$$\begin{aligned} E^{(0)}_{\mathbf{\Sigma}}\left[\mathrm{tr}\hat{\mathbf{\Sigma}}(\mathbf{\Phi})\mathbf{\Sigma}^{-1}\right] &= E^{(1)}_{\mathbf{\Sigma}}\Big[2\textstyle\sum_{i\neq j}\frac{\Phi_i(L)}{l_i-l_j} + 2\sum_i \frac{\partial\Phi_i(L)}{\partial l_i} \\ &\quad +(n-p-1)\textstyle\sum_i \frac{\Phi_i(L)}{l_i}\Big] \\ &= E^{(1)}_{\mathbf{\Sigma}}\Big[2\textstyle\sum_{i>j}\frac{\Phi_i(L)-\Phi_j(L)}{l_i-l_j} + 2\sum_i \frac{\partial\Phi_i(L)}{\partial l_i} \\ &\quad +(n-p-1)\textstyle\sum_i \frac{\Phi_i(L)}{l_i}\Big]. \ \square \end{aligned}$$

Lemma 4.5. *Let* $\mathbf{S} = \mathbf{HLH}'$, $\mathbf{L} = \mathrm{diag}(l_1, \ldots, l_p)$ *and* $l_1 \geqslant \cdots \geqslant l_p$. *Moreover, let* $\mathbf{H}$ *be* $p \times p$, *orthogonal matrix and also, consider the estimator* $\hat{\mathbf{\Sigma}}_{SDS} = \mathbf{HDLH}'$ *and* $\mathbf{D} = \mathrm{diag}(d_1, \ldots, d_p)$. *In addition, let* $\mathbf{HDH}'$ *be a* $p \times p$ *matrix of constant, under the conditions of Lemma 2.2. in [7], we have*

$$\begin{aligned} &\text{i)}E^{(0)}_{\mathbf{\Sigma}}\left[\mathrm{tr}(\hat{\mathbf{\Sigma}}_{SDS}\mathbf{\Sigma}^{-1})\right] = E^{(1)}_{\mathbf{\Sigma}}\left[2\sum_{i>j}\frac{d_i-d_j}{l_i-l_j}\right] + p\gamma(1), \\ &\text{ii)}E^{(0)}_{\mathbf{\Sigma}}\left[\mathrm{tr}(\hat{\mathbf{\Sigma}}_{SDS}\mathbf{\Sigma}^{-1})^2\right] = \gamma(2)\Bigg[n^2\sum_{i=1}^{p} d_i^2 + n(\sum_{i=1}^{p} d_i)^2 \\ &\qquad\qquad + n\mathrm{tr}(\mathbf{\Sigma}^{-1}\mathbf{HDH}'\mathbf{\Sigma}\mathbf{HDH}')\Bigg]. \end{aligned}$$

Proof. Under the given condition, using Lemma 4.4. with $\mathbf{\Phi}(\mathbf{L}) = \mathbf{DL}$.

We can also see that

$$\begin{aligned} E^{(0)}_{\mathbf{\Sigma}}[\mathrm{tr}(\mathbf{HDLH'\Sigma}^{-1}] &= E^{(1)}_{\mathbf{\Sigma}}\Big[2\textstyle\sum_{i>j}\frac{d_il_i-d_jl_j}{l_i-l_j} \\ &+2\textstyle\sum_i\frac{\partial d_il_i}{\partial l_i}+(n-p-1)\sum_i\frac{d_il_i}{l_i}\Big] \\ &= E^{(1)}_{\mathbf{\Sigma}}\Big[2\textstyle\sum_{i>j}\frac{d_il_i-d_jl_j}{l_i-l_j} \\ &+2\textstyle\sum_i^p d_i+(n-p-1)\sum_i^p d_i] \,. \end{aligned}$$

