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Algebraic Properties of $C_c(X)$

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Dedicated to Professor A.R. Aliabad on the occasion of his 70th birthday

Abstract. The article is focuses on the study of the theory of rings of continuous functions with countable images on a topological space X; the algebraic properties of $C_c(X)$. The connection between the properties of X and the algebraic properties of $C_c(X)$ is investigated.

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1 Introduction

All topological spaces in this paper are infinite Tychonoff. The ring of all real-valued continuous functions on a space X is denoted by $C(X)$, and the subring of all bounded functions in $C(X)$, is denoted by $C^*(X)$. The topological study of $C(X)$ started in the 1920s. Later, some researchers were interested in studying the algebraic structure of $C(X)$.

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They attempted to understand more precisely the connection between the topological properties of X and the algebraic properties of $C(X)$. Undoubtedly, the valuable book "Rings of Continuous Functions" by Gillman and Jerison is an excellent example of what has been happening in this regard. The ring of continuous functions, the topic that algebra and topology play a role simultaneously, is still a dynamic topic for research. We refer the interesting reader to [5, 6, 28, 55, 62] for more background information on algebraic properties of rings of continuous functions.

In 2013, $C_c(X)$ a genuinely proper subring of $C(X)$ was introduced in [26] and also, the pointfree version of $C_c(X)$ has been introduced in [40, 60]. The structure of $C_c(X)$ is studied in [7, 10, 25, 26]. To see the story of the creation of $C_c(X)$ and some results obtained so far, we refer the interesting reader to [57]. In this paper, we focus on the algebraic properties of $C_c(X)$ and aim to obtain some results regarding $C_c(X)$. We try to compare the algebraic behavior of $C_c(X)$ and $C(X)$ and state the contrasts and similarities.

In [26], it has been shown that for every topological space X, there exists a zero-dimensional space (a space with a base consisting of clopens) Y, which is a continuous image of X and $C_c(X) \cong C_c(Y)$. This result means that we could consider only zero-dimensional spaces by studying the ring $C_c(X)$ without loss of generality. The reader can also see the pointfree version of these topics in [22, 60] and the structure of the pointfree version of $C_c(X)$ is studied in [1, 20, 21, 23].

This article can be structured into eight sections. The current section serves as the introductory part. In Section 2, we prove that $C_c(X)$ and the classical ring of quotients of $C_c(X)$ are always clean and Marot. In Section 3, we concentrate on $C_c(X)$ when it is a von Neumann regular ring, a classical ring, or a von Neumann local ring. In Section 4, in contrast to $C(X)$, we show that a Baer ring $C_c(X)$ may not be an I-ring. We also show that there is a fraction-dense ring $C_c(X)$ that is not strongly fraction-dense. In Section 5, in contrast to $C(X)$, we prove that $C_c(X)$ is an almost p.p. ring if and only if it is a p.f. ring. In Section 6, we focus on the notions of self-injective rings and continuous rings. In contrast to $C(X)$, we show that a continuous ring $C_c(X)$ may not be self-injective. In Section 7, we prove that in contrast to $C(X)$,

$C_c(X)$ is a Bézout ring if and only if it is an elementary divisor ring. In the last section, we conclude the paper with a diagram of implications to summarize the relations between the algebraic properties we studied in the previous sections.

In this article, all rings are commutative with $1 \neq 0$. The set $\mathrm{Ann}(X) = \{r \in R | rX = 0\}$ is the annihilator of a subset X of a ring R. We also use $\mathrm{Ann}(x)$ for $\mathrm{Ann}(\{x\})$. $x \in R$ is called a zero-divisor, if $\mathrm{Ann}(x) \neq 0$, otherwise, a non zero-divisor (or regular). For a ring R, $Q_{cl}(R)$ (resp., $Q_{max}(R)$) denote the classical ring of quotients (resp., maximal quotient ring) of R. For each $f \in C_c(X)$; the *zero-set* of f; denoted by $Z(f)$; is the set of zeros of f and $X \setminus Z(f)$ is the *cozero-set* of f; denoted by $\mathrm{coz}\, f$. The set of interior points in a set A is denoted by A° or $\mathrm{int}\, A$. Other conventions in $C(X)$ and ring theory follow those in [29], [43], and [44].

2 Clean Rings and Marot Rings

A ring R is called a *clean ring* if every element in R can be expressed as the sum of a unit and an idempotent. The notion of a clean ring was defined by Nicholson in [58]. A history of commutative clean rings is found in [54]. In [7, Corollary 2.8], it is shown that, for any space X, the ring $C_c(X)$ is always a clean ring. We now state a result in this direction.

Theorem 2.1. *The rings $C_c(X)$ and $Q_{cl}(C_c(X))$ are always clean.*

Proof. By [7, Corollary 2.8], it suffices to show that $Q_{cl}(C_c(X))$ is clean. By [10, Theorem 2.2], $Q_{cl}(C_c(X))$ is a direct limit of the rings $C_c(K)$, where K is a dense σ-clopen set of X. In [14, Proposition 2.4], it is shown that the directed limits of clean rings are always clean. Since $C_c(K)$ is clean for any space K, we infer that $Q_{cl}(C_c(X))$ is clean. □

Before presenting our next observation, we recall that an element is called *regular* if it is not a zero-divisor. It is easy to see that $f \in C_c(X)$ is regular if and only if $Z^\circ(f) = \emptyset$. Recall that a ring R is *additively regular* if for each regular element $f \in R$ and each $g \in R$, there is an element $t \in R$ such that $g + ft$ is regular. Additively regular rings were named by Gilmer and Huckaba [30]. For more information, the reader is referred to [45].

