Common Fixed Point Theorem in Metric Spaces of Fisher and Sessa

A.R Valipour Baboli, M.B Ghaemi*

Department of Mathematics, College of Basic Sciences, Karaj Branch, Islamic Azad Univercity, Alborz, Iran.

Abstract

In this note it is shown that T and I have a unique common fixed point on a compact subset C of a metric space X, where T and I are two self maps on C, I is non-expansive and the pair $(T, I)$is weakly commuting. further we show this result by replacing compatibility instead of weakly commutativity pair (T, I) and continuity instead of non-expansiveness of I.

Keywords: Common fixed point, commuting and compatible maps, compact space.
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1. Introduction and Preliminaries

Many authors have written some papers in which two self maps on a closed convex set has a unique fixed point for example [1], [3] and [9]. In 1986 , Fisher and Sessa proved a fixed point theorem for two self maps on a subset of a Banach space which is closed convex[3]. Sessa in [9] generalized a result of Das and Naik [1]. They defined two maps T and I on a metric space (X, d) into itself to be weakly commuting iff

$$d(TIx, ITx) \leq d(Ix, Tx) \tag{1.1}$$

for all x in X .A self map I on a metric space X is said to be non-expansive provied that $d(Ix, Iy) \leq d(x, y)$, holds for all x, y in X. Two commuting maps clearly satisfy (1.1) but the converse is not generally true as is shown with the following example.

Example 1.1. Let $X = [0, 1]$. Suppose X is endowed with the Euclidean metric . Define T and I by $Tx = \dfrac{x}{x+4}, Ix = \dfrac{x}{2}$ for any x in X. Then

$$\begin{aligned} d(TIx, ITx) &= \frac{x}{x+8} - \frac{x}{2x+8} = \frac{x^2}{2(x+8)(x+4)} \\ &\leq \frac{x^2+2x}{2(x+4)} = \frac{x}{2} - \frac{x}{x+4} = d(Ix, Tx). \end{aligned}$$

But for any $x \neq 0, TIx = \dfrac{x}{x+8} > \dfrac{x}{2x+8} = ITx$.

Fisher and Sessa proved the following theorem.

Theorem 1.1. [3] *Let T and I be two weakly commuting mappings of C into itself satisfying the inequality*

$$d(T(x), T(y)) \leq ad(I(x), I(y)) + (1-a)max\{d(T(x), I(x)), d(T(y), I(y))\}, \tag{1.2}$$

for all x, y in C , where $0 < a < 1$ and C is a closed convex subset of Banach space X. If I is linear , non-expansive in C and IC contains TC, then $T$and I have a unique common fixed point in C.

*Corresponding author: A.R Valipour Baboli Tel. 09356742864. M.B Ghaemi. Tel. 09128204188 . Fax +9821-73228403.
Email addresses: a.valipour@umz.ac.ir (A.R Valipour Baboli), mghaemi@iust.ac.ir (M.B Ghaemi)

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2. Main results

Our aim is to modification of theorem 1.1.

Theorem 2.1. *Let T and I be two self maps and weakly commuting on C into satisfying* 1.1*, where C is a compact subset of X. If I is non-expansive on C and IC contains TC, then T and I have a unique common fixed point in C.*

PROOF. Let $x = x_0$ be an arbitrary point in C and for any $n \in N$ choose x_{n+1} such that $Tx_n = Ix_{n+1}$. Since C is compact so $\{x_n\}$ has a convergence subsequence $\{y_k\}_{k=1}^{\infty}$, (which we show each y_k with y_n^k where it represent k'th member $\{y_n\}$ and n'th element of $\{x_n\}$) , such that $y_n^k \longrightarrow x^*$, where $x^* \in C$. Now we show $d(Tx^*, Ix^*) = 0$.

$$\begin{aligned}
d(Tx^*, Ix^*) &\leq \overline{lim} d(Tx^*, Ty_n^k) + \overline{lim} d(Ty_n^k, Iy_n^k) + \overline{lim} d(Iy_n^k, Ix^*) \\
&\leq \overline{lim} d(Ix^*, Iy_n^k) + \overline{lim}(1-a)max\{d(Tx^*, Ix^*), d(Ty_n^k, Iy_n^k)\} \\
&+ \overline{lim} d(Ty_n^k, Iy_{n^k}) + \overline{lim} d(Iy_n^k, Ix^*).
\end{aligned}$$

