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Exploring the Geometric Characteristics of Harmonic Functions Using the Second-Order Jackson q-Derivative

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Abstract. In this paper, we introduce a new subclass of harmonic functions defined by Jackson's second-order q-derivative. This subclass, denoted as $\mathcal{AH}_q(\sigma, \tau, \rho)$, is characterized by a higher-order differential inequality involving both the first and second q-derivatives. We establish the necessary and sufficient conditions for a function to belong to this class and investigate its geometric properties, such as coefficient bounds, distortion bounds, and closure under convolution. Additionally, we analyze the relationship between $\mathcal{AH}_q(\sigma, \tau, \rho)$ and a related subclass of analytic functions, denoted as $\mathcal{A}_q(\sigma, \tau, \rho)$). Furthermore, we explore the radii of convexity and starlikeness for the functions in these classes, providing significant contributions to the field of geometric function theory. Our results generalize and extend existing works on harmonic mappings defined by higher-order differential operators.

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1 Introduction

Let $\mathbb{C}$ represent the complex plane, and consider the open unit disk $\mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}$. For a harmonic function $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}}$ to be locally univalent and sense-preserving in $\mathbb{E}$, it must satisfy the condition $|\mathfrak{v}'(z)| < |\mathfrak{u}'(z)|$ for all $z \in \mathbb{E}$ (see [8]).

The class $\mathcal{SH}$ consists of harmonic, sense-preserving, univalent functions that are normalized such that $\mathfrak{f}(0) = 0$ and $\mathfrak{f}_z(0) = 1$ in the unit disk $\mathbb{E}$. A subclass of $\mathcal{SH}$, denoted by $\mathcal{SH}^0$, contains functions $\mathfrak{f} \in \mathcal{SH}$ for which $\mathfrak{v}'(0) = 0$. The harmonic components $\mathfrak{u}$ and $\mathfrak{v}$ are analytic in $\mathbb{E}$ and can be written as:

$$\mathfrak{u}(z) = z + \sum_{s=2}^{\infty} u_s z^s, \quad \mathfrak{v}(z) = \sum_{s=2}^{\infty} v_s z^s. \tag{1}$$

If $\mathfrak{v}(z) = 0$, the function reduces to the well-known class $\mathcal{S}$, which consists of normalized, analytic, univalent functions. Thus, we have the inclusions $\mathcal{S} \subset \mathcal{SH}^0 \subset \mathcal{SH}$.

The subclass $\mathcal{K}$ of $\mathcal{S}$ represents functions that map $\mathbb{E}$ onto convex domains, while the subclass $\mathcal{S}^*$ represents those mapping onto starlike domains. Similarly, the subclasses of $\mathcal{SH}^0$ corresponding to these geometric properties are denoted by $\mathcal{KH}^0$ and $\mathcal{SH}^{0,*}$. For further details, see [8, 21].

The q-derivative for a function $\psi \in \mathcal{S}$, introduced by Jackson [19], is defined as follows for $0 < q < 1$:

$$D_q\psi(z) = \begin{cases} \frac{\psi(z)-\psi(qz)}{(1-q)z}, & \text{if } z \neq 0, \\ \psi'(0), & \text{if } z = 0. \end{cases} \tag{2}$$

As q approaches 1, the q-derivative converges to the standard derivative, $D_q\psi(z) \to \psi'(z)$. For the expansions in (1) and using (2), the q-derivative becomes:

$$D_q\mathfrak{u}(z) = 1 + \sum_{s=2}^{\infty} [s]_q u_s z^{s-1}, \quad D_q^2\mathfrak{u}(z) = 1 + \sum_{s=2}^{\infty} [s]_q^2 u_s z^{s-1},$$

and

$$D_q\mathfrak{v}(z) = \sum_{s=2}^{\infty}[s]_q v_s z^{s-1}, \quad D_q^2\mathfrak{v}(z) = \sum_{s=2}^{\infty}[s]_q^2 v_s z^{s-1},$$

where $[s]_q = \frac{1-q^s}{1-q}$ is the q-number, which approaches s as q approaches 1.

Jackson also introduced the q-integral [20], defined as:

$$\int_0^z \psi(\zeta)\, d_q\zeta = z(1-q)\sum_{k=0}^{\infty} q^k \psi(zq^k),$$

provided the series converges.

For more detailed information on fractional works, the reader may refer to works such as [25, 24, 5, 28, 4, 14].

In 2019, Ahuja and Çetinkaya [1] introduced the class $\mathcal{SH}_q$ of q-harmonic, univalent, and sense-preserving functions $\mathfrak{f} = \mathfrak{u}+\overline{\mathfrak{v}}$. A function $\mathfrak{f}$ belongs to $\mathcal{SH}_q$ if and only if $\left|\frac{D_q\mathfrak{v}(z)}{D_q\mathfrak{u}(z)}\right| < 1$. As $q \to 1$, this class reduces to the class $\mathcal{SH}$.

The class $\mathcal{SH}_q^*(\alpha)$, consisting of q-starlike harmonic functions of order α, is defined by the inequality

$$\operatorname{Re}\left\{\frac{zD_q\mathfrak{u}(z) - \overline{zD_q\mathfrak{v}(z)}}{\mathfrak{u}(z) + \overline{\mathfrak{v}}(z)}\right\} > \alpha,$$

for $0 \leq \alpha < 1$. Similarly, the class $\mathcal{KH}_q(\alpha)$ consists of q-convex harmonic functions of order α, defined by the condition

$$\operatorname{Re}\left\{\frac{zD_q^2\mathfrak{u}(z) - \overline{zD_q^2\mathfrak{v}(z)}}{zD_q\mathfrak{u}(z) - \overline{zD_q\mathfrak{v}(z)}}\right\} > \alpha,$$

where $0 \leq \alpha < 1$. For more information, see [2, 3, 22, 29, 32].

