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Boundary Stabilization of a Compactly System of Wave Equations

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Abstract. We obtain decay estimates of the energy of solutions to a compactly system of wave equations with a nonlinear boundary dissipation which is weak asu_t tends to infinity.

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1. Introduction

In this paper we are concerned with the decay property of the solutions to the evolutionary system

$$\partial_{tt}u_1 - \Delta u_1 + \Psi(u_1 - u_2) = 0 \quad \text{in } \Omega \times \mathbb{R}_+, \tag{1.1}$$

$$\partial_{tt}u_2 - \Delta u_2 + \Psi(u_2 - u_1) = 0 \quad \text{in } \Omega \times \mathbb{R}_+, \tag{1.2}$$

$$u_i = 0 \quad \text{on } \Gamma_0 \times \mathbb{R}_+, \ \ i = 1, 2, \tag{1.3}$$

$$\frac{\partial u_i}{\partial \nu} + \varphi_i u_i + f_i(\partial_t u_i) = 0 \quad \text{on } \Gamma_1 \times \mathbb{R}_+, \ \ i = 1, 2, \tag{1.4}$$

$$u_i(x, 0) = u_{i0} \quad \text{and} \quad \partial_t u_i(x, 0) = u_{i1} \quad \text{in } \Omega, \ i = 1, 2, \tag{1.5}$$

where Ω be a bounded open domain in $\mathbb{R}^n$, Γ_0, Γ_1 is a partition of boundary Γ, ν is the outward unit normal vector to Γ, $\Psi : \Omega \to \mathbb{R}$, $\varphi_i : \Gamma_1 \to \mathbb{R}$ for $i = 1, 2$, and $f_i : \mathbb{R} \to \mathbb{R}$. The problem of proving the energy decay rates for solutions of systems of evolution equations with dissipation at the boundary has been treated by several authors. Indeed, in the case of wave or plate equations we can mention Conrad-Rao ([1]), Komornik-Zuazua ([3,6]),Lagnese ([7]), Lasiecka ([8]), Lasiecka-Tataru ([9]), Lions ([10]) and Zuazua ([13]) among others. Very little is known for compactly wave equations. To our knowlege, uniform decay estimates for the one-dimensional case and applying a linear boundary feedback was studied by Najafi-Sahrangi-Wang ([12]), and quite recently Komornik-Rao ([5]) have obtained exponential decay in the multi-dimensional case when the boundary dissipation satisfies:

$$c_1|v|^p \leqslant |f_i(v)| \leqslant c_2|v|^{\frac{1}{p}} \quad \text{if } \ |v| \leqslant 1, \tag{1.6}$$

$$c_3|v| \leqslant |f_i(v)| \leqslant c_4|v| \quad \text{if } \ |v| \geqslant 1, \tag{1.7}$$

where c_1, c_2, c_3 and c_4 are four positive constants. These works, [11] and [5], have a serious drawback from the point of view of physical applications: they never apply for bounded functions f_i because $c_3 > 0$ in (1.7). The purpose of this paper is to obtain a variant of Najafi and Komornik-Rao results for functions such that

$$-\infty < \lim_{v\to-\infty} f_i(v) < \lim_{v\to+\infty} f_i(v) < +\infty. \tag{1.8}$$

If $f_i(v)$ satisfies at most (1.8), the dissipation effect of $f_i(\partial_t u_i)$ is weak as $|\partial_t u_i|$ is large and for convenience we call such a term weak dissipation. The most typical example is $f_i(v) = \frac{v}{\sqrt{1+v^2}}$. Let us note that the case of single wave equation with internal damping $f(v)$ satisfying (1.8) was studied by Nakao ([12]).

Through the paper we shall make the following assumptions:

(A1) The domain Ω is of class C^2.

(A2) The partition of Γ satisfies the condition $\overline{\Gamma_0} \bigcap \overline{\Gamma_1} = \emptyset$.

(A3) There exists a point $x_0 \in \mathbb{R}^n$ such that, putting $m(x) = x - x_0$, we have

$$m \cdot \nu \leqslant 0 \quad \text{on } \Gamma_0 \quad \text{and} \quad \inf_{\Gamma_1} m \cdot \nu > 0.$$

(A4) The coefficients φ_i are non-negative and they belong to $C^1(\Gamma_1)$. Moreover, either

$$\Gamma_0 \neq \emptyset \quad \text{or} \quad \inf_{\Gamma_1} \varphi_i > 0,\ i = 1, 2.$$

(A5) The function Ψ is non-negative and belong to $L_\infty(\Omega)$.