Theorem 2.2. *The ring $C_c(X)$ is always an additively regular ring.*

Proof. Assume that $g \in C_c(X)$ and f is a regular in $C_c(X)$. By definition, we have $\mathrm{Im}(\frac{g}{f}|_{\mathrm{coz}\, f}) \neq \mathbb{R}$. Take $r \in \mathbb{R} \setminus \mathrm{Im}(\frac{g}{f}|_{\mathrm{coz}\, f})$. Clearly, $g(x) - rf(x) \neq 0$ for every $x \in \mathrm{coz}\, f$. Therefore, $Z^\circ(g - rf) \subseteq Z^\circ(f) = \emptyset$. This means that $g - rf$ is a regular element in $C_c(X)$, as desired. □

We can draw the following conclusion from Theorem 2.2. Before stating, let us recall some definitions and facts. An ideal I of a ring R is called *regular* if it contains a regular element. An ideal is *regularly generated* if it can be generated by a set of regular elements. A ring R is called *Marot* if every regular ideal of R is regularly generated, see [47]. It is easy to see that every additively regular ring is Marot. However, the converse is not true, see [51]. An *overring of* R is a ring between R and $Q_{cl}(R)$. It is known that each overring of a Marot ring is Marot, see for example [35, Corollary 7.3].

Corollary 2.3. *The rings $C_c(X)$ and $Q_{cl}(C_c(X))$ are always Marot.*

3 Von Neumann Regular, Von Neumann Local and Classical Rings

An element a in a ring R is called *von Neumann regular*, if there exists $b \in R$ such that $a = a^2 b$. It is well-known that if a is a von Neumann regular element of a ring R, then there are a unit $u \in R$ and an idempotent $e \in R$ such that $a = ue$, see [19, Corollary 1] for example. A ring R is called von Neumann regular if each of its elements is von Neumann regular. Following [26], a space X is called a *countably P-space* (briefly, *CP-space*) if every zero-set in $Z_c(X)$ (i.e., the family of all zero-sets of $C_c(X)$) is open. Also, a point $p \in X$ is called a *CP-point* if $f(p) = 0$ (where $f \in C_c(X)$) implies that $p \in \mathrm{int}\, Z(f)$. Every P-space is a CP-space. However, the converse is not true, see [26]. It is proved that a zero-dimensional space X is a CP-space if and only if it is a P-space, see [26, Corollary 5.7]. Some algebraic and topological characterizations of CP-spaces are stated in [26, Theorem 5.8]. We present another characterization in the following theorem.

Theorem 3.1. *The following statements are equivalent.*

1. X *is a* CP*-space.*
2. $C_c(X)$ *is a von Neumann regular ring.*
3. *For* $f \in C_c(X)$*,* $Z^\circ(f) \neq \emptyset$ *implies* $Z(f)$ *is open.*

Proof. (1) $\Leftrightarrow$ (2) It follows from Theorem 5.8 in [26].
(2) $\Rightarrow$ (3) It is clear.
(3) $\Rightarrow$ (2) Assume (3). Thus, there is a nontrivial idempotent $e \in C_c(X)$. Take $f \in C_c(X)$ such that $Z^\circ(f) \neq \emptyset$. From $Z^\circ(fe) \neq \emptyset$, we have $Z(fe)$ is clopen. Hence, there is a $g \in C_c(X)$ such that $fe = (fe)^2 g$. Since $Z^\circ(f) \neq \emptyset$, we infer that $e = feg$. This implies that $Z(f) \subseteq Z(e)$. By a similar argument, $Z(f(1-e))$ is clopen and so there is a $h \in C_c(X)$ such that $1 - e = f(1-e)h$. This implies that $Z(f) \subseteq Z(1-e)$. From $Z(e) \cap Z(1-e) = \emptyset$, we conclude that $Z(f) = \emptyset$ and so f is a unit. This means that $C_c(X)$ is a von Neumann regular ring. □

The following fact is the counterpart of [56, Theorem 3.3], we present it for the sake of the reader.

Proposition 3.2. *Let* $A(X)$ *be a ring such that* $C_c^*(X) \subseteq A(X) \subseteq C_c(X)$*. If* $A(X)$ *is a von Neumann regular ring then* $A(X) = C_c(X)$*.*

Proof. Assume, for a contradiction, $f \in C_c(X) \setminus A(X)$. Define $g = f \vee 0$ and $h = -f \vee 0$. From $f = g - h$, we infer that either $g \notin A(X)$ or $h \notin A(X)$. Without loss of generality, we may assume that $g \notin A(X)$. This implies that $1 + g \notin A(X)$. From $\frac{1}{1+g} \in C_c^*(X)$, we deduce that $\frac{1}{1+g} \in A(X)$. This means that $\frac{1}{1+g}$ is neither unit nor zero-divisor. This implies that $A(X)$ is not a von Neumann regular ring, a contradiction. □

Clearly, the set of von Neumann regular elements of a ring R is multiplicatively closed. The following shows the sum of two von Neumann regular elements in $C_c(X)$ need not be von Neumann regular.

Example 3.3. Take $X = \mathbb{Q} \setminus \{0\}$ as a subspace of $\mathbb{Q}$. Consider the ring $C_c(X)$ and define

$$f(x) = \begin{cases} 0 & x < 0 \\ x^2 & x > 0. \end{cases} \qquad g(x) = \begin{cases} 0 & x < 0 \\ -x & x > 0. \end{cases}$$

Since $Z(f) = Z(g)$ is open, f and g are von Neumann regular. It is easy to check that $Z(f+g) = ((-\infty, 0) \cup \{1\}) \cap \mathbb{Q}$ that is not open. Thus, $f+g$ is not von Neumann regular in $C_c(X)$.

The above example suggests the following.

Theorem 3.4. *Let R be the set of all von Neumann regular elements of $C_c(X)$. The following statements are equivalent.*

1. *R is a von Neumann regular ring.*
2. *The sum of two units of $C_c(X)$ is a von Neumann regular element.*
3. *For any $f, g \in C_c(X)$, if $Z(f)$ and $Z(g)$ are open then $Z(f+g)$ is open.*
4. *For $f, g, h, k \in R$, $Z(f) = Z(g) = Z(h) = Z(k) = \emptyset$ implies $Z(f+g+h+k)$ is open.*