A function $\mathfrak{f} : \mathbb{E} \to \mathbb{C}$ is said to be subordinate to another function $\mathfrak{g} : \mathbb{E} \to \mathbb{C}$, written as $\mathfrak{f}(z) \prec \mathfrak{g}(z)$, if there exists a function $\omega : \mathbb{E} \to \mathbb{C}$ such that $\omega(0) = 0$ and $\mathfrak{f}(z) = \mathfrak{g}(\omega(z))$ (see [15]).

We now introduce a new class of functions based on the Jackson q-derivative.

Definition 1.1. The class $\mathcal{AH}_q(\sigma,\tau,\rho)$ is the set of functions $\mathfrak{f}=\mathfrak{u}+\overline{\mathfrak{v}}\in\mathcal{SH}^0$ that satisfy the condition:

$$\operatorname{Re}\left\{\sigma D_q\mathfrak{u}(z)+\tau D_q^2\mathfrak{u}(z)-\rho\right\}>\left|\sigma D_q\mathfrak{v}(z)+\tau D_q^2\mathfrak{v}(z)\right|$$

for $\tau>0$, $0\le\rho<\sigma$, and $z\in\mathbb{E}$.

For $q\to1$ and specific choices of the parameters, we recover known classes such as $W_H^0(\alpha)$ studied by Ghosh and Vasudevarao [18], and $W_H^0(\delta,\lambda)$ explored by Rajbala and Prajapat [23]. For additional details on harmonic classes defined by higher-order differential inequalities, see [6, 7, 9, 10, 11, 12, 13, 16, 30, 31].

Definition 1.2. The class $\mathcal{A}_q(\sigma,\tau,\rho)$ consists of analytic functions $\psi\in\mathcal{S}$ that satisfy the inequality:

$$\operatorname{Re}\left\{\sigma D_q\psi(z)+\tau D_q^2\psi(z)-\rho\right\}>0,\quad z\in\mathbb{E}.$$

This paper aims to investigate geometric properties such as distortion bounds, coefficient bounds, and radii of starlikeness and convexity for the new class $\mathcal{AH}_q(\sigma,\tau,\rho)$. Furthermore, we explore sufficient conditions under which $\mathfrak{f}\in\mathcal{AH}_q(\sigma,\tau,\rho)$ belongs to the class of close-to-convex functions. The results obtained here extend known findings in the field of geometric function theory.

2 Geometric Properties of the Class $\mathcal{AH}_q(\sigma,\tau,\rho)$

In this section, we first establish a connection between the class $\mathcal{AH}_q(\sigma,\tau,\rho)$ and the class $\mathcal{A}_q(\sigma,\tau,\rho)$. Next, we explore the geometric properties of the class $\mathcal{AH}_q(\sigma,\tau,\rho)$. We derive necessary and sufficient conditions for a function to belong to this class and obtain bounds on the coefficients of functions in this class. These results provide insights into the geometric structure and constraints of harmonic functions related to the q-derivative operator.

Theorem 2.1. *A function $\mathfrak{f}=\mathfrak{u}+\overline{\mathfrak{v}}$ is in the class $\mathcal{AH}_q(\sigma,\tau,\rho)$ if and only if for every complex number ε with $|\varepsilon|=1$, the function $\Gamma_\varepsilon=\mathfrak{u}+\varepsilon\mathfrak{v}$ belongs to the class $\mathcal{A}_q(\sigma,\tau,\rho)$.*

Proof. Suppose $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}}$ is in the class $\mathcal{AH}_q(\sigma, \tau, \rho)$. For each complex number ε with $|\varepsilon| = 1$, consider $\Gamma_\varepsilon = \mathfrak{u} + \varepsilon\mathfrak{v}$. Then we have:

$$\begin{aligned}
&\operatorname{Re}\left\{\sigma D_q \Gamma_\varepsilon(z) + \tau D_q^2 \Gamma_\varepsilon(z)\right\} \\
&= \operatorname{Re}\left\{\sigma D_q \mathfrak{u}(z) + \tau D_q^2 \mathfrak{u}(z) + \varepsilon\left[\sigma D_q \mathfrak{v}(z) + \tau D_q^2 \mathfrak{v}(z)\right]\right\} \\
&> \operatorname{Re}\left\{\sigma D_q \mathfrak{u}(z) + \tau D_q^2 \mathfrak{u}(z)\right\} - \left|\sigma D_q \mathfrak{v}(z) + \tau D_q^2 \mathfrak{v}(z)\right| \\
&> \rho.
\end{aligned}$$

Thus, $\Gamma_\varepsilon(z)$ belongs to $\mathcal{A}_q(\sigma, \tau, \rho)$. Conversely, if $\Gamma_\varepsilon(z) \in \mathcal{A}_q(\sigma, \tau, \rho)$ for all $|\varepsilon| = 1$, then:

$$\operatorname{Re}\left\{\sigma D_q \mathfrak{u}(z) + \tau D_q^2 \mathfrak{u}(z)\right\} > \operatorname{Re}\left\{-\varepsilon\left[\sigma D_q \mathfrak{v}(z) + \tau D_q^2 \mathfrak{v}(z)\right]\right\} + \rho$$

By choosing ε appropriately, we find:

$$\operatorname{Re}\left\{\sigma D_q \mathfrak{u}(z) + \tau D_q^2 \mathfrak{u}(z) - \rho\right\} > \left|\sigma D_q \mathfrak{v}(z) + \tau D_q^2 \mathfrak{v}(z)\right|.$$

Thus, $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}}$ is in $\mathcal{AH}_q(\sigma, \tau, \rho)$. □

Theorem 2.2. *Let $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}}$ be in $\mathcal{AH}_q(\sigma, \tau, \rho)$. For $s \geq 2$, the coefficient v_s satisfies the following bound:*

$$|v_s| \leq \frac{\sigma + \tau - \rho}{[s]_q\left(\sigma + \tau[s]_q\right)}.$$

The function $\mathfrak{f}(z) = z + \frac{\sigma+\tau-\rho}{[s]_q(\sigma+\tau[s]_q)}\bar{z}^s$ achieves equality.