Additionally, by using

$$\frac{d_il_i-d_jl_j}{l_i-l_j} = \tfrac{d_il_i-d_jl_i+d_jl_i+d_jl_j}{l_i-l_j} = \tfrac{d_i-d_j}{l_i-l_j}l_i+d_j,$$

we arrive at

$$\begin{aligned} E^{(0)}_{\mathbf{\Sigma}}[\mathrm{tr}(\mathbf{HDLH'\Sigma}^{-1})] &= E^{(1)}_{\mathbf{\Sigma}}[2\textstyle\sum_{i>j}\frac{d_i-d_j}{l_i-l_j}l_i \\ &\quad +2\textstyle\sum_{i>j}d_j+(n-p+1)\sum_i^p d_i]. \end{aligned}$$

By using the equation

$$\sum_{i>j}d_j=\sum_{i=1}^{p}\sum_{j=1}^{i-1}d_j=\sum_{j=1}^{p}\sum_{i=j+1}^{p}d_j=\sum_{j=1}^{p}(p-j)d_j,$$

we have

$$\begin{aligned} &E^{(0)}_{\mathbf{\Sigma}}[\mathrm{tr}(\mathbf{HDLH'\Sigma}^{-1})] \\ &\qquad = E^{(1)}_{\mathbf{\Sigma}}\left[2\sum_{i>j}\frac{d_i-d_j}{l_i-l_j}l_i+\sum_i(n+p+1-2i)d_i\right] \\ &\qquad = E^{(1)}_{\mathbf{\Sigma}}\left[2\sum_{i>j}\frac{d_i-d_j}{l_i-l_j}l_i+p\right] \\ &\qquad = E^{(1)}_{\mathbf{\Sigma}}\left[2\sum_{i>j}\frac{d_i-d_j}{l_i-l_j}l_i\right]+p\gamma(1), \end{aligned}$$

which completes the proof.

Proof. Using the equation $\mathbf{S} = \mathbf{HLH}' and \mathbf{HH}' = \mathbf{I}_p$, we arrive at

$$\begin{aligned} tr(\hat{\mathbf{\Sigma}}_{SDS}\mathbf{\Sigma}^{-1})^2 &= tr(\hat{\mathbf{\Sigma}}_{SDS}\mathbf{\Sigma}^{-1}\hat{\mathbf{\Sigma}}_{SDS}\mathbf{\Sigma}^{-1}) \\ &= \mathrm{tr}(\mathbf{HDLH}'\mathbf{\Sigma}^{-1}\mathbf{HDLH}'\mathbf{\Sigma}^{-1}) \\ &= \mathrm{tr}(\mathbf{HDH}'\mathbf{HLH}'\mathbf{\Sigma}^{-1}\mathbf{HDH}'\mathbf{HLH}'\mathbf{\Sigma}^{-1}) \\ &= \mathrm{tr}(\mathbf{HDH}'\mathbf{S}\mathbf{\Sigma}^{-1}\mathbf{HDH}'\mathbf{S}\mathbf{\Sigma}^{-1}) \\ &= \mathrm{tr}(\mathbf{\Sigma}^{-1}\mathbf{HDH}'\mathbf{S}\mathbf{\Sigma}^{-1}\mathbf{HDH}'\mathbf{S}). \end{aligned}$$

Now, using Lemma 4.1. with $\mathbf{Q} = \mathbf{\Sigma}^{-1}\mathbf{HDH}'$, we arrive at