There are two cases , if $\overline{lim} d(Tx^*, Ix^*) \geq \overline{lim} d(Ty_n^k, Iy_n^k)$, then

$$\begin{aligned}
ad(Tx^*, Ix^*) &\leq (a+1)\overline{lim} d(x^*, y_n^k) + \overline{lim} d(Ty_n^k, Iy_n^k) \\
&= \overline{lim} d(Ty_n^k, Iy_n^k) \\
&\leq \overline{lim} d(Ty_n^k, Ix_{n+1}) + \overline{lim} d(x_{n+1}, y_n^k) \\
&\leq \overline{lim} d(x_{n+1}, y_n^k) = 0,
\end{aligned}$$

then $d(Tx^*, Ix^*) = 0$. But if $\overline{lim} d(Ty_n^k, Iy_n^k) \geq d(Tx^*, Ix^*)$, then

$$\begin{aligned}
d(Tx^*, Ix^*) &\leq (a+1)\overline{lim} d(x^*, y_n^k) + (2-a)\overline{lim} d(Ty_n^k, Iy_n^k) \\
&= (2-a)\overline{lim} d(Ty_n^k, Iy_n^k) \leq (2-a)\overline{lim}(d(Ty_n^k, Ix_{n+1}) + \overline{lim} d(Ix_{n+1}, Iy_n^k)) \\
&\leq (2-a)\overline{lim} d(x_{n+1}, y_n^k) = 0,
\end{aligned}$$

so $d(Tx^*, Ix^*) = 0$. Set $K_n = \{x \in C : d(Tx, Ix) \leq \frac{1}{n}\}$ and $H_n = \{x \in C : d(Tx, Ix) \leq \frac{a+1}{a.n}\}$. Clearly for each $n, K_n \neq \emptyset$ and $K_1 \supseteq K_2 \supseteq ... \supseteq K_n \supseteq ....$ Thus each of the sets $\overline{TK_n}$, where $\overline{TK_n}$ denotes the closure of TK_n, must be non-empty for $n = 1, 2, ...$ and $\overline{TK_1} \supseteq \overline{TK_2} \supseteq ... \supseteq \overline{TK_n} \supseteq ....$ Further, for arbitrary $x, y \in K_n$,

$$\begin{aligned}
d(Tx, Ty) &\leq ad(Ix, Iy) + (1-a)max\{d(Tx, Ix), d(Ty, Iy)\} \\
&\leq a[d(Ix, Tx) + d(Tx, Ty) + d(Ty, Iy)] + \frac{(1-a)}{n} \leq \frac{(a+1)}{n} + ad(Tx, Ty)
\end{aligned}$$

and so $d(Tx, Ty) \leq \dfrac{(a+1)}{(1-a)n}$. Thus $\lim_{n \longrightarrow \infty} diam(\overline{TK_n}) = 0$. It follows, by a well known result of Cantor(see, e.g [2],p. 156) the intersection $\bigcap_{n=1}^{\infty} \overline{TK_n}$ contains exactly one point w. Now let y be an arbitrary point in $\overline{TK_n}$. Then for arbitrary $\epsilon \geq 0$ there is a point y' in K_n such that $d(Ty', y) \leq \epsilon$.

Using the weak commutativity of T and I ,non-expansiveness of I and applying (2.1) and (2.2) we have

$$\begin{aligned}
&d(Ty, Iy)\\
&\leq d(Ty, TIy') + d(TIy', ITy') + d(ITy', Iy)\\
&\leq ad(Iy, I^2y') + (1-a)max\{d(Ty, Iy), d(TIy', I^2y')\} + d(TIy', ITy') + d(ITy', Iy)\\
&\leq ad(y, Iy') + (1-a)max\{d(Ty, Iy), d(TIy', ITy') + d(ITy', I^2y)\} + d(Iy', Ty') + d(Ty', y)\\
&\leq a[d(y, Ty') + d(Ty', Iy')] + (1-a)max\{d(Ty, Iy), d(Iy', Ty') + d(Ty', Iy')\} + \frac{1}{n} + \epsilon\\
&\leq a(\epsilon + \frac{1}{n}) + (1-a)max\{d(Ty, Iy), \frac{1}{n} + \frac{1}{n}\} + \frac{1}{n} + \epsilon\\
&\leq a(\epsilon + \frac{1}{n}) + \frac{1}{n} + \epsilon + (1-a)max\{d(Ty, Iy), \frac{2}{n}\}\\
&\leq (1+a)\epsilon\frac{(a+1)}{n} + (1-a)max\{d(Ty, Iy), \frac{2}{n}\}.
\end{aligned}$$

Since ϵ is arbitrary it follows that

$$d(Ty, Iy) \leq \frac{(a+1)}{n} + (1-a)max\{d(Ty, Iy), \frac{2}{n}\}. \tag{2.1}$$

That are two possible .If $d(Ty, Iy) \leq \frac{a+1}{n} + (1-a)d(Ty, Iy)$ so $d(Ty, Iy) \leq \frac{(a+1)}{a.n}$.In both cases y lies in H_n . Thus $\overline{K_n} \subseteq H_n$ and so the point w must be in H_n for $n = 1, 2, ...$ It follows that $d(Tw, Iw) \leq \frac{(a+1)}{a.n}$, for $n = 1, 2, ...$ and so $Tw = Iw$.Since (1.1) holds, we also have $ITw = TIw$.Thus $d(T^2w, Tw) \leq ad(ITw, Iw) + (1-a)max\{d(T^2w, ITw), d(Tw, Iw)\} = ad(T^2w, Tw)$, so $T^2w = ITw = TIw$ and $Tw = w'$is a fixed point of T for $a < 1$. Further, $Iw' = ITw = TIw = TTw = Tw' = w'$ and so w' is also a fixed point of I. uniqueness, suppose w'' is a common fixed point too .Then

$$\begin{aligned}
d(w', w'') &= d(Tw', Tw'')\\
&\leq ad(Iw', Iw'') + (1-a)max\{d(Tw', Iw'), d(Tw'', Iw'')\} \leq ad(w', w')
\end{aligned}$$

and the uniqueness of the common fixed point follows since $a < 1$.