Proof. Assume $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}} \in \mathcal{AH}_q(\sigma, \tau, \rho)$, with $\mathfrak{v}(re^{i\theta})$ expressed as a series, where $0 \leq \rho < 1$ and $\theta \in \mathbb{R}$. We then have:

$$\begin{aligned}
&\rho^{s-1}[s]_q\left(\sigma + \tau[s]_q\right)|v_s| \\
&\leq \frac{1}{2\pi}\int_0^{2\pi}\left|\sigma D_q \mathfrak{v}(\rho e^{i\theta}) + \tau D_q^2 \mathfrak{v}(\rho e^{i\theta})\right| d\theta \\
&< \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re}\{\sigma D_q \mathfrak{u}(\rho e^{i\theta}) + \tau D_q^2 \mathfrak{u}(\rho e^{i\theta}) - \rho\} d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re}\left[\sigma + \tau - \rho + \sum_{s=2}^{\infty}[s]_q\left(\sigma + \tau[s]_q\right) u_s \rho^{s-1} e^{i(s-1)\theta}\right] d\theta \\
&= \sigma + \tau - \rho.
\end{aligned}$$

As ρ approaches 1 from below, the bound (2.2) is achieved. □

Theorem 2.3. *Let* $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}} \in \mathcal{AH}_q(\sigma, \tau, \rho)$. *For* $s \geq 2$, *the following inequality holds:*

$$|u_s| + |v_s| \leq \frac{2(\sigma + \tau - \rho)}{[s]_q\,(\sigma + \tau[s]_q)}$$

Equality is achieved for the function $\mathfrak{f}(z) = z + \sum_{s=2}^{\infty} \frac{2(\sigma+\tau-\rho)}{[s]_q(\sigma+\tau[s]_q)} z^s$.

Proof. Consider $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}} \in \mathcal{AH}_q(\sigma, \tau, \rho)$. By Theorem 2.1, $\Gamma_\varepsilon = \mathfrak{u} + \varepsilon\mathfrak{v}$ belongs to $\mathcal{A}_q(\sigma, \tau, \rho)$ for all ε where $|\varepsilon| = 1$. Hence, we have:

$$\mathrm{Re}\{\sigma D_q(\mathfrak{u}(z) + \varepsilon\mathfrak{v}(z)) + \tau D_q^2(\mathfrak{u}(z) + \varepsilon\mathfrak{v}(z))\} > \rho$$

for all $z \in \mathbb{E}$. In the unit disk $\mathbb{E}$, there exists an analytic function Φ with positive real part that can be written as $\Phi(z) = 1 + \sum_{s=1}^{\infty} \phi_s z^s$. This implies:

$$\sigma D_q(\mathfrak{u}(z) + \varepsilon\mathfrak{v}(z)) + \tau D_q^2(\mathfrak{u}(z) + \varepsilon\mathfrak{v}(z)) = \rho + (\sigma + \tau - \rho)\Phi(z).$$

Comparing coefficients, we obtain:

$$[s]_q\,(\sigma + \tau[s]_q)\,(u_s + \varepsilon v_s) = (\sigma + \tau - \rho)\phi_{s-1} \quad \text{for } s \geq 2.$$

Since the real part of $\Phi(z)$ is positive, we have $|\phi_s| \leq 2$ for $s \geq 1$. By selecting ε with $|\varepsilon| = 1$, we can confirm that the inequality is valid. The function $\mathfrak{f}(z) = z + \sum_{s=2}^{\infty} \frac{2(\sigma+\tau-\rho)}{[s]_q(\sigma+\tau[s]_q)} z^s$ shows that the inequality is sharp and exact. □

Theorem 2.4. *If* $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}}$ *belongs to the class* $\mathcal{SH}_q^0$, *where*

$$\sum_{s=2}^{\infty} [s]_q\,(\sigma + \tau[s]_q)\,(|u_s| + |v_s|) \leq \sigma + \tau - \rho,$$

then $\mathfrak{f}$ *is in the class* $\mathcal{AH}_q(\sigma, \tau, \rho)$.