Proof. (1) $\Rightarrow$ (2) It is clear.
(2) $\Rightarrow$ (1) It follows from the fact that every element of $C_c(X)$ is a sum of two units of $C_c(X)$. For example for each $f \in C_c(X)$, we have $f = \frac{1}{2}(|f| + f + 1) + \frac{1}{2}(-|f| + f - 1)$.
(1) $\Leftrightarrow$ (3) It follows from the fact that $f \in C_c(X)$ is a von Neumann regular element if and only if $Z(f)$ is open, see [26].
(3) $\Rightarrow$ (4) Let $f, g, h, k \in R$ and $Z(f) = Z(g) = Z(h) = Z(k) = \emptyset$. From (3), we deduce $Z(f+g)$ and $Z(h+k)$ are open. Again by assuming (3), we infer that $Z(f+g+h+k)$ is open.
(4) $\Rightarrow$ (3) (We translate the proof of Theorem 2.11 in [3] to the language of rings of continuous functions.) Assume $Z(f)$ and $Z(g)$ are open for $f, g \in C_c(X)$. There are units $h, k \in C_c(X)$ such that fh and gk are idempotents. It is clear that $Z(2fh-1) = Z(2gk-1) = \emptyset$. Hence, we have $Z(f - \frac{h^{-1}}{2}) = Z(g - \frac{k^{-1}}{2}) = Z(\frac{h^{-1}}{2}) = Z(\frac{k^{-1}}{2}) = \emptyset$. From (4), we conclude that

$$Z(f+g) = Z(f - \frac{h^{-1}}{2} + g - \frac{k^{-1}}{2} + \frac{h^{-1}}{2} + \frac{k^{-1}}{2})$$

is open, as desired. $\square$

A ring is *classical* if it coincides with its classical ring of quotients. Equivalently, each of its elements is a unit or a zero-divisor, see [44, pp.

320-322] for more details. It is clear that every von Neumann regular ring is classical.

Following [41], a space X is called an *almost CP-space* if for each nonempty $Z(f) \in Z_c(X)$, we have $\operatorname{int} Z(f) \neq \emptyset$. This is equivalent to saying that every element of $C_c(X)$ is either a unit or a zero divisor, i.e., $C_c(X)$ is classical. Just for the record, we make the following fact.

Theorem 3.5. *The following statements are equivalent.*

1. *X is an almost CP-space.*
2. *$C_c(X)$ is a classical ring.*

Corollary 3.6. *For any ring $C_c(X)$, we have*

$$\text{von Neumann regular ring} \Rightarrow \text{classical ring.}$$

However, the implication is not reversible.

Proof. The first statement is clear. For the second statement, let X be the one-point compactification of an uncountable discrete space. It is clear that X is an almost CP-space that is not a CP-space. This means that $C_c(X)$ is a classical ring that is not von Neumann regular. $\square$

Following [16], an element a in a ring R is called *von Neumann local*, if either a or $1 - a$ is von Neumann regular. A ring R is called von Neumann local if each of its elements is von Neumann local. According to [46], a space X is called an *essential CP-space* whenever all points except at most one point of X are CP-point. It is not hard to see that $C_c(X)$ is a von Neumann local if and only if X is an essential CP-space. Conditions equivalent to a space being an essential CP-space are given in [46, Proposition 2.4.]. We present some other characterizations in the following proposition.

Proposition 3.7. *Let X be a topological space. The following statements are equivalent.*

1. *If $f^2 + g^2$ is a unit in $C_c(X)$, then either f or g is von Neumann regular.*
2. *If $f + g$ is a unit in $C_c(X)$, either f or g is von Neumann regular.*

3. $C_c(X)$ is a von Neumann local ring.

Proof. (1) $\Rightarrow$ (2) Assume that $f, g \in C_c(X)$ and $f+g$ is unit. Since $Z(f^2+g^2) \subseteq Z(f+g)$, we infer that f^2+g^2 is unit. From (1), we deduce that either f or g is von Neumann regular, as desired.
(2) $\Rightarrow$ (3) Let $f \in C_c(X)$. From (2) and $f+(1-f)=1$, we infer that either f or $1-f$ is von Neumann regular, as desired.
(3) $\Rightarrow$ (1) Assume that $f, g \in C_c(X)$ and f^2+g^2 is unit. Write $f^2u+g^2u=1$ where u is unit. From (3), we have that either f^2u or $1-f^2u$ is von Neumann regular. This means that either f^2u or g^2u is von Neumann regular. Since $Z(f^2u)=Z(f)$ and $Z(g^2u)=Z(g)$, we conclude that either f or g is von Neumann regular and we are done. $\square$

The following shows the sum of two von Neumann local elements in $C_c(X)$ need not be von Neumann local.

Example 3.8. Take $X = \mathbb{Q} \setminus \{0\}$ as a subspace of $\mathbb{Q}$. Consider the ring $C_c(X)$ and define

$$f(x) = \begin{cases} 0 & x<0 \\ 1+\cos^2 x & x>0. \end{cases} \qquad g(x) = \begin{cases} 0 & x<0 \\ -1-\sin^2 x & x>0. \end{cases}$$

Since $Z(f)$ and $Z(g)$ are open, f and g are von Neumann regular. On the other hand,

$$f(x)+g(x) = \begin{cases} 0 & x<0 \\ \cos 2x & x>0. \end{cases} \qquad 1-(f(x)+g(x)) = \begin{cases} 1 & x<0 \\ 2\sin^2 x & x>0. \end{cases}$$

It is easy to verify that neither $f+g$ nor $1-(f+g)$ is not von Neumann regular in $C_c(X)$.

The following shows the product of two von Neumann local elements in $C_c(X)$ need not be von Neumann local.

Example 3.9. Take $X = \mathbb{Q} \setminus \{0\}$ as a subspace of $\mathbb{Q}$. Consider the ring $C_c(X)$ and define

$$f(x) = \begin{cases} 0 & x<0 \\ -1-\cos^2 x & x>0. \end{cases} \qquad g(x) = \begin{cases} 0 & x<0 \\ -\sin^2 x & x>0. \end{cases}$$

Since $Z(f)$ and $Z(1-g)$ are open, f and g are von Neumann local. On the other hand,

$$f(x)g(x) = \begin{cases} 0 & x < 0 \\ \sin^2 x(1+\cos^2 x) & x > 0. \end{cases} \qquad 1-f(x)g(x) = \begin{cases} 1 & x < 0 \\ \cos^4 x & x > 0. \end{cases}$$

It is easy to check that neither fg nor $1-fg$ is not von Neumann regular in $C_c(X)$.

