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A Note on Amenability Modulo an Ideal of Unitial Banach Algebras

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Abstract. In this paper we are continuing the study of the concept of amenability modulo an ideal of Banach algebras which is an extension of the usual notion of amenability. Analogous to the amenability of Banach algebra, we show the relation between amenability modulo an ideal of Banach algebra and its unitization.

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1. Introduction

The concept of the amenability of (discrete) groups was considered by J. Von Neumann in 1929 [19]. Since the 1940s, amenability has become an important concept in abstract harmonic analysis. But in terms of amenability for the topological groups and semigroups was used by Day in 1950 and 1957 respectively in Articles [6] and [7]. A Hausdorff and locally compact group G is called to be amenable when there exists a left invariant mean on $L^1(G)$. This concept comes to the attention of

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mathematicians and since then they wrote numerous articles about this theory. The concept of amenability of Banach algebras was introduced by Barry Johnson in 1972 [12]. He showed that for a Hausdorff and locally compact group G, G is amenable (in the usual sense) if and only if (resp. $l^1(G)$) $L^1(G)$ is amenable. In fact, he defined the amenability of a Banach algebra A through vanishing of first cohomology groups of A with coefficients in X^* for each Banach A-bimodule X, where X^* denotes the first dual space of X, which is a Banach A-bimodule in the usual way. This fact fails for semigroups in general. If S is E-unitary inverse semigroup with the set of idempotents E, amenability of $l^1(S)$ implies that E is finite [8], but there are many amenable E-unitary inverse semigroups (including the bicyclic semigroup and many Clifford semigroups) with an infinite set of idempotents.

As in the above it is mentioned, theorem of Johnson fails to be true for discrete semigroups. Under what conditions Johnson's theorem holds for semigroup algebra? For semigroup S, some necessary and sufficient conditions (in especial cases) for amenability of semigroup algebra $l^1(S)$ was introduced. For example, when E is finite, amenability of $l^1(S)$ is equivalent to amenability of all maximal subgroups of S [8, 5]. So it seems that the expression of another concept for amenability of Banach algebras in dealing with the concept of amenability of semigroup is essential.

The concept of amenability modulo an ideal introduced and initiated by the first author and M. Amini in [1]. They proved that for semigroup S, amenability of $l^1(S)$ modulo ideals that induced by certain classes of group congruences on S is equivalent to the amenability of S. This could be considered as a restoring the Johnson's theorem for a large class of semigroups. Basic properties of amenability modulo an ideal such as virtual and approximate diagonal modulo an ideal, contractible modulo an ideal was studied in [14] by the authors. They showed that for the semigroup S, the semigroup algebra $l^1(S)$ is contractible modulo an ideal if and only if $\frac{S}{\sigma}$ is finite, restoring the Selivanov's theorem for a large class of semigroups [18].

In this paper, we show that amenability (contractibility) modulo an ideal of Banach algebra A and amenability (contractibility) modulo an ideal of the unitization $A^\sharp$ of A, are equivalent. Also by using these results we give some hereditary property of amenability modulo an ideal of Banach algebras.

2. Preliminaries

Let A be a Banach algebra. A Banach space X which is also a A-bimodule is said to be a Banach A-bimodule if there is $C > 0$ such that

$$\|a \cdot x\| \leqslant C\|a\|\|x\| \ , \ \|x \cdot a\| \leqslant C\|a\|\|x\| \quad (a \in A, x \in X).$$

The minimum constant C that can occur in above inequalities is denoted by C_X. Let A be a Banach algebra and X be a Banach A-bimodule, then X^* is a Banach A-bimodule under the actions:

$$\langle a \cdot f, x\rangle = \langle f, x \cdot a\rangle \ , \ \langle f \cdot a, x\rangle = \langle f, a \cdot x\rangle, \quad (a \in A \, , \, x \in X \, , \, f \in X^*).$$