Proof. Assume $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}}$ is in $\mathcal{SH}_q^0$. Using condition (2.4), consider the following:

$$\begin{aligned}
\operatorname{Re}\left\{\sigma D_q \mathfrak{u}(z) + \tau D_q^2 \mathfrak{u}(z) - \rho\right\} &= \operatorname{Re}\left[\sigma + \tau - \rho + \sum_{s=2}^{\infty} [s]_q \left(\sigma + \tau [s]_q\right) u_s z^{s-1}\right] \\
&> \sigma + \tau - \rho - \sum_{s=2}^{\infty} [s]_q \left(\sigma + \tau [s]_q\right) |u_s| \\
&\geq \sum_{s=2}^{\infty} [s]_q \left(\sigma + \tau [s]_q\right) |v_s| \\
&> \left| \sum_{s=2}^{\infty} [s]_q \left(\sigma + \tau [s]_q\right) v_s z^{s-1} \right| \\
&= \left| \sigma D_q \mathfrak{v}(z) + \tau D_q^2 \mathfrak{v}(z) \right|.
\end{aligned}$$

This confirms that $\mathfrak{f}$ belongs to $\mathcal{AH}_q(\sigma, \tau, \rho)$. □

Theorem 2.5. *Let $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}} \in \mathcal{AH}_q(\sigma, \tau, \rho)$. Then the following inequality holds:*

$$|z| - \frac{2(\sigma + \tau - \rho)}{[2]_q \left(\sigma + \tau [2]_q\right)} |z|^2 \leq |\mathfrak{f}(z)| \leq |z| + \frac{2(\sigma + \tau - \rho)}{[2]_q \left(\sigma + \tau [2]_q\right)} |z|^2 .$$

Proof. Consider $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}} \in \mathcal{AH}_q(\sigma, \tau, \rho)$. By taking the modulus of $\mathfrak{f}$ and applying Theorem 2.3, we obtain:

$$\begin{aligned}
|\mathfrak{f}(z)| &\leq |z| + \sum_{s=2}^{\infty} \left(|u_s| + |v_s|\right) |z|^s \\
&\leq |z| + \sum_{s=2}^{\infty} \frac{2(\sigma + \tau - \rho)}{[s]_q \left(\sigma + \tau [s]_q\right)} |z|^s \\
&\leq |z| + \frac{2(\sigma + \tau - \rho)}{[2]_q \left(\sigma + \tau [2]_q\right)} |z|^2 .
\end{aligned}$$

The other side of the inequality can be obtained by a similar method. □

3 Radii of Convexity and Starlikeness

In this section, we determine the radii of convexity and starlikeness for functions in the class $\mathcal{AH}_q(\sigma, \tau, \rho)$. We derive explicit formulas for these radii and explore their dependence on the parameters of the functions, employing established lemmas and theorems.

Lemma 3.1. *[26, Theorem 1] Consider* $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}}$*, where* $\mathfrak{u}$ *and* $\mathfrak{v}$ *are defined by (1). If the following condition holds:*

$$\sum_{s=2}^{\infty}[s]_q\left(|u_s| + |v_s|\right) \leq 1,$$

then $\mathfrak{f}$ *is harmonic, univalent, sense-preserving in* $\mathbb{E}$*, and* $\mathfrak{f} \in \mathcal{SH}_q^*$.

Lemma 3.2. *[26] Consider* $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}}$*, where* $\mathfrak{u}$ *and* $\mathfrak{v}$ *are defined by (1). If the following condition is satisfied:*

$$\sum_{s=2}^{\infty}[s]_q^2\left(|u_s| + |v_s|\right) \leq 1,$$

then $\mathfrak{f}$ *is harmonic, univalent, sense-preserving in* $\mathbb{E}$*, and* $\mathfrak{f} \in \mathcal{KH}_q$.

The next theorem addresses the radius of starlikeness for functions in $\mathcal{AH}_q(\sigma, \tau, \rho)$.

Theorem 3.3. *Let* $\mathfrak{f}$ *be a function in the class* $\mathcal{AH}_q(\sigma, \tau, \rho)$*. Then* $\mathfrak{f}$ *is q-starlike harmonic within the disk* $|z| < \mathfrak{r}_\star$*, where*

$$\mathfrak{r}_\star = \inf_{s\geq 2}\left(\frac{\sigma + \tau[s]_q}{2(\sigma + \tau - \rho)s(s-1)}\right)^{\frac{1}{s-1}}.$$

Proof. Let $\mathfrak{f} \in \mathcal{AH}_q(\sigma, \tau, \rho)$. For a fixed $0 < \mathfrak{r} < 1$, consider the scaled function

$$\mathfrak{f}_\mathfrak{r}(z) = \mathfrak{r}^{-1}\mathfrak{f}(\mathfrak{r}z) = \mathfrak{r}^{-1}\mathfrak{u}(\mathfrak{r}z) + \mathfrak{r}^{-1}\overline{\mathfrak{v}(\mathfrak{r}z)}.$$

This function $\mathfrak{f}_\mathfrak{r}$ remains within the class $\mathcal{AH}_q(\sigma, \tau, \rho)$. We express $\mathfrak{f}_\mathfrak{r}$ as

$$\mathfrak{f}_\mathfrak{r}(z) \quad = \quad z + \sum_{s=2}^{\infty}\mathfrak{u}_s\mathfrak{r}^{s-1}z^s + \overline{\sum_{s=2}^{\infty}\mathfrak{v}_s\mathfrak{r}^{s-1}z^s}.$$

To establish that $\mathfrak{f}_\mathfrak{r}$ is q-starlike, we need to show that

$$\sum_{s=2}^{\infty}[s]_q\left(|u_s| + |v_s|\right)\mathfrak{r}^{s-1} \leq 1.$$

From Theorem (2.3), we have

$$\sum_{s=2}^{\infty}[s]_q\left(|u_s|+|v_s|\right)\mathfrak{r}^{s-1}\leq\sum_{s=2}^{\infty}\frac{2(\sigma+\tau-\rho)}{\sigma+\tau[s]_q}\mathfrak{r}^{s-1}. \tag{3}$$

Given that

$$1=\sum_{s=2}^{\infty}\frac{1}{s(s-1)},$$

the inequality (3) can be rewritten as

$$\sum_{s=2}^{\infty}\frac{2(\sigma+\tau-\rho)}{\sigma+\tau[s]_q}\mathfrak{r}^{s-1}\leq\sum_{s=2}^{\infty}\frac{1}{s(s-1)}.$$