The following example shows that condition of theorem 2.1 can be take placed and is diffrerent of result by Sessa 1.1 because C is non-convex.

Example 2.1. Choosing $C = [0, \frac{1}{2}] \bigcup \{1\}, Ix = \frac{x}{2}$and $Tx = \frac{x}{x+4}$then$TC = [0, \frac{1}{9}] \bigcup \{\frac{1}{5}\} \subseteq\subseteq [0, \frac{1}{4}] \bigcup \{\frac{1}{2}\} = IC$,I is non-expansive and the pair $(I, T)$is weakly commuting, where both of them are self maps.Further , I and T have a unique common fixed point which we know it is 0.

Let I be the identity map in Theorem 2.1,we have the following corollary which extends Theorem 1.1[3].

Corollary 2.2. *Let T be a mapping of C into itself satisfying the inequality*

$$d(T(x), T(y)) \leq ad(I(x), I(y)) + (1-a)max\{dT(x), I(x)), d(T(y), I(y))\},$$

for all $x, y \in C$, where $0 < a < 1$. Then T has a unique fixed point.

The result of this corollary was given in [4]. We note that the weak commutativity in Theorem 2.1 is a necessary condition. It suffices to consider the following example.

Example 2.2. Let $X = R$ and let $C = [0, 1]$. Define T and I by $Tx = \frac{1}{3}, Ix = \frac{x}{2}$ for any $x \in C$. It is easily seen that all the conditions of Theorem 2.1 are satisfied except that of weak commutativity since with $x = \frac{1}{2}, d(TI(\frac{1}{2}), IT(\frac{1}{2})) = \frac{1}{6} > d(T(\frac{1}{2}), I(\frac{1}{2}))$. However T and I do not have a common fixed point.

In 1990 , G. Jungck extended a fixed point theorem of Fisher and Sessa by replacing the requirements of weak commutativity and non-expansiveness by compatibility and continuty respectively.
G.Jungck[7] defined two self maps to be compatible iff whenever (x_n) is a sequence in X such that $Tx_n, Ix_n \longrightarrow t$for some $t \in X$, then $d(ITx_n, TIx_n) \longrightarrow 0$. Clearly, commuting maps are weakly commuting, and weakly commuting maps are compatible.

Lemma 2.3. *(Proposition2.2, [7]) . Let $f, g : (X, d) \longrightarrow (X, d)$ be compatible.*
1. If $f(t) = g(t)$, then $fg(t) = gf(t)$.
2. suppose that $lim_n f(x_n) = lim_n g(x_n) = t$ for some t in X.
(a) If f is continuous at t , $lim_n gf(x_n) = f(t)$.
b If f and g are continuous at t, then $f(t) = g(t)$ and $fg(t) = gf(t)$.

Lemma 2.4. [6].*Let T and I be compatible self maps of a metric space (X, d) with I continuous. Suppose there exist real number $r > 0$ and $a \in (0, 1)$ such that for all $x, y \in X$,*

$$d(Tx, Ty) \leq rd(Ix, Iy) + amax\{d(Tx, Ix), d(Ty, Iy)\} \tag{2.2}$$

Then $Tw = Iw$ for some $w \in X$iff $A = \bigcap\{cl(T(K_n)) : n \in N\} \neq \emptyset$, where $k_n = \{x \in X : d(Tx, Ix) \leq \frac{1}{n}\}$.

Using lemmas 2.1, 2.2 the following corollary concludes.

Corollary 2.5. *Let T and I be two compatible self maps of a compact subset C of a complete metric space X .Suppose that I is continuous , linear and $TC \subseteq IC$. If there exists $a \in (0, 1)$ such that for all $x, y \in C, T$ and I satisfy the following inequality*

$$d(T(x), T(y)) \leq ad(I(x), I(y)) + (1 - a)max\{d(T(x), I(x)), d(T(y), I(y))\}.$$

Then $T$and I have a unique common fixed point in C.

Example 2.3. Let $X = [0, 1]$ and $C = [0, 1]$ with the Euclidean metric and define I and T by $Ix = \frac{x}{2}, Tx = \frac{x}{x+3}$ for any $x \in C$. Now C is compact and $I, T : C \longrightarrow C$, where $TC = [0, \frac{1}{4}] \subset [0, \frac{1}{2}] = IC$ and I is linear and continuous. Clearly I and T are compatible on C and so satisfy in inequality(2.1) . Then $x = 0$ is a unique common fixed point in C.

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