Journal of Mathematical Extension
Vol. 1, N0. 2, (2007), 149-155

The Characterization of the Spectrum of a Class of Relations

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Abstract. Hereditary, directed subsets of a group and a semigroup and some of their properties are discussed. A class of relations in terms of the range projections of a partial representation of a discrete group is introduced. It is shown that the spectrum of these relations is homeomorphic to the set of all characters of the diagonal subalgebra of the Toeplitz algebra.

AMS Subject Classification: 46L99.
Keywords and Phrases: hereditary subset, directed subset, spectrum, relation, quasi-lattice.

1. Introduction

In ([3]), the concept of a hereditary, directed subset of a semigroup P is introduced. Also, by a partial representation u of a group G on a Hilbert space H, we mean a map $u : G \longrightarrow B(H)$ with the following properties:

(i) $u_e = 1$

(ii) $u_{t^{-1}} = u_t^*$

(iii) $u_s u_t u_{t^{-1}} = u_{st} u_{t^{-1}}, \;\; s, t \in G.$

Let $u_t u_t^*$ satisfy the special relations $\mathcal{R}$ which will be defined later. The spectrum of the relations $\mathcal{R}$ is defined in ([1]).

On the other hand, Nica, in ([3]), has introduced the spectrum of the diagonal subalgebra of the Toeplitz algebra, denoted by $sp(\mathcal{D})$. In this article, we want to make a homeomorphism between $sp(\mathcal{D})$ and the spectrum of the relations $\mathcal{R}$. For this purpose, first, we bring some terminologies.

A *partially ordered group* is a pair (G, P) where G is a discrete group, and P is a subsemigroup of G. We denote $P^{-1} = \{x^{-1} : x \in P\}$ and always assume that $P \cap P^{-1} = \{e\}$.

For $x, y \in G$, define

$$x \leqslant y \Longleftrightarrow x^{-1} y \in P.$$

The relation $'' \leqslant''$, which is called the *left invariant order relation* induced by P, is a partial order relation. Obviously,

$$P = \{x \in G : e \leqslant x\}, \;\; P^{-1} = \{x \in G : x \leqslant e\}.$$

Also, $x \in PP^{-1}$ if and only if x has an upper bound in P.

The ordered group (G, P) is called *quasi-lattice ordered group* if for any $n \geqslant 1$, any $x_1, \cdots, x_n$ in G which have common upper bounds in P, also have a least common upper bounded in P. The least common upper bound of x and y is denoted by $x \vee y$. If $x, y \in G$ have no common upper bound in P, then, by convention, we write $x \vee y = \infty$.

Definition 1.1. *A subset w of G is hereditary if $xP^{-1} \subseteq w$ for every $x \in w$. It is called directed, if every $x, y \in w$ have an upper bound in $w \cap P$.*

We remark that every directed subset of G is contained in PP^{-1}, because every two element in it have an upper bound in P.

Lemma 1.2. *Suppose $w \subseteq G$ is hereditary. Then w is directed if and only if for every $x, y \in w, x \vee y$ exists and is in w.*

Proof. First, suppose that w is directed and take $x, y \in w$. Then there exists an element z in $w \cap P$ such that $x \leqslant z$ and $y \leqslant z$. The quasi-lattice condition implies that the least upper bound of x and $y, x \vee y$, exists and is in P. It remains to prove $x \vee y \in w$. Clearly, $x \vee y \leqslant z$, and so $z^{-1}(x \vee y) \in P^{-1}$, which implies that $x \vee y \in zP^{-1}$. But since w is hereditary and $z \in w$, $zP^{-1} \subseteq w$, and so $x \vee y \in w$.

The converse is clear by taking $z = x \vee y$. □

2. Main Result

Recall that a subset w of P is called *hereditary* if

$$s, t \in P,\ s \leqslant t,\ t \in w \Longrightarrow s \in w.$$

Also, it is called *directed* if any two elements of w have a common upper bound in w. Let Ω denote the set of all nonempty, hereditary directed subsets of P. Consider $w \in \Omega$, and take $t \in w$. Obviously, $e \leqslant t$ and so $e \in w$, because w is hereditary. Furthermore, identifying every subset of P with its characteristic function and considering the product topology on $\{0,1\}^P$, we observe that Ω is a compact, Hausdorff space ([3]).

let (G, P) be a quasi-lattice ordered group. Consider the compact, Hausdorff space $X = \Pi_{t\in G}\{0,1\}$ which can be identified with $\mathcal{P}(G)$, the collection of all subset of G, or with $\{0,1\}^G$. The subset $X_G := \{w \in X : e \in w\}$ is a compact, Hausdorff space with the relative topology inherited from $\{0,1\}^G$. For each $t \in G$, let $X_t = \{w \in X_G : t \in w\}$, and denote the characteristic function on X_t by χ_t.

Define a partial homeomorphism $\theta_t : X_{t^{-1}} \to X_t$ by $\theta_t(w) = tw$. Then $(\{X_t\}_{t\in G}, \{\theta_t\}_{t\in G})$ is a partial action, in the sense of [2] and [4].