The above examples suggest the following.

Theorem 3.10. *Let R be the set of all von Neumann local elements of $C_c(X)$. The following statements are equivalent.*

1. *R is a von Neumann local ring.*

2. *$R = C_c(X)$.*

Proof. We shall only prove (1) $\Rightarrow$ (2). We show that every element of $C_c(X)$ is von Neumann local. By Theorem 2.1 (or [7, Corollary 2.8]), the ring $C_c(X)$ is always a clean ring. Write $f = u + e$ where $u \in C_c(X)$ is a unit and $e \in C_c(X)$ is an idempotent. Then $f = u+e = u(1+u^{-1}e)$. We note that $1 + u^{-1}e$ and u are von Neumann local. From (1), R is multiplicatively closed. Hence, f is von Neumann local and we are done. $\square$

4 I-rings and Baer Rings

A ring R is called *Baer* if every annihilator in R is of the form eR for some idempotent $e \in R$. We say a space X is *countably extremally disconnected* (for short, *c-extremally disconnected*) if $\bigcap_{f\in A} Z(f)$ has a closed interior for any subset $A \subseteq C_c(X)$.

The following fact shows that $C_c(X)$ is a Baer ring if and only if X is a c-extremally disconnected space. Although its proof is routine, we present it for the sake of the reader.

Theorem 4.1. *The following statements are equivalent.*

1. *$C_c(X)$ is a Baer ring.*

2. X is a c-extremally disconnected space.

Proof. (1) $\Rightarrow$ (2) Assume $S \subseteq C_c(X)$. Put $Y = \bigcap_{f\in S} Z(f)$. We have $\mathrm{Ann}(S) = \bigcap_{f\in S} \mathrm{Ann}(f)$. From (1), there exists $e^2 = e \in C_c(X)$ such that $\mathrm{Ann}(S) = (e)$. We claim that $\mathrm{int}\, Y = Z(1-e)$. First, we show that $\mathrm{int}\, Y \subseteq Z(1-e)$. Let $x \in \mathrm{int}\, Y$. With the help of Proposition 4.4 in [26], there is a $g \in C_c(X)$ such that $g(x) = 1$ and $g(X \setminus \mathrm{int}\, Y) = 0$. This yields that $x \in X\setminus Z(g) \subseteq \mathrm{int}\, Y \subseteq \bigcap_{f\in S} Z(f)$. Hence, we have $X\setminus Z(g) \subseteq Z(f)$ for each $f \in S$. This means that $g \in \mathrm{Ann}(S) = (e)$. From $g(x) = 1$, we have $x \notin Z(e)$ and so $x \in Z(1-e)$. Thus, $\mathrm{int}\, Y \subseteq Z(1-e)$. Now, we show that $Z(1-e) \subseteq \mathrm{int}\, Y$. Let $x \in Z(1-e)$. Proposition 4.4 in [26] implies that there is a $k \in C_c(X)$ such that $k(x) = 1$ and $k(X \setminus Z(1-e)) = 0$. It is clear that $(1-e)k = 0$ and so $k \in \mathrm{Ann}(S)$. This means that $kf = 0$ for each $f \in S$. Thus, we have $X \setminus Z(k) \subseteq \bigcap_{f\in S} Z(f)$. From this, $X \setminus Z(k) \subseteq \mathrm{int} \bigcap_{f\in S} Z(f)$ and so $x \in \mathrm{int} \bigcap_{f\in S} Z(f)$. Hence, we deduce that $Z(1-e) \subseteq \mathrm{int}\, Y$, as desired.
(2) $\Rightarrow$ (1) Let A be a subset of $C_c(X)$. Let I be the ideal generated by A. We show that $\mathrm{Ann}(I)$ is generated by an idempotent. Without loss of generality, we may assume that $\mathrm{int} \bigcap_{f\in I} Z(f) \neq \emptyset$. Put $Y = \mathrm{int} \bigcap_{f\in I} Z(f)$. From (2), there exists $e \in C_c(X)$ such that $e(Y) = 1$ and $e(X \setminus Y) = 0$. We claim that $\mathrm{Ann}(I) = (e)$. Obviously, $eI = 0$ and so $(e) \subseteq \mathrm{Ann}(I)$. Take $g \in \mathrm{Ann}(I)$. It is easy to see that $\mathrm{coz}\, g \subseteq Y$ and so $ge = g$. This means that $g \in (e)$. Hence, we deduce that $\mathrm{Ann}(A) = \mathrm{Ann}(I) = (e)$, as desired. $\square$

Before presenting our next observation, we need the following useful lemma. We just give the proof for the sake of the reader. First, let us recall some definitions.

According to [17], a ring R is said to be an *I-ring* if each subring of $Q_{max}(R)$ containing R is integrally closed in $Q_{max}(R)$.

An ideal in a ring R is said to be *projective* provided it is a projective R-module. We refer the reader to [44, §2] for projective modules. A ring R is *semihereditary* if every finitely generated ideal of R is projective. If I is an ideal of R, and $I^{-1} = \{r \in Q_{cl}(R) : rI \subset R\}$ (the inverse of I), where I is an ideal of R, then I is *invertible* if $II^{-1} = R$. A ring R is called *Prüfer* if every finitely generated regular ideal is invertible. Obviously, every classical ring is Prüfer.

Lemma 4.2. *Every reduced I-ring is a semihereditary Baer ring.*

Proof. Let R be a reduced I-ring. With the help of Theorems 6 and 9 in [17], we deduce that R is a Prüfer ring and $Q_{cl}(R) = Q_{max}(R)$. Since R is a reduced ring, R has an injective maximal quotient ring. This implies that $Q_{cl}(R)$ is self-injective, and so R is a semihereditary ring. Thus, every idempotent of $Q_{cl}(R)$ belongs to R. Now since $Q_{cl}(R)$ is a Baer ring, we infer that R is Baer, see [12, Proposition 3.1.5(ii)]. □

In [48, Theorem 2.7], Martinez proved that $C(X)$ is a Baer ring if and only if it is an I-ring. To see an algebraic proof of Martinez's result, the reader refer to [27]. In contrast to $C(X)$, we show that a Baer ring $C_c(X)$ may not be an I-ring.