Thus, if

$$\mathfrak{r}^{s-1}\leq\frac{\sigma+\tau[s]_q}{2(\sigma+\tau-\rho)s(s-1)},$$

for all $s\geq 2$, then

$$\sum_{s=2}^{\infty}[s]_q\left(|u_s|+|v_s|\right)\leq 1.$$

This implies

$$\mathfrak{r}_\star=\inf_{s\geq 2}\left(\frac{\sigma+\tau[s]_q}{2(\sigma+\tau-\rho)s(s-1)}\right)^{\frac{1}{s-1}}.$$

□

Finally, we determine the radius of convexity for functions in $\mathcal{AH}_q(\sigma, \tau,\rho)$.

Theorem 3.4. *Let $\mathfrak{f}$ be a function in the class $\mathcal{AH}_q(\sigma,\tau,\rho)$. Then $\mathfrak{f}$ is q-convex harmonic within the disk $|z|<\mathfrak{r}_c$, where*

$$\mathfrak{r}_c=\inf_{s\geq 2}\left(\frac{\sigma+\tau[s]_q}{2[s]_q(\sigma+\tau-\rho)s(s-1)}\right)^{\frac{1}{s-1}}.$$

Proof. Let $\mathfrak{f}\in\mathcal{AH}_q(\sigma,\tau,\rho)$. For a fixed $0<\mathfrak{r}<1$, consider the function

$$\mathfrak{f}_\mathfrak{r}(z)=\mathfrak{r}^{-1}\mathfrak{f}(\mathfrak{r}z)=\mathfrak{r}^{-1}\mathfrak{u}(\mathfrak{r}z)+\mathfrak{r}^{-1}\overline{\mathfrak{v}(\mathfrak{r}z)}.$$

This function is still in $\mathcal{AH}_q(\sigma,\tau,\rho)$. We write

$$\mathfrak{f}_{\mathfrak{r}}(z) \quad = \quad z+\sum_{s=2}^{\infty}\mathfrak{u}_s\mathfrak{r}^{s-1}z^s+\overline{\sum_{s=2}^{\infty}\mathfrak{v}_s\mathfrak{r}^{s-1}z^s}.$$

To show $\mathfrak{f}_{\mathfrak{r}}$ is q-convex, we need to show that

$$\sum_{s=2}^{\infty}[s]_q^2\left(|u_s|+|v_s|\right)\mathfrak{r}^{s-1}\leq 1.$$

From Theorem 2.3, we have

$$\sum_{s=2}^{\infty}[s]_q^2\left(|u_s|+|v_s|\right)\mathfrak{r}^{s-1}\leq \sum_{s=2}^{\infty}\frac{2(\sigma+\tau-\rho)}{\sigma+\tau[s]_q}\mathfrak{r}^{s-1}. \tag{4}$$

Given that

$$1=\sum_{s=2}^{\infty}\frac{1}{s(s-1)},$$

the inequality (4) can be rewritten as

$$\sum_{s=2}^{\infty}\frac{2(\sigma+\tau-\rho)}{\sigma+\tau[s]_q}\mathfrak{r}^{s-1}\leq\sum_{s=2}^{\infty}\frac{1}{s(s-1)}.$$

Thus, if

$$\mathfrak{r}^{s-1}\leq\frac{\sigma+\tau[s]_q}{2[s]_q(\sigma+\tau-\rho)s(s-1)},$$

for all $s\geq 2$, then

$$\sum_{s=2}^{\infty}[s]_q^2\left(|u_s|+|v_s|\right)\leq 1.$$

This implies

$$\mathfrak{r}_c=\inf_{s\geq 2}\left(\frac{\sigma+\tau[s]_q}{2[s]_q(\sigma+\tau-\rho)s(s-1)}\right)^{\frac{1}{s-1}}.$$

□

4 Closure Properties of the Class $\mathcal{AH}_q(\sigma, \tau, \rho)$

This section explores the closure properties of the class $\mathcal{AH}_q(\sigma, \tau, \rho)$, focusing on its behavior under convolution and convex combinations. We investigate how these operations affect the class and provide key insights into its structural stability and mathematical consistency.

Theorem 4.1. *The class $\mathcal{AH}_q(\sigma, \tau, \rho)$ is closed under convex combinations.*

Proof. Consider functions $\mathfrak{f}_k = \mathfrak{u}_k + \overline{\mathfrak{v}_k}$ belonging to the class $\mathcal{AH}_q(\sigma, \tau, \rho)$ for each $k = 1, 2, \ldots, n$. Suppose that the weights φ_k satisfy $\sum_{k=1}^{n} \varphi_k = 1$ and $0 \leq \varphi_k \leq 1$. The convex combination of these functions is given by:

$$\mathfrak{f}(z) = \sum_{k=1}^{n} \varphi_k \mathfrak{f}_k(z) = \mathfrak{u}(z) + \overline{\mathfrak{v}(z)},$$

where

$$\mathfrak{u}(z) = \sum_{k=1}^{n} \varphi_k \mathfrak{u}_k(z) \quad \text{and} \quad \mathfrak{v}(z) = \sum_{k=1}^{n} \varphi_k \mathfrak{v}_k(z).$$