(A6) The function f_i are continuous, non-decreasing and

$$f_i(v) = 0 \Leftrightarrow v = 0,\ i = 1, 2.$$

Remarks.

1. *If $n = 1$, then Ω is bounded open interval, say $\Omega = (x_1, x_2) \subset \mathbb{R}$, and hence hypothesis (A1) is always satisfied. Furthermore, hypothesis*

(A2)-(A3) are satisfied in each of the following cases:

i) $\Gamma_0 = \emptyset, \quad \Gamma_1 = \{x_1, x_2\}$,

ii) $\Gamma_0 = \{x_1\}, \quad \Gamma_1 = \{x_2\}$,

iii) $\Gamma_0 = \{x_2\}, \quad \Gamma_1 = \{x_1\}$.

2. *If* $n \geqslant 2$ *and* Ω *is star-shaped with respect to some point* $x_0 \in \Omega$ *that is* $m \cdot \nu \geqslant 0$ *on* Γ*, then hypothesis (A2)-(A3) are satisfied with* $\Gamma_0 = \emptyset$, $\Gamma_1 = \Gamma$.

3. *If* $n \geqslant 2$ *and* $\Omega = \Omega_1 \setminus \Omega_0$ *where* Ω_0, Ω_1 *are star-shaped domains with respect to some point* $x_0 \in \Omega_0$ *and* $\Omega_0 \subset \Omega_1$*, then hypothesis (A2)-(A3) are satisfied with* $\Gamma_0 = \partial\Omega_0$, $\Gamma_1 = \partial\Omega_1$.

4. *If* $n \geqslant 2$ *and* Ω *is not of the form mentioned in the preceding two examples, then in general there is no point* x_0 *satisfying simultaneously (A2) and (A3). By applying an approximatinal method of Grisvard [2], one could considerably weaken assumptions (A2)-(A3), at least in dimensions* $n = 2, 3$*, by adapting an analogous argument given in Komornik-Zuazua [6] for the wave equation.*

5. *The condition* $\inf_{\Gamma_1} m \cdot \nu > 0$ *of (A3) is unnecessary if we use the weight* $m \cdot \nu$ *in the boundary damping as in Komornik-Zuazua [6].*

2. Main Result

If $u = (u_1, u_2)$ is a solution of the problem (1.1)-(1.5), then we define its energy $E : [0, \infty) \to [0, \infty)$ by the following formulas:

$$2E(t) = \int_\Omega \{(\sum_{i=1}^{2} (\partial_t u_i)^2 + |\nabla u_i|^2) + \Psi(u_1 - u_2)^2\}dx + \int_{\Gamma_1} \sum_{i=2}^{2} \varphi_i u_i^2 d\gamma \quad (2.1)$$

It is well-known (see [5, Theorem 1]) that under hypothesis (A1)-(A6), given $(u_{i0}, u_{i1}) \in H^1_{\Gamma_0} \times L^2(\Omega)$ for $i = 1, 2$ arbitrarily, the problem (1.1)-(1.5) has a unique weak solution satisfying

$$u_i \in C([0, \infty), H^1_{\Gamma_0}(\Omega)) \bigcap C^1([0, \infty), L^2(\Omega)), \quad i = 1, 2 \quad (2.2)$$

where we set

$$H^1_{\Gamma_0}(\Omega) = \{v \in H^1(\Omega) : \quad v = 0 \quad \text{on} \quad \Gamma_0\}.$$

Our main result is the following

Theorem. *Let* $(u_{i0}, u_{i1}) \in (H^2(\Omega) \bigcap H^1_{\Gamma_0}(\Omega)) \times H^1_{\Gamma_0}(\Omega), \quad i = 1, 2$ *such that*

$$\frac{\partial u_{i0}}{\partial \nu} + \varphi_i u_{i0} + f_i(u_{i1}) = 0, \quad i = 1, 2 \quad \text{on} \ \Gamma_1. \quad (2.3)$$