Corollary 4.3. *For any ring $C_c(X)$, we have*

$$I\text{-}ring \Rightarrow Baer\ ring.$$

However, the implication is not reversible.

Proof. The first statement follows from Lemma 4.2. For the second statement, let X be an uncountable discrete space. We follow the constructions of [26, Remark 7.4]. As was mentioned in [26, Remark 7.4, line 17], $C_c(X)$ is a von Neumann regular Baer ring. This yields that $Q_{cl}(C_c(X)) = C_c(X)$. As was noted in [26, Remark 7.4, line 11], $C_c(X)$ is not self-injective and so $C_c(X) \neq Q_{max}(C_c(X))$ (note, $Q_{max}(R)$ is self-injective for each reduced ring R, see [44, Corollary 13.37]). As mentioned in the proof of Lemma 4.2, we have $Q_{cl}(R) = Q_{max}(R)$ for every I-ring R. This implies that $C_c(X)$ is not an I-ring. □

Remark 4.4. According to [33], $C(X)$ is called *fraction-dense* if $Q_{cl}(C(X))$ and $Q_{max}(C(X))$ have the same idempotents. A ring $C(X)$ is called *strongly fraction-dense* if $Q_{cl}(C(X)) = Q_{max}(C(X))$, see [8, 11, 33] for more details. Obviously, $C(X)$ is fraction-dense whenever it is strongly fraction-dense. The question of whether there is a space in which $C(X)$ is fraction-dense but not strongly fraction-dense is an open problem in [33, p. 983]. In [27], it is shown that X is a fraction-dense space if and only if $Q_{cl}(X)$ is a continuous ring.

It would be interesting to know that the answer is negative if we replace $C(X)$ with $C_c(X)$. To see this, let $C_c(X)$ be the same as Corollary

4.3. As we have observed, $Q_{max}(C_c(X)) \neq Q_{cl}(C_c(X))$. This shows that $C_c(X)$ is not strongly fraction-dense. On the other hand, $Q_{cl}(C_c(X))$ is a Baer ring since $Q_{cl}(C_c(X)) = C_c(X)$ is a Bear ring. Using [9, Theorem 4.3((4) $\Leftrightarrow$ (5))], we deduce $Q_{cl}(C_c(X))$ and $Q_{max}(C_c(X))$ have the same idempotents. This means that $C_c(X)$ is fraction-dense.

5 p.p. Rings, Almost p.p. Rings and p.f. Rings

R is called a *p.p. ring* if its principal ideals are projective, or equivalently, if the annihilator of each element is generated by an idempotent. It is known that R is a p.p. ring if every element of R can be written as the product of a non zero-divisor and an idempotent, see [18], [38] and [53] for examples.

Following [46], a space X is called *countably basically disconnected* (for short, c-basically disconnected) if every $\operatorname{cl}\operatorname{coz} f$, where $f \in C_c(X)$, is open. Every c-extremally disconnected space is c-basically disconnected. The following example shows that there is a c-basically disconnected space that is not c-extremally disconnected.

Example 5.1. Let X be the one-point Lindelöfization of an uncountable discrete space, (See [63] for more information). Suppose that D is an uncountable discrete space, and let p be a point not in D. Let $X = D \cup \{p\}$, and say that $U \subseteq X$ is open if and only if either $p \notin U$, or $p \in U$ and $X \setminus U$ is countable. It is not hard to see that G_δ-sets in X are open. This means that X is a CP-space, and so X is a c-basically disconnected. Now let $X \setminus \{p\} = U \cup V$, where U and V are uncountable and disjoint. Obviously, U and V are open, but $\operatorname{cl} U = U \cup \{p\}$ and $\operatorname{cl} V = V \cup \{p\}$ are not disjoint, so X is not c-extremally disconnected.

Theorem 5.2. *The following statements are equivalent.*

1. *$C_c(X)$ is a p.p. ring.*
2. *X is a c-basically disconnected space.*

Proof. (1) $\Rightarrow$ (2) Let $f \in C_c(X)$. From (1), there is an idempotent $e \in C_c(X)$ such that $\operatorname{Ann}(f) = (e)$. This implies that $\operatorname{coz} f \subseteq Z(e)$. It suffices to show that $\operatorname{cl}\operatorname{coz} f = Z(e)$. Assume, for a contradiction, there

exists $x \in Z(e) \setminus \operatorname{cl}\operatorname{coz} f \subseteq Z(e)$. By the complete regularity of X, there exists $g \in C_c(X)$ such that $g(x) = 1$ and $g(\operatorname{cl}\operatorname{coz} f) = 0$. It is clear that $g \in \operatorname{Ann}(f)$ and $g \notin (e)$. That is a contradiction. Hence $\operatorname{cl}\operatorname{coz} f = Z(e)$. This means that $\operatorname{cl}\operatorname{coz} f$ is open, as desired.
(2) $\Rightarrow$ (1) Let $f \in C_c(X)$. From (2), $\operatorname{cl}\operatorname{coz} f$ is open. Consider the idempotent $e \in C_c(X)$ with $e(x) = 0$ for all $x \in \operatorname{cl}\operatorname{coz} f$ and $e(x) = 1$ otherwise. It is clear that $e \in \operatorname{Ann}(f)$. Take $h \in \operatorname{Ann}(f)$. Thus, we have $\operatorname{coz} f \subseteq Z(h)$ and so $\operatorname{cl}\operatorname{coz} f \subseteq Z(h)$. This implies $Z(e) \subseteq Z(h)$. By [26, Lemma 2.4], h is a multiple of e. This yields that $\operatorname{Ann}(f) = (e)$, as desired. $\square$

According to [2], a ring R is called an *almost p.p. ring* if for each $a \in R$, the annihilator ideal $\operatorname{Ann}(a)$ is generated by its idempotents (the terms *almost weak Baer* [59] and *feebly Baer* [42] are also used). A ring R is called a *p.f. ring* if every principal ideal of R is flat (also known as *PIF* [50]). It is known that a ring R is a p.f. ring if and only if $\operatorname{Ann}(a)$ is a pure ideal for each $a \in R$, see [2] or [5]. An ideal I of a ring R is called *pure* if $I \cap J = IJ$ for every ideal J of R, or equivalently, if for any $a \in I$, there exists $b \in I$ such that $a = ab$. Conditions equivalent to a ring being a p.f. ring are given in [50, Proposition 2.1].