Both $\mathfrak{u}$ and $\mathfrak{v}$ are analytic within the unit disk $\mathbb{E}$ and satisfy the conditions $\mathfrak{u}(0) = \mathfrak{v}(0) = D_q\mathfrak{u}(0) - 1 = D_q\mathfrak{v}(0) = 0$. Additionally,

$$\begin{aligned}
\operatorname{Re}\left\{\sigma D_q \mathfrak{u}(z) + \tau D_q^2 \mathfrak{u}(z) - \rho\right\} &= \operatorname{Re}\left[\sum_{k=1}^{n} \varphi_k \{\sigma D_q \mathfrak{u}_k(z) + \tau D_q^2 \mathfrak{u}_k(z) - \rho\}\right] \\
&> \sum_{k=1}^{n} \varphi_k \left|\sigma D_q \mathfrak{v}_k(z) + \tau D_q^2 \mathfrak{v}_k(z)\right| \\
&\geq \left|\sigma D_q \mathfrak{v}(z) + \tau D_q^2 \mathfrak{v}(z)\right|,
\end{aligned}$$

which confirms that $\mathfrak{f}$ belongs to $\mathcal{AH}_q(\sigma, \tau, \rho)$. □

Definition 4.2. A sequence $\{a_s\}_{s=0}^{\infty}$ of non-negative real numbers is called a "convex null sequence" if a_s approaches 0 as $s \to \infty$ and the sequence satisfies

$$a_0 - a_1 \geq a_1 - a_2 \geq a_2 - a_3 \geq \ldots \geq a_{s-1} - a_s \geq \ldots \geq 0.$$

Lemma 4.3. *(see [17]) The function*

$$A(z) = \frac{a_0}{2} + \sum_{s=1}^{\infty} a_s z^s$$

defined by a convex null sequence $\{a_s\}_{s=0}^{\infty}$ is analytic and has a positive real part within the unit disk $\mathbb{E}$.

Lemma 4.4. *(see [27]) If the function $\Phi(z)$ is analytic in $\mathbb{E}$, with $\Phi(0) = 1$ and $\mathrm{Re}\{\Phi(z)\} > \frac{1}{2}$ for all $z \in \mathbb{E}$, then for any analytic function Γ defined in $\mathbb{E}$, the convolution $\Phi * \Gamma$ maps values within the convex hull of the image of Γ.*

Lemma 4.5. *If $\Gamma \in \mathcal{A}_q(\sigma, \tau, \rho)$, then*

$$\mathrm{Re}\left(\frac{\Gamma(z)}{z}\right) > \frac{1}{2}.$$

Proof. Consider $\Gamma \in \mathcal{A}_q(\sigma, \tau, \rho)$, given by $\Gamma(z) = z + \sum_{s=2}^{\infty} U_s z^s$. The inequality

$$\mathrm{Re}\left[\sigma + \tau - \rho + \sum_{s=2}^{\infty} [s]_q \left(\sigma + \tau [s]_q\right) U_s z^{s-1}\right] > 0 \quad \text{for } z \in \mathbb{E},$$

is equivalent to $\mathrm{Re}\{\Phi(z)\} > \frac{1}{2}$ within $\mathbb{E}$, where

$$\Phi(z) = 1 + \frac{1}{2(\sigma + \tau - \rho)} \sum_{s=2}^{\infty} [s]_q \left(\sigma + \tau [s]_q\right) U_s z^{s-1}.$$

The sequence $\{a_s\}_{s=0}^{\infty}$ is a convex null sequence if

$$a_0 = 2 \text{ and } a_{s-1} = \frac{2(\sigma + \tau - \rho)}{[s]_q \left(\sigma + \tau [s]_q\right)} \text{ for } s \geq 2.$$

Applying Lemma 4.3, the function

$$A(z) = 1 + \sum_{s=2}^{\infty} \frac{2(\sigma + \tau - \rho)}{[s]_q \left(\sigma + \tau [s]_q\right)} z^{s-1}$$

is analytic with $\mathrm{Re}\{A(z)\} > 0$ in $\mathbb{E}$. Therefore,

$$\frac{\Gamma(z)}{z} = \Phi(z) * \left(1 + \sum_{s=2}^{\infty} \frac{2(\sigma+\tau-\rho)}{[s]_q\,(\sigma+\tau[s]_q)} z^{s-1}\right),$$

and by Lemma 4.4, we conclude that $\mathrm{Re}\left\{\frac{\Gamma(z)}{z}\right\} > \frac{1}{2}$ for $z \in \mathbb{E}$. □

Lemma 4.6. *The class $\mathcal{A}_q(\sigma,\tau,\rho)$ is closed under convolution.*

Proof. Let $\Gamma_1(z) = z + \sum_{s=2}^{\infty} \mathfrak{U}_s z^s$ and $\Gamma_2(z) = z + \sum_{s=2}^{\infty} \mathfrak{V}_s z^s$. The convolution of these functions is given by:

$$\Gamma(z) = (\Gamma_1 * \Gamma_2)(z) = z + \sum_{s=2}^{\infty} \mathfrak{U}_s \mathfrak{V}_s z^s.$$

Applying the q-derivative and convolution properties, we have:

$$\begin{aligned} D_q\Gamma(z) &= D_q\Gamma_1(z) * \frac{\Gamma_2(z)}{z}, \\ D_q^2\Gamma(z) &= D_q^2\Gamma_1(z) * \frac{\Gamma_2(z)}{z}. \end{aligned}$$

Substituting these into the formula, we obtain:

$$\frac{\sigma D_q\Gamma(z) + \tau D_q^2\Gamma(z) - \rho}{\sigma+\tau-\rho} = \left(\frac{\sigma D_q\Gamma_1(z) + \tau D_q^2\Gamma_1(z) - \rho}{\sigma+\tau-\rho}\right) * \frac{\Gamma_2(z)}{z}. \tag{5}$$