Theorem 2.1. *([1]) The set of hereditrary, directed subsets of G containing e, which is denoted by H, is invariant under the partial action θ on X_G; i.e., $\theta_z(H \cap X_{z^{-1}}) \subseteq H$ for every $z \in G$.*

A corollary to this theorem runs as follows:

Corollary 2.2. *Suppose $w \in X_{t^{-1}}$ is hereditary and directed, then so is tw.*

Proof. Clearly, $w \in H$. Since $t^{-1} \in w$, we have $w \in X_{t^{-1}}$. Thus,

$w \in H \cap X_{t^{-1}}$, and so the above theorem implies that

$$tw = \theta_t(w) \in \theta_t(H \cap X_{t^{-1}}) \subseteq H. \square$$

Suppose that the range projections $u_t u_t^{-1} = u_t u_t^*$ of a partial representation u, [1], satisfy the relations $\mathcal{R}$ given by

(i) $u_t^* u_t = 1$, for any $t \in P$;

(ii) $u_t u_t^* u_s u_s^* = u_{t \vee s} u_{t \vee s}^*$, for any $t, s \in G$.

Define the spectrum of the relations $\mathcal{R}$ by

$$\Omega_{\mathcal{R}} = \{w \in X_G : f(t^{-1}w) = 0, \text{ for all } f \in \mathcal{R}, t \in w\}.$$

It is shown that $\Omega_{\mathcal{R}}$ is a compact, Hausdorff space ([1, Proposition 4.1]).

Suppose that $\mathcal{D}$ is the diagonal subalgebra of the Toeplitz algebra $\tau(G, P)$ as introduced in [3]. Indeed, $\mathcal{D}$ consists of all linear operators T on $\ell^2(P)$ whose matrices relative to the canonical basis of $\ell^2(P)$ are diagonal. By the *spectrum of* $\mathcal{D}$, denoted by $sp(\mathcal{D})$, we mean the set of all characters of D. Nica has shown that there is a homeomorphism between $sp(\mathcal{D})$ and Ω. It is worthy of attention to remark that from his homeomorphism, we can obtain the form of each set in Ω; in fact, if $T_t (t \in P)$, are the generators of the Toeplitz algebra then every nonempty, hereditary directed subset of P is of the form

$$A_\gamma = \{t \in P : \gamma(T_t T_t^*) = 1\}$$

where $\gamma \in sp(\mathcal{D})$.

In the remaining, our aim is to identify $\Omega_{\mathcal{R}}$ with $sp(\mathcal{D})$.

Theorem 2.3. *The spaces* Ω *and* $\Omega_{\mathcal{R}}$ *are homeomorphic.*

Proof. By Theorem 6.4 of [1], $\Omega_{\mathcal{R}}$ is the set of hereditary, directed subsets of G which contain the identity element. Take $w \in \Omega_{\mathcal{R}}$. Clearly, $w \cap P$ is a nonempty directed subset of P. Suppose $s, t \in P$, $s \leqslant t$, and $t \in w \cap P$. Then $s \in tP^{-1}$, and so $s \in w \cap P$, because w is a hereditary subset of G. Consequently, $w \cap P \in \Omega$ for every $w \in \Omega$. Now, define $\psi : \Omega_{\mathcal{R}} \to \Omega$ by $\psi(w) = w \cap P$. First, we show that ψ is continuous. Suppose that $\{w_i\}_i$ is a net in $\Omega_{\mathcal{R}}$ and $w_i \to w$ is $\Omega_{\mathcal{R}}$ as $i \to \infty$. Identifying each w in X_G with χ_w, the characteristic function of w, we have $x_{w_i} \to x_w$ pointwise as $i \to \infty$, and $\chi_{w_i}\chi_P \to \chi_w \chi_P$ pointwise as $i \to \infty$; that is, $\chi_{w_i \cap P} \to \chi_{w \cap P}$ pointwise as $i \to \infty$; equivalently, $w_i \cap P \to w \cap P$ as $i \to \infty$. Since $\Omega_{\mathcal{R}}$ and Ω are compact Hausdorff spaces, to show that ψ is a homeomorphism, it remains to prove that it is a bijection. So let $w_1, w_2 \in \Omega_{\mathcal{R}}, w_1 \cap P = w_2 \cap P$, but $w_1 \neq w_2$. Assume that $x \in w_1 - w_2$. Since w_1 is a directed subset of G, there exists $z \in w_1 \cap P = w_2 \cap P$ so that $x \leqslant z$, and so $p = x^{-1}z \in P$. Therefore, $zP^{-1} \subseteq w_2$, because w_2 is hereditary. This, in turn, implies that $x = zP^{-1} \in w_2$, which is a contradiction. Hence, ψ is one-to-one. Finally, for every $w' \in \Omega$, consider $w = w'P^{-1}$. Then it can be easily seen that $w \cap P = w'$ and $w \in \Omega_{\mathcal{R}}$. $\square$

Corollary 2.4. *There is a homeomorphism between the spaces* $sp(\mathcal{D})$ *and* $\Omega_{\mathcal{R}}$.

Acknowledgment. The authors thank Professor B. Tabatabaie for his valuable comments.

References

[1] R. Exel, M. Laca, and J. Quigg, Partial dynamical systems and C^*-algebras generated by partial isometries, *J. Operator Theory*, 47 (2002), 169-186.

[2] K. McClanahan, K-theory for partial crossed products by discrete groups, *J. Funct. Anal.*, 130 (1995), 77-117.

[3] A. Nica, C^*-algebras generated by isometries and Wiener-Hopf operators, *J. Operator Theory*, 27 (1992), 17-52.

[4] J. C. Quigg and I. Raeburn, Characterisation of crossed products by partial actions, *J. Operator Theory*, 37 (1997), 311-340.

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