Remark 5.3. For any ring $C(X)$, we have

$$\text{p.p. ring} \Rightarrow \text{almost p.p. ring} \Rightarrow \text{p.f. ring}.$$

However, neither implication is reversible, see [2] for example.

Following [7], a space X (not necessarily zero-dimensional) is called an *F_c-space* if every localization $C_c(X)_P$ of $C_c(X)$ at a prime ideal P is a domain. Every c-basically disconnected space is an F_c-space. However, the converse is not true. As mentioned in [7, Remark 6.7.], $\beta\mathbb{N} \setminus \mathbb{N}$ is an F_c-space that is not c-basically disconnected.

Theorem 5.4. *The following statements are equivalent.*

1. *$C_c(X)$ is an almost p.p. ring.*
2. *$C_c(X)$ is a p.f. ring.*
3. *X is an F_c-space.*

4. *For each $f \in C_c(X)$, there exists a unit $u \in C_c(X)$ such that $f = u|f|$.*

Proof. (1) $\Rightarrow$ (2) It is clear.
(2) $\Rightarrow$ (1) It follows from Theorem 2.1 and the fact that every pure ideal of a clean ring is generated by idempotents, see [54, Theorem 1.7(1) $\Leftrightarrow$ (15)].
(2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) It follows from Theorem 5.9 in [7]. □

Example 5.5. The space $\mathbb{Q}$ with the usual topology is a strongly zero-dimensional space (i.e., for every $Z_1, Z_2 \in Z(\mathbb{Q})$, with $Z_1 \cap Z_2 = \emptyset$ there exists a clopen set F such that $Z_1 \subseteq F$ and $Z_2 \cap F = \emptyset$).

It is not hard to see that a metric strongly zero-dimensional F_c-space is discrete. By Theorem 5.4, $C_c(\mathbb{Q})$ is not a p.f. ring.

6 Self-injective Rings and Continuous Rings

A ring R is said to be *self-injective* whenever any homomorphism $g : I \to R$, for each arbitrary ideal I of R, can be extended to R, i.e., there exists r in R such that $f(x) = rx$ for every x in I, see [44, §3] for additional information regarding this concept.

Theorem 6.1. *For a reduced ring R the following are equivalent.*

1. *R is a von Neumann regular I-ring.*
2. *R is a classical I-ring.*
3. *R is a self-injective ring.*

Proof. (1) $\Rightarrow$ (2) It is clear.
(2) $\Rightarrow$ (3) Assume R is an I ring. Thus, we have $Q_{cl}(R) = Q_{max}(R)$ by Theorem 9 in [17]. Since R is a classical ring, we deduce that $R = Q_{max}(R)$ that implies that R is self-injective.
(3) $\Rightarrow$ (1) It is clear. □

An ideal I is said to be *essential* in a ring R, if its intersection with any nonzero ideal is nonzero, in this case we write $I \leq_{ess} R$. It goes without saying that I in a reduced ring R is essential if and only if $\mathrm{Ann}(I) = 0$.

The next result is the counterpart of [4, Theorem 3.1].

Lemma 6.2. *The following statements are equivalent.*

1. *I is an essential ideal in $C_c(X)$.*
2. *$\bigcap Z[I]$ is a nowhere dense subset of X.*

Lemma 6.3. *Let I be an ideal of $C_c(X)$. The following statements are equivalent.*

1. *I is not an essential ideal in $C_c(X)$.*
2. *$I \subseteq (e)$ for some idempotent $e \in C_c(X)$.*
3. *$\bigcap Z[I]$ contains a clopen set.*

Proof. (1) $\Rightarrow$ (2) Assume (1). Thus, $I \cap J = 0$ for some ideal J in $C_c(X)$. It is known that every ideal of a clean ring R with $J(R) = 0$ contains a nonzero idempotent. Since $C_c(X)$ is a clean ring, we have $I \cap (e) = 0$ for some $e^2 = e \in J$. This yields that $I \subseteq (1-e)$, as desired.
(2) $\Rightarrow$ (3) Assume (2). We deduce that $Z(e) \subseteq \bigcap Z[I]$, as desired.
(3) $\Rightarrow$ (1) It follows from Lemma 6.2. $\square$

Let I be an ideal of a ring R. By a *complement of I* in R, we mean an ideal J of R, maximal with respect to the property $I \cap J= 0$. Zorn's Lemma guarantees the existence of complements of ideals in a ring. In other words, every ideal of a ring has a complement. Recall that an ideal I of R is a *direct summand* of R if there is an ideal J of R with $R = I \oplus J$.

Lemma 6.4. *For any ideal I of $C_c(X)$, the following are equivalent.*

1. *$I = (e)$, where $e \in C_c(X)$ is an idempotent.*
2. *I is a direct summand of $C_c(X)$.*
3. *(a) $\bigcap_{f\in I} Z(f)$ is clopen.*
   (b) For any $g \in C_c(X)$, $\bigcap_{f\in I} Z(f) \subseteq Z(g)$ implies $g \in I$.

Proof. (1) $\Leftrightarrow$ (2) It follows from [43, §1, Exercise 7].
(2) $\Rightarrow$ (3) Suppose that $I \oplus J = C_c(X)$. Write $1 = e_1 + e_2$, where $e_1 \in I$ and $e_2 \in J$. Left multiplying by e_1, we have $e_1 = e_1e_1 + e_1e_2$. This show that $e_1^2 = e_1$ and $e_1e_2 = 0$. For any $x \in I$, we also have

$x = xe_1 + xe_2$. So $x = xe_1 \in (e_1)$. This shows that $I = (e_1)$. This yields $\bigcap_{f\in I} Z(f) = Z(e_1)$ and so $\bigcap_{f\in I} Z(f)$ is clopen. Now, suppose that $g \in C_c(X)$ such that $Z(e_1) \subseteq Z(g)$. Using [26, Lemma 2.4], we infer that g is a multiple of e_1. Thus, we have $g \in I$ as desired.
(3) $\Rightarrow$ (2) Define $e \in C_c(X)$ as follows.

$$e(x) = \begin{cases} 0 & x \in \bigcap_{f\in I} Z(f) \\ 1 & x \in X \setminus \bigcap_{f\in I} Z(f) \end{cases}$$

By assumption, we have $e \in I$ and so $(e) \subseteq I$. On the other hand, we have $k = ke$ for any $k \in I$. This shows that $I = (e)$. This implies that I is a direct summand of $C_c(X)$. $\square$

Following [32], we say that an ideal I is *essentially closed* in a ring R provided I has no proper essential extensions within R. In short, I is essentially closed in R if and only if $I \leq_{ess} J \leq R$ always implies $J = I$.