Since $\Gamma_1 \in \mathcal{A}_q(\sigma,\tau,\rho)$, we have $\mathrm{Re}\left[\frac{\sigma D_q\Gamma_1(z)+\tau D_q^2\Gamma_1(z)-\rho}{\sigma+\tau-\rho}\right] > 0$ for $z \in \mathbb{E}$. Additionally, by Lemma 4.5, $\mathrm{Re}\left[\frac{\Gamma_2(z)}{z}\right] > \frac{1}{2}$ in $\mathbb{E}$. Applying Lemma 4.4 to (5), we conclude that $\mathrm{Re}\left[\frac{\sigma D_q\Gamma(z)+\tau D_q^2\Gamma(z)-\rho}{\sigma+\tau-\rho}\right] > 0$ in $\mathbb{E}$. Hence, $\Gamma = \Gamma_1 * \Gamma_2 \in \mathcal{A}_q(\sigma,\tau,\rho)$. □

Using Lemma 4.6, we now demonstrate that the class $\mathcal{AH}_q(\sigma,\tau,\rho)$ is preserved under the convolution operation.

Theorem 4.7. *For $m = 1, 2$, let $\mathfrak{f}_m \in \mathcal{AH}_q(\sigma,\tau,\rho)$. Then, the convolution $\mathfrak{f}_1 * \mathfrak{f}_2$ is also a member of $\mathcal{AH}_q(\sigma,\tau,\rho)$.*

Proof. Assume that $\mathfrak{f}_m = \mathfrak{u}_m + \overline{\mathfrak{v}_m} \in \mathcal{AH}_q(\sigma,\tau,\rho)$ for $m = 1, 2$. The convolution $\mathfrak{f}_1 * \mathfrak{f}_2$ is given by $\mathfrak{u}_1 * \mathfrak{u}_2 + \overline{\mathfrak{v}_1 * \mathfrak{v}_2}$. To establish that this convolution function belongs to $\mathcal{AH}_q(\sigma,\tau,\rho)$, we must show that the function $F_\varepsilon = \mathfrak{u}_1 * \mathfrak{u}_2 + \varepsilon(\mathfrak{v}_1 * \mathfrak{v}_2)$ is within $\mathcal{A}_q(\sigma,\tau,\rho)$ for any ε where $|\varepsilon| = 1$.

According to Lemma 4.6, since $\mathcal{A}_q(\sigma,\tau,\rho)$ is closed under convex combinations, and both $\mathfrak{u}_m+\varepsilon\mathfrak{v}_m$ for $m = 1, 2$ are in $\mathcal{A}_q(\sigma,\tau,\rho)$, it follows that the functions $U_1 = (\mathfrak{u}_1-\mathfrak{v}_1)*(\mathfrak{u}_2-\varepsilon\mathfrak{v}_2)$ and $U_2 = (\mathfrak{u}_1+\mathfrak{v}_1)*(\mathfrak{u}_2+\varepsilon\mathfrak{v}_2)$ also belong to $\mathcal{A}_q(\sigma,\tau,\rho)$.

Since $\mathcal{A}_q(\sigma,\tau,\rho)$ is closed under convex combinations, the function

$$U_\varepsilon = \frac{1}{2}(U_1 + U_2) = \mathfrak{u}_1 * \mathfrak{u}_2 + \varepsilon(\mathfrak{v}_1 * \mathfrak{v}_2)$$

is also within $\mathcal{A}_q(\sigma,\tau,\rho)$. Thus, it follows that the class $\mathcal{AH}_q(\sigma,\tau,\rho)$ remains invariant under convolution, implying that the convolution of any two functions in $\mathcal{AH}_q(\sigma,\tau,\rho)$ will also be in $\mathcal{AH}_q(\sigma,\tau,\rho)$. □

5 Examples of Functions in the Class $\mathcal{AH}_q(\sigma, \tau, \rho)$

In this section, we provide examples to facilitate a better understanding of the theoretical concepts.

Example 5.1. Consider the function $\mathfrak{f}(z) = z + 0.35z^3 - 0.2\overline{z}^4$. According to Theorem 2.4, $\mathfrak{f}$ belongs to the class $\mathcal{AH}_q(0.25, 2, 1, 1)$. Additionally, the function $\mathfrak{f}$ is a q-starlike harmonic function by virtue of the Lemma 3.1. The image of the unit disk and concentric circle within it under the function $\mathfrak{f}$ is shown in Figure 1.

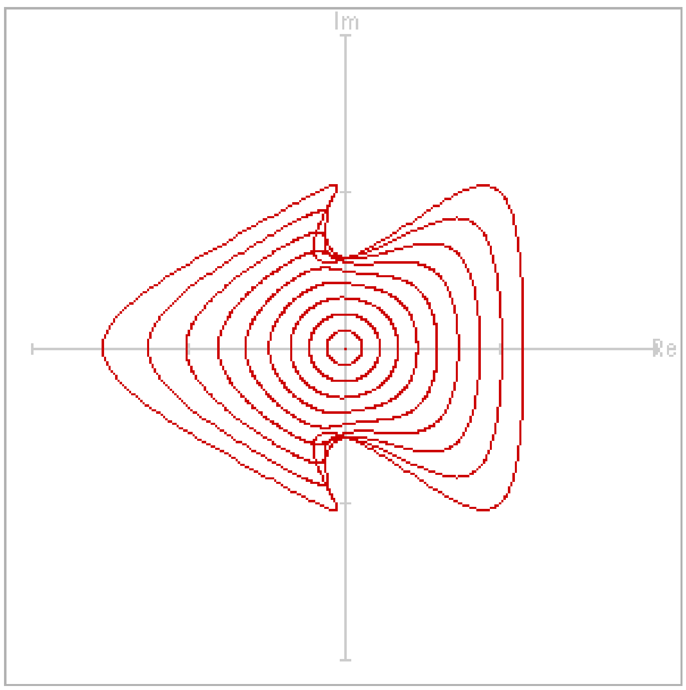

Figure 1: Under the map $\mathfrak{f}(z) = z + 0.35z^3 - 0.2\overline{z}^4$, the image of concentric circles inside the unit disk.