Assume that there exists a number p *such that*

$$p = 1 \quad \text{if} \ n = 1, \quad (2.4)$$

$$p > 1 \quad \text{if} \ n = 2, \quad (2.5)$$

$$p \geqslant n - 1 \quad \text{if} \ n \geqslant 3, \quad (2.6)$$

and four positive constants $c_1,\ c_2,\ c_3,\ c_4$ *such that*

$$c_1|s|^p \leqslant |f_i(s)| \leqslant c_2|s|^{\frac{1}{p}} \quad \textit{if } \ |s| \leqslant 1, \tag{2.7}$$

$$c_3|s| \leqslant |f_i(s)| \leqslant c_4|s| \quad \textit{if } \ |s| \geqslant 1, \tag{2.8}$$

then the strong solution of (1.1)-(1.5) satisfies the estimates:

$$E(t) \leqslant cE(0)e^{-\alpha t}, \quad \textit{for all } \ t \geqslant 0 \quad \textit{if } \ p = 1, \tag{2.9}$$

$$E(t) \leqslant \frac{c}{(1+t)^{\frac{2}{p-1}}}, \quad \textit{for all } \ t \geqslant 0 \quad \textit{if } \ p > 1, \tag{2.10}$$

where c, α *are positive constants depending on initial data.*

To end this section, let us recall the following useful lemma.

Lemma. *([4, Theorem 9.1]). Let* $E : \mathbb{R}_+ \to \mathbb{R}_+$ *be a non-increasing function and assume that there are two constants* $\beta > 0$ *and* $T > 0$ *such that*

$$\int_t^{\infty} E^{\beta+1}(\tau)d\tau \leqslant T\beta E(0)E(t), \quad \textit{for all } \ t \in [0, \infty). \tag{2.11}$$

Then we have

$$E(t) \leqslant E(0)(\frac{T+\beta t}{T+\beta T})^{\frac{-1}{\beta}}, \quad \textit{for all } \ t \geqslant T. \tag{2.12}$$

3. Proof of the Theorem

Multiplying (1.1) with $\partial_t u_1$, (1.2) with $\partial_t u_2$, integrating by parts their sum in $\Omega \times (0, T)$ and finally eliminating the normal derivatives by using the boundary conditions (1.3) and (1.4) we obtain easily that

$$E(0) - E(T) = \int_0^T \int_{\Gamma_1} (\partial_t u_1 f_1(\partial_t u_1) + \partial_t u_2 f_2(\partial_t u_2)) d\gamma dt \qquad (3.1)$$

for every positive number T. Being the primitive of an integrable function, hence E is locally absolutely continuous and

$$E^{'} = -\int_{\Gamma_1} (\partial_t u_1 f_1(\partial_t u_1) + \partial_t u_2 f_2(\partial_t u_2)) d\gamma \qquad (3.2)$$

is satisfied almost everywhere in $\mathbb{R}_+$. We are going to prove that the energy of the strong solution

$$\begin{aligned} u_i \quad \in \quad & L^\infty(\mathbb{R}_+, H^2(\Omega) \bigcap H^1_{\Gamma_0}(\Omega)) \\ & \bigcap W^{1,\infty}(\mathbb{R}_+, H^1_{\Gamma_0}(\Omega)) \bigcap W^{2,\infty}(\mathbb{R}_+, L^2(\Omega)) \end{aligned}$$

for $i = 1, 2$ satisfies the estimate

$$\int_S^T E^{\frac{p+1}{2}}(t) dt \leqslant cE(S) \qquad (3.3)$$

for all $0 \leqslant S < T < +\infty$ and where c is a positive constant depending on p and initial data. Here and in the sequel we shall denote by c various positive constants, depending only on the initial energy $E(0)$ and on the four constants c_i appearing in assumptions (2.7)-(2.8). According to Komornik-Rao ([5, Lemma 4.1]) we have by multiplying the equation (1.1) by

$$E^{\frac{p-1}{2}}(t)(2(x - x_0) \cdot \nabla u_1 + (n - \epsilon) u_1)$$

that

$$\int_S^T E^{\frac{p+1}{2}}(t)dt \leqslant cE(S) + c\sum_{i=1}^{2}\int_S^T (E^{\frac{p-1}{2}}(t) \int_{\Gamma_1}\Big((\partial_t u_i)^2 + f_i(\partial_t u_i)^2 + (u_i)^2\Big)d\gamma dt. \quad (3.4)$$