Lemma 6.5. *Let I be an ideal of $C_c(X)$. The following statements are equivalent.*

1. *I is a essentially closed ideal.*
2. *For any $g \in C_c(X)$, $\operatorname{int}\bigcap_{f\in I} Z(f) \subseteq Z(g)$ implies $g \in I$.*

Proof. (1) $\Rightarrow$ (2) Assume that $g \in C_c(X)$ such that $\operatorname{int}\bigcap_{f\in I} Z(f) \subseteq Z(g)$. This implies that $I \leq_{ess} I + (g)$. From (1), we infer that $g \in I$, as desired.
(2) $\Rightarrow$ (1) Assume that, for a contradiction, $I \leq_{ess} J$ and $h \in J \setminus I$. From (2), $x_0 \in \operatorname{int}\bigcap_{f\in I} Z(f) \setminus Z(h)$. By Proposition 4.4 in [26], there is a $k \in C_c(X)$ such that $k(x_0) = 1$ and $k(x) = 0$ for $x \notin \operatorname{int}\bigcap_{f\in I} Z(f) \setminus Z(h)$. It is clear that $kh \neq 0$ and $(kh)f = 0$ for each $f \in I$. That is a contradiction. $\square$

Consider the following conditions on a ring R:

(C1) Every nonzero ideal is essential in a direct summand of R.

(C2) Every ideal that is isomorphic to a direct summand of R is also a direct summand of R.

(C3) If $eR \cap fR = 0$ where e and f are idempotents in R, then $eR \oplus fR$ is a direct summand of R.

R is called *continuous* if it satisfies conditions (C1) and (C2) (and hence C3), and a CS-*ring* if it satisfies condition (C1) only. It is clear that a ring R is CS if and only if every essentially closed ideal is a direct summand of R, see [12, §2] for more details.

We need the following facts.

Remark 6.6. For any ring R, we have

$$\text{self-injective} \Rightarrow \text{continuous} \Rightarrow \text{CS}.$$

But, we have to notice that the above implications are not reversible, see [12, Example 2.1.14].

At this point, we need the following known theorem, a proof for it is included for the reader's convenience.

Theorem 6.7. *Let R be a commutative ring. The following statements are equivalent.*

1. *R is a von Neumann regular Baer ring.*
2. *R is a reduced continuous ring.*

Proof. (1) $\Rightarrow$ (2) It follows from Proposition 3.1 in [24].
(2) $\Rightarrow$ (1) We first recall that a result of Yousif [65], which states R is continuous if and only if $R/J(R)$ is von Neumann regular and R is a CS-ring where its the singular ideal is its Jacobson radical. Note that the singular ideal of a commutative reduced ring is zero. Now, suppose that R is commutative reduced continuous. Hence R is a von Neumann regular CS-ring. It is known that a commutative reduced ring R is a CS-ring if and only if it is Baer, see [12, Theorem 3.3.1]. Hence R is a von Neumann regular Baer ring. □

With the help of Theorem 6.7 and [6, Theorem 4.6(2) $\Leftrightarrow$ (5)], we deduce that $C(X)$ is a self-injective ring if and only if it is a continuous ring. In the following, we show that a continuous ring $C_c(X)$ may not be self-injective.

Corollary 6.8. *For any ring $C_c(X)$, we have*

$$\textit{self-injective} \Rightarrow \textit{continuous}.$$

However, the implication is not reversible.

Proof. The first statement is clear by Remark 6.6. For the second statement, let X be a discrete space, which is uncountable. We follow the constructions of [26, Remark 7.4]. As noted in [26, Remark 7.4, line 17], $C_c(X)$ is a von Neumann regular Baer ring. By Theorem 6.7, we infer that $C_c(X)$ is a continuous ring. On the other hand, the authors in [26, Remark 7.4, line 11] asserted that $C_c(X)$ is not self-injective. $\square$

7 Elementary Divisor, Bézout and Arithmetical Rings

In this section we deal with $C_c(X)$ when it is a Bézout ring (i.e., every finitely generated ideal is principal).

First, let us recall some definitions. Following Kaplansky [39], a ring R is said to be an *elementary divisor ring* if each matrix over R is equivalent to a diagonal one. A ring R is called *Hermite* if each 1 by 2 and 2 by 1 matrix over R is equivalent to a diagonal matrix. A ring R is called *arithmetical* if the lattice of ideals of R is distributive.

We need the following observations.

Remark 7.1. For any ring R, we have

$$\text{elementary divisor} \Rightarrow \text{Hermite} \Rightarrow \text{Bézout} \Rightarrow \text{arithmetical}.$$

However, neither implication is reversible, see [28] and [37].

Remark 7.2. For any ring $C(X)$, we have

$$\text{elementary divisor} \Rightarrow \text{Hermite} \Rightarrow \text{Bézout}.$$

However, neither implication is reversible, see [28]. It is known that $C(X)$ is a Bézout ring if and only if it is arithmetical ring, see [5] for example.

Remark 7.3. For a ring R w.gl.dim R is the smallest non-negative integer n (if it exists) such that Tor_{n+1}^{R} is the 0-functor. Otherwise, w.gl.dim $R = \infty$. Thus, w.gl.dim $R \leq 1$ if every ideal or equivalently, if every finitely generated ideal of R is flat.