Example 5.2. Consider the function $\mathfrak{g}(z) = z + 0.45z^3 + 0.25z^5$. By Theorem 8, $\mathfrak{g}$ is a member of the $\mathcal{AH}_q(0.25, 2, 1, 1)$ class. Moreover, based on the Lemma 3.2, $\mathfrak{g}$ is a q-convex harmonic function. The images of the unit disk and concentric disks within it under the mapping of $\mathfrak{g}$ are illustrated in Figure 2.

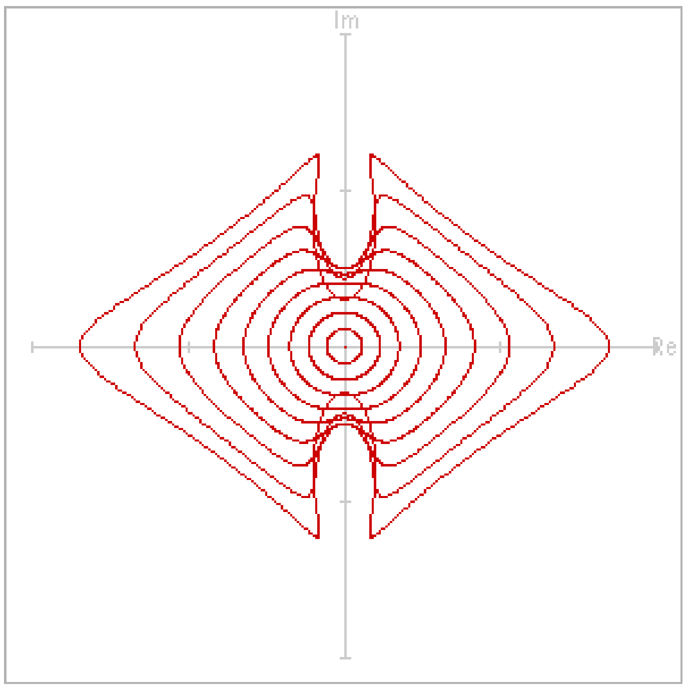

Figure 2: Under the map $\mathfrak{g}(z) = z + 0.45z^3 + 0.25z^5$, the image of concentric circles inside the unit disk.

Example 5.3. Let $\mathfrak{h}(z) = \mathfrak{f}(z) * \mathfrak{g}(z) = z + 0.1575z^3$, where $\mathfrak{f}$ and $\mathfrak{g}$ are the functions given in Example 5.1 and Example 5.2. According to Theorem 2.4, the function $\mathfrak{h}(z)$ belongs to the class $\mathcal{AH}_q(0.25, 2, 1, 1)$. Furthermore, $\mathfrak{h}$ is a q-starlike harmonic function according to Lemma 3.1. Figure 2 illustrates the image of concentric circles inside the unit disk $\mathbb{E}$ under the transformation defined by $\mathfrak{h}(z) = z + 0.1575z^3$.

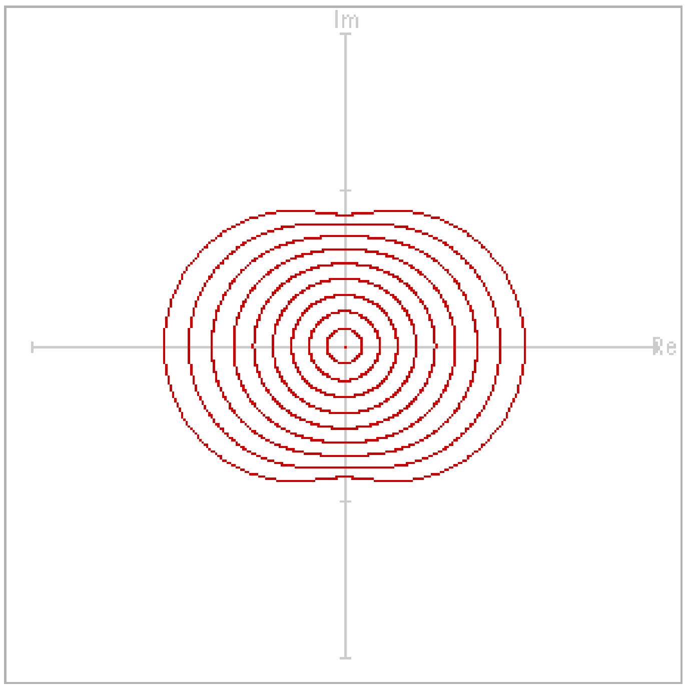

Figure 3: Under the map $\mathfrak{h}(z) = z + 0.1575z^3$, the image of concentric circles inside the unit disk.

6 Conclusions

In this study, we introduced a new subclass of harmonic functions defined by Jackson's second-order q derivative. We derived the necessary coefficient conditions for functions to belong to this class, along with geometric properties such as coefficient bounds, distortion bounds, and closure under convolution. Additionally, we obtained the radii of convexity and starlikeness. These results provide a broader perspective on harmonic mappings and improve some previously established results.

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