$$\begin{aligned} E^{(0)}_{\mathbf{\Sigma}} \quad & [\mathrm{tr}(\hat{\mathbf{\Sigma}}_{SDS}\mathbf{\Sigma}^{-1})^2] = E^{(0)}_{\mathbf{\Sigma}}[\mathrm{tr}(\mathbf{\Sigma}^{-1}\mathbf{HDH}'\mathbf{S}\mathbf{\Sigma}^{-1}\mathbf{HDH}'\mathbf{S})] \\ & = \gamma(2)[n^2\mathrm{tr}(\mathbf{\Sigma}^{-1}\mathbf{HDH}'\mathbf{HDH}'\mathbf{\Sigma}) + n\mathrm{tr}(\mathbf{\Sigma}^{-1}\mathbf{HDH}'\mathbf{\Sigma}\mathbf{HDH}') \\ & + n\mathrm{tr}(\mathbf{D})\ \mathrm{tr}(\mathbf{\Sigma}^{-1}\mathbf{HDH}'\mathbf{\Sigma})] = \gamma(2)[n^2\mathrm{tr}(\mathbf{HD}^2\mathbf{H}'\mathbf{\Sigma}\mathbf{\Sigma}^{-1}) \\ & + n\mathrm{tr}(\mathbf{\Sigma}^{-1}\mathbf{HDH}'\mathbf{\Sigma}\mathbf{HDH}') + n\mathrm{tr}(\mathbf{D})\mathrm{tr}(\mathbf{HDH}'\mathbf{\Sigma}\mathbf{\Sigma}^{-1})] \\ & = \gamma(2)[n^2\mathrm{tr}(\mathbf{D}^2\mathbf{H}'\mathbf{H}) \\ & + n\mathrm{tr}(\mathbf{\Sigma}^{-1}\mathbf{HDH}'\mathbf{\Sigma}\mathbf{HDH}') + n\mathrm{tr}(\mathbf{D})\ \mathrm{tr}(\mathbf{DH}'\mathbf{H})] \\ & = \gamma(2)[n^2\mathrm{tr}(\mathbf{D}^2) + n\mathrm{tr}(\mathbf{\Sigma}^{-1}\mathbf{HDH}'\mathbf{\Sigma}\mathbf{HDH}') + n\mathrm{tr}(\mathbf{D}^2)] \\ & = \gamma(2)[n^2\textstyle\sum_i^p d_i^2 + n(\sum_i^p d_i)^2 + n\mathrm{tr}(\mathbf{\Sigma}^{-1}\mathbf{HDH}'\mathbf{\Sigma}\mathbf{HDH}')]. \ \square \end{aligned}$$

4.2. Proofs of the Results in Section 3

Proof of Theorem 3.1. The risk difference of the estimators $\hat{\mathbf{\Sigma}}_{JS}$ and $\hat{\mathbf{\Sigma}}_{UB}$ relative to squared loss is written as

$$\begin{aligned} \Delta_1 &= R(\hat{\mathbf{\Sigma}}_{UB}, \mathbf{\Sigma}) - R(\hat{\mathbf{\Sigma}}_{JS}, \mathbf{\Sigma}) \\ &= E^{(0)}_{\mathbf{\Sigma}}[\mathrm{tr}(\hat{\mathbf{\Sigma}}_{UB}\mathbf{\Sigma}^{-1})^2 - 2\mathrm{tr}(\hat{\mathbf{\Sigma}}_{UB}\mathbf{\Sigma}^{-1}) - \mathrm{tr}(\hat{\mathbf{\Sigma}}_{JS}\mathbf{\Sigma}^{-1})^2 + 2\mathrm{tr}(\hat{\mathbf{\Sigma}}_{JS}\mathbf{\Sigma}^{-1})] \\ &= E^{(0)}_{\mathbf{I}}[\mathrm{tr}(n^{-1}\mathbf{S})^2 - 2\mathrm{tr}(n^{-1}\mathbf{S}) - \mathrm{tr}(\mathbf{TDT}')^2 + 2\mathrm{tr}(\mathbf{TDT}')] \\ &= \frac{1}{n^2}E^{(0)}_{\mathbf{I}}[\mathrm{tr}\mathbf{S}^2] - \frac{2}{n}E^{(0)}_{\mathbf{I}}(\mathrm{tr}\mathbf{S}) - E^{(0)}_{\mathbf{I}}[\mathrm{tr}(\mathbf{TDT}')^2] + 2E^{(0)}_{\mathbf{I}}[\mathrm{tr}(\mathbf{TDT}')]. \end{aligned}$$