Let A be a reduced ring. With the help of [31, Corollary 4.2.6] and

[36, Theorem, p. 952], w.gl.dim $A \leq 1$ if and only if A is an arithmetical p.f. ring, or equivalently, if and only if A is an arithmetical ring.

Corollary 7.2 of [7] states if $C_c(X)$ is a Bézout ring, then X is an F_c-space. In [7, p. 382], the authors pointed out that "we are undecided about the converse of this result". We hope that Theorem 7.4 sheds some light on this statement.

Theorem 7.4. *The following statements are equivalent.*

1. *$C_c(X)$ is an elementary divisor ring.*
2. *$C_c(X)$ is a Hermite ring.*
3. *$C_c(X)$ is a Bézout ring.*
4. *$C_c(X)$ is an arithmetical ring.*
5. *w.gl.dim $C_c(X) \leq 1$.*
6. *$C_c(X)$ is an arithmetical p.f ring.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) It follows from Remark 7.1.
(3) $\Rightarrow$ (1) Assume (3). In [52], it is proved that Bézout rings whose proper homomorphic images all have stable range 1 are elementary divisor rings. It is known that every clean ring has stable range 1. Since $C_c(X)$ is a clean ring, every homomorphic image of $C_c(X)$ is clean and so has stable range 1 (i.e., a ring R is said to have *stable range 1* if, for any $a, b \in R$, $Ra + Rb = R$ implies that $a + bx$ is a unit for some $x \in R$). It is well-known that every clean ring has stable range 1, see [64] for more details. This implies that $C_c(X)$ is an elementary divisor ring.
(4) $\Rightarrow$ (3) In view of Lemma 1.7 in [61], it suffices that every Pierce stalk of the ring is a Bézout ring. Let $S(C_c(X))$ be the nonempty set of all the proper ideals of a ring $C_c(X)$ generated by idempotents. By Zorn's Lemma, $S(C_c(X))$ contains maximal elements. If P is a maximal element of the set $S(C_c(X))$, then the factor ring $C_c(X)/P$ is called a *Pierce stalk* of $C_c(X)$. Let $C_c(X)/P$ be a Pierce stalk of $C_c(X)$. From (4), we infer that $C_c(X)/P$ is an arithmetical ring. With the help of [15, Proposition 1.2] and the fact that $C_c(X)$ is a clean ring, we deduce that $C_c(X)/P$ is a local ring. This implies that $C_c(X)/P$ is a Bézout ring,

see [37, Theorem 5], as desired.
(4) ⇔ (5) ⇔ (6) It follows from Remark 7.3 and the fact that $C_c(X)$ is a reduced ring. □

In a natural generalization of the Bézout property, a ring R is called a *quasi-Bézout ring* (the terms *regular Bézout* [45] and *almost Bézout* [13] are also used) if each finitely generated regular ideal of R is principal. Clearly, every quasi-Bézout ring is Prüfer. However, the converse is not true. For example the domain $\mathbb{Z}[\sqrt{-5}]$ is a Prüfer ring that is not quasi-Bézout, see [49] for example.

We can draw the following conclusion from [49, Theorem 2].

Corollary 7.5. *$C_c(X)$ is a quasi-Bézout ring if and only if it is a Prüfer ring,*

We end this paper with a result on regular ideals of rings of continuous functions with countable images.

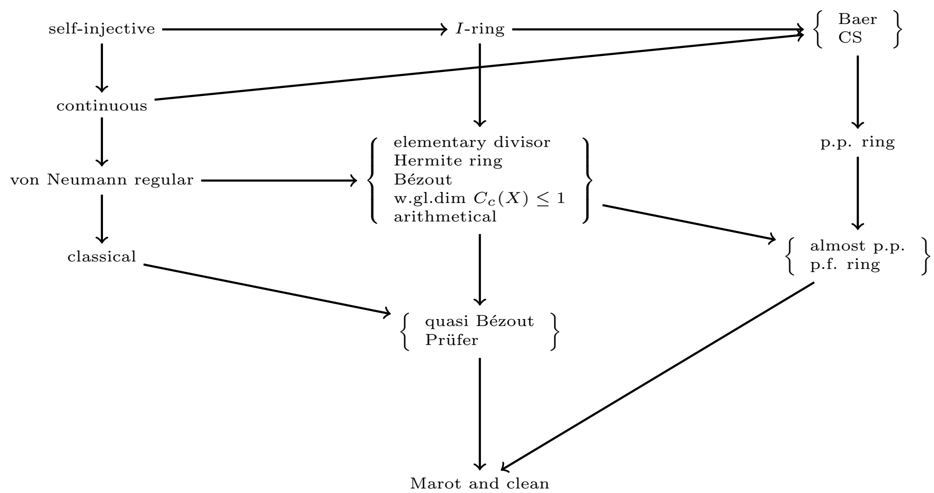
Corollary 7.6. *Let $C_c(X)$ be a quasi-Bézout ring. The following are equivalent for an ideal I of $C_c(X)$.*

1. *Every element of I is a multiple of a regular element of I.*
2. *I is regularly generated.*
3. *I contains a regular element.*

Proof. (1) ⇒ (2) ⇒ (3) is trivial.
(3) ⇒ (2) It follows from Corollary 2.3.
(2) ⇒ (1) Let $f \in I$. From (2), $f = \sum_{i=1}^{n} r_i g_i$ where r_i's are regular. Since $C_c(X)$ is quasi-Bézout, the ideal $J = (r_1, r_2, ..., r_n)$ is principal, say $J = (r')$. From $f \in J = (r')$, we deduce that f is a multiple of r', as desired. □

8 A Taxonomy of $C_c(X)$

We summarize the relationship between the properties we have studied in this paper for $C_c(X)$ by the following diagram (note, the concepts placed in each bracket are equivalent for $C_c(X)$):



1. Baer ring $\not\Rightarrow$ I-ring.
2. Continuous ring $\not\Rightarrow$ Self-injective ring.
3. Classical ring $\not\Rightarrow$ von Neumann regular ring.
4. p.p. ring $\not\Rightarrow$ Baer ring.
5. p.f. ring $\not\Rightarrow$ p.p. ring.
6. $C_c(X)$ is not always a p.f. ring.
7. $C_c(X)$ is always clean and Marot.

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