In order to get rid of the term u_i^2 in estimate (3.4), we apply a method introduced by Conrad-Pao ([1]). Thanks to [5, pp 353-355] we arrive at:

$$\begin{aligned}\int_S^T E^{\frac{p+1}{2}}(t)\int_{\Gamma_1}|u|^2 d\gamma dt \leqslant\ & cE(S) + c(\int_S^T E^{\frac{p-1}{2}}(t) \\ & \int_{\Gamma_1}|u_1 f_1(\partial_t u_1)| + |u_2 f_2(\partial_t u_2)|)d\gamma dt \\ & + c\int_S^T E^{\frac{p}{2}}\int_{\Gamma_1}|\partial_t u|^2 d\gamma dt \qquad (3.5)\end{aligned}$$

where $|u|^2 = u_1^2 + u_2^2$ and $|\partial_t u|^2 = (\partial_t u_1)^2 + (\partial_t u_2)^2$. Now, we want to estimate the last term of right-hand side of the inequality (3.5). Using the growth assumption (2.7), we have

$$\begin{aligned}\int_{|\partial_t u_1|\leqslant 1}(\partial_t u_1)^2 d\gamma &\leqslant \int_{|\partial_t u_1|\leqslant 1}(\partial_t u_1 f_1(\partial_t u_1))^{\frac{2}{p+1}}d\gamma \\ &\leqslant c(\int_{|\partial_t u_1|\leqslant 1}\partial_t u_1 f_1(\partial_t u_1)d\gamma)^{\frac{2}{p+1}} \\ &\leqslant c(\int_{\Gamma_1}\partial_t u_1 f_1(\partial_t u_1)d\gamma)^{\frac{2}{p+1}}. \qquad (3.6)\end{aligned}$$

On the other hand, if $n = 1$, we have $\partial_t u_1 \in H^1_{\Gamma_0}(\Omega) \subset L^\infty(\Omega)$ and then

$$\begin{aligned}\int_{|\partial_t u_1|\geqslant 1}(\partial_t u_1)^2 d\gamma &\leqslant c\int_{|\partial_t u_1|\geqslant 1}(\partial_t u_1)^2 f_1(\partial_t u_1)d\gamma \\ &\leqslant c\|\partial_t u_1\|_{L_\infty}\int_{\Gamma_1}\partial_t u_1 f_1(\partial_t u_1)d\gamma \\ &\leqslant c|\partial_t E|. \qquad (3.7)\end{aligned}$$

If $n \geqslant 2$, set $\|\cdot\|_r := \|\cdot\|_{L^r(\Gamma_1)}$, $s := \frac{2}{p+1}$ and $\alpha := \frac{2-s}{1-s}$, we have $0 < s < 1$ and $\alpha = \frac{2p}{p-1} > 2$. Then

$$\begin{aligned}\int_{|\partial_t u_1|\geqslant 1} (\partial_t u_1)^2 d\gamma &\leqslant c\int_{\Gamma_1} |\partial_t u_1|^{2-s}(\partial_t u_1 f_1(\partial_t u_1))^s d\gamma \\ &\leqslant \| \, |\partial_t u_1|^{2-s}\|_{\frac{1}{1-s}} \|(\partial_t u_1 f_1(\partial_t u_1))^s\|_{\frac{1}{s}} \\ &\leqslant c|\partial_t E|^{\frac{1}{p+1}}, \end{aligned} \tag{3.8}$$

in the last step we used the fact $H^1(\Omega) \subset L^{\frac{2p}{p-1}}(\Gamma)$ following from (2.6). We obtain similar inequalities for u_2. Now, for any fixed $\epsilon > 0$ we have

$$c|u_i f_i(\partial_t u_i)| \leqslant \epsilon u_i^2 + c\epsilon^{-1} f_i(\partial_t u_i)^2, \tag{3.9}$$

and

$$cE^{\frac{p}{2}}\|\partial_t u_i\|_{L^2(\Gamma_1)} \leqslant \epsilon E^{\frac{p+1}{2}} + c\epsilon^{-p}|\partial_t E|. \tag{3.10}$$