From Lemmas 4.2. and 4.3., we have

$$\Delta_1 \ = \gamma(2)[\tfrac{p}{n}(n+p+1) - (n^2+n)\textstyle\sum_{i=1}^p d_i^2 - n(\sum_{i=1}^p d_i)^2].$$

To complete the proof, it is enough to show that

$$\frac{p}{n}(n+p+1)-(n^2+n)\sum_{i=1}^{p} d_i^2 - n(\sum_{i=1}^{p} d_i)^2 \geqslant 0 \ .$$

From $i \leqslant p \leqslant n+1, i=1,\ldots,p$, we have $d_i \leqslant (n-p+1)^{-1}$,

$$\begin{array}{ll} \sum_{i=1}^{p} d_i^2 & \leqslant p(n-p+1)^{-2}, \\ (\sum_{i=1}^{p} d_i)^2 & \leqslant p^2(n-p+1)^{-2}. \end{array}$$

Therefore,

$$\begin{array}{ll} \frac{p}{n}(n+p+1)-(n^2+n) & \\ & \sum_{i=1}^{p} d_i^2 - n(\sum_{i=1}^{p} d_i)^2 \geqslant \frac{p}{n}(n+p+1) \\ & -np(n-p+1)^{-2}(n+p+1) \\ & \geqslant \frac{p}{n}(n+p+1)\big[1-(\frac{n}{n-p+1})^2\big]. \end{array}$$

The assumption $p \leqslant 1$ or $p=1$ completes the proof. □

Proof of Theorem 3.2. Using Lemmas 4.3. and 4.5., we can write the risk difference of estimators $\hat{\mathbf{\Sigma}}_{SDS}$ and $\hat{\mathbf{\Sigma}}_{JS}$ as

$$\begin{aligned} \Delta_2 &= R(\hat{\mathbf{\Sigma}}_{SDS},\mathbf{\Sigma}) - R(\hat{\mathbf{\Sigma}}_{JS},\mathbf{\Sigma}) \\ &= E_{\mathbf{\Sigma}}^{(0)}[\mathrm{tr}(\hat{\mathbf{\Sigma}}_{SDS}\mathbf{\Sigma}^{-1})^2 - 2\mathrm{tr}(\hat{\mathbf{\Sigma}}_{SDS}\mathbf{\Sigma}^{-1}) - \mathrm{tr}(\hat{\mathbf{\Sigma}}_{JS}\mathbf{\Sigma}^{-1})^2 \\ &+ 2\mathrm{tr}(\hat{\mathbf{\Sigma}}_{JS}\mathbf{\Sigma}^{-1})] \\ &= \gamma(2)[n^2\sum_{i=1}^{p} d_i^2 + n(\sum_{i=1}^{p} d_i)^2 + n\mathrm{tr}(\mathbf{\Sigma}^{-1}\mathbf{HDH}'\mathbf{\Sigma}\mathbf{HDH}')] \\ &- 4E_{\mathbf{\Sigma}}^{(1)}\big[\sum_{i>j}\frac{d_i-d_j}{l_i-l_j}\big] \\ &- 2p\gamma(1) - \gamma(2)[(n^2+n)\sum_{i=1}^{p} d_i^2 + n(\sum_{i=1}^{p} d_i)^2] + 2p\gamma(1) \\ &= n\gamma(2)\mathrm{tr}(\mathbf{\Sigma}^{-1}\mathbf{HDH}'\mathbf{\Sigma}\mathbf{HDH}') - 4E_{\mathbf{\Sigma}}^{(1)}\big[\sum_{i>j}\frac{d_i-d_j}{l_i-l_j}\big] - n\gamma(2)\sum_{i=1}^{p} d_i^2. \end{aligned}$$

For $i > j$, $d_i > d_j$ and $l_i < l_j$, we infer that

$$-4E_{\mathbf{\Sigma}}^{(1)}\Big[\sum_{i>j}\frac{d_i - d_j}{l_i - l_j}\Big] \geqslant 0. \qquad (4)$$

Finally, from (3) and (4), we have $\Delta_2 \leqslant 0$, where the proof is complete. □

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