From (3.8)-(3.10), and by choosing ϵ small enough we have

$$\begin{aligned} &\int_S^T E^{\frac{p-1}{2}}(t)\int_{\Gamma_1} |u|^2 d\gamma dt \\ \leqslant\ & \delta\int_S^T E^{\frac{p-1}{2}}(t)dt + cE(S) \\ &+c\int_0^T E^{\frac{p-1}{2}}\int_{\Gamma_1}(f_1(\partial_t u_1)^2 + f_2(\partial_t u_2)^2)d\gamma dt \end{aligned} \tag{3.11}$$

when $\delta > 0$ is an arbitrary real number. By adding (3.4) and (3.11) we obtain easily

$$\begin{aligned}\int_S^T E^{\frac{p+1}{2}}(t)dt &\leqslant cE(S) + c\int_S^T E^{\frac{p-1}{2}}(t) \\ &\qquad \int_{\Gamma_1}(|\partial_t u|^2 + f_1(\partial_t u_1)^2 + f_2(\partial_t u_2)^2)d\gamma dt \end{aligned} \tag{3.12}$$

for all $0 \leqslant S < T < \infty$.

From (3.8) and the conditions (2.7)-(2.8) we have

$$\|f_i(\partial_t u_i)\|_{L^2(\Gamma_1)} \leqslant c|\partial_t E|^{\frac{1}{p+1}}, \quad i = 1, 2. \tag{3.13}$$

Substituting into the right-hand side of (3.12), we obtain that

$$\int_S^T E^{\frac{p+1}{2}}(t)dt \leqslant cE(S) + c\int_S^T E^{\frac{p-1}{2}}|\partial_t E|^{\frac{2}{p+1}}dt. \tag{3.14}$$

Using the Young's inequality, for any fixed $\epsilon > 0$ we have

$$cE^{\frac{p-1}{2}}|\partial_t E|^{\frac{2}{p+1}} \leqslant \epsilon E^{\frac{p+1}{2}} + c\epsilon^{\frac{1-p}{2}}|\partial_t E|. \tag{3.15}$$

Therefore,

$$(1-\epsilon)\int_S^T E^{\frac{p+1}{2}}(t)dt \leqslant c(1+\epsilon^{\frac{1-p}{2}})E(S), \tag{3.16}$$

and choosing $0 < \epsilon < 1$, it follows that

$$\int_S^T E^{\frac{p+1}{2}}(t)dt \leqslant cE(S), \tag{3.17}$$

with $p = 1$ if $n = 1$. Thanks to Lemma, we deduce (2.9)-(2.10).

References

[1] F. Conrad and B. Rao, Decay of solutions of wave equations in a star-shaped domain with nonlinear boundary feedback, *Asymptotic Anal.*, 7 (1993), 159-177.

[2] P. Grisvard, Controlabilite exacte des solutions de equation des ondes en presence des singularites, *J. Math. Pures et Appl.*, 68 (1989), 215-259.

[3] V. Komornik, On the nonlinear boundary stabilization of the wave equation, *Chin. Ann. of Math.*, 14B(2) (1993), 153-164.

[4] V. Komornik, *Exact controllability and stabilization, The multiplier method*, Masson, Paris, 1994.

[5] V. Komornik and B. Rao, Boundary stabilization of compactly coupled wave equations, *Asymptotic Anal.*, 14 (1997), 339-359.

[6] V. Komornik and E. Zuazua, A direct method for the boundary stabilization of the wave equation, *J. Math. Pures et Appli.*, 69 (1990), 33-59.

[7] J. Lagnese, *Boundary stabilization of thin plates*, SIAM, Philadelphia, 10, (1989).

[8] I. Lasiecga, Stabilization of wave and plate-like equation with nonlinear dissipation the boundary, *J. Diff. Equa.*, 79 (1989), 340-381.

[9] I. Lasiecka and D. Tataru, Uniform boundary stabilization of semi-linear wave equation with nonlinear boundary damping, *Diff. Integ. Equa.*, 6(3), (1993), 507-533.

[10] J. Lion, Exact controllability stabilization and perturbations of distributed systems, *SIAM Rev.*, 30 (1998), 1-68.

[11] M. Najafi, G. R. Shrangi, and H. Wang, The study of the stability of the coupled wave equations under various end conditions, *Proc. 31st Conf. Decision and Control, Tucson*, AZ, December 1992.

[12] M. Nakao, Energy decay for the wave equation with a nonlinear weak damping, *Diff. Integ. Equa.*, 8 (3) (1995), 681-688.

[13] E. Zuazua, Uniform stabilization of the wave equation by nonlinear boundary feedback, *SIAM J. Control and Optim.*, 28 (1990), 466-478.

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