A bounded linear map $D : A \to X$ is called a *derivation* if for all $a, b \in A$, $D(ab) = a \cdot D(b) + D(a) \cdot b$. The derivation D is said to be inner if there exists $x \in X$ such that $D(a) = a.x - x.a$ for each $a \in A$. We Denote by $\mathcal{Z}^1(A, X)$ the space of all bounded derivations from A into X and by $\mathcal{B}^1(A, X)$ the space of all inner derivations from A into X. The quotient space $\mathcal{H}^1(A, X) = \mathcal{Z}^1(A, X)/\mathcal{B}^1(A, X)$ is called the first cohomology group of A with coefficients in X. A Banach algebra A is said to be amenable if $\mathcal{H}^1(A, X^*) = \{0\}$ and A is called contractible (super-amenable) if $\mathcal{H}^1(A, X) = 0$ for every Banach A-bimodule X. There are many alternative formulations of the notion of amenability, of which we note the following, for further details see [2,3,4,11,13,15,16,17]. In the following we recall the concept of amenability modulo an ideal and contractibility modulo an ideal of [1,14].

Definition 2.1. [1, Definition 1] *Let I be a closed ideal of A. A is called amenable modulo I if for every Banach A-bimodule X such that $I \cdot X = X \cdot I = 0$, every derivation D from A into X^* is inner on the set theoretic difference $A \setminus I := \{a \in A : a \notin I\}$.*

We denote the space of all bounded derivation from A into X such that $I \cdot X = X \cdot I = 0$ by $\mathcal{Z}^1_I(A, X)$, the space of all inner derivation on $A \setminus I$

by $\mathcal{B}_I^1(A,X)$ and the first Hochschild cohomology group of A modulo I with coefficients in X by

$$\mathcal{H}_I^1(A,X) := \mathcal{Z}_I^1(A,X)/\mathcal{B}_I^1(A,X).$$

Definition 2.2. [14, Definition 3.1] *A Banach algebra A is contractible modulo I if for every Banach A-bimodule X such that $I \cdot X = X \cdot I = 0$, every bounded derivation D from A into X is an inner derivation on the set theoretic difference $A \setminus I := \{a \in A : a \notin I\}$.*

3. Main Results

One could modify the proof of the following Lemma (which is essentially due to Ghahramani and Loy [10]) to get the same result for modulo an ideal.

Lemma 3.1. *Let A be an unital Banach algebra amenable modulo I, X be an A-bimodule such that $I \cdot X = X \cdot I = 0$ and $D : A \to X^*$ a derivation. Then there are $\phi \in e \cdot X^* \cdot e$, and $\eta \in X^*$, such that*

1. *$\|\eta\| \leqslant 2C_X\|D\|$;*
2. *$D = ad_\phi + ad_\eta$ on $A \setminus I$.*

Proof. As in Lemma 2.3 [10], there is a derivation $D_1 : A \to e{\cdot}X^*{\cdot}e$, $\eta \in X^*$ such that $\|\eta\| \leqslant 2C_X\|D\|$ and $D = D_1 + ad_\eta$. Now $e{\cdot}X^*{\cdot}e \cong (e{\cdot}X{\cdot}e)^*$ isometrically, A is amenable modulo I and $I \cdot (e \cdot X \cdot e) = (e \cdot X \cdot e) \cdot I = 0$ then there is a $\phi \in e \cdot X^* \cdot e$ such that $D_1 = ad_\phi$ on $A \setminus I$. Thus $D = ad_\phi + ad_\eta$ on $A \setminus I$. □

We recall that for a normed algebra A, the unitization of A over $\mathbb{C}$ is the normed algebra consisting of the set $A \times \mathbb{C}$ with product defined by

$$(a,\lambda)(b,\mu) = (ab + \lambda b + \mu a, \lambda\mu)\,(a,b \in A,\ \lambda,\mu \in \mathbb{C})$$

and norm defined by $\|(a,\lambda)\| = \|a\| + |\lambda|$. The unitization of A is denoted by $A^\sharp = A \oplus \mathbb{C}$ and is a normed algebra with unit element $(0,1)$.

Theorem 3.2. *A is amenable modulo I if and only if $A^\sharp$ is amenable modulo I.*

Proof. Let X be an $A^\sharp$-bimodule such that $I \cdot X = X \cdot I = 0$ and $D : A^\sharp \to X^*$ be a derivation. Since $A^\sharp$ is an unital algebra with unit $e = (0, 1)$, Lemma 2.3 [10] implies that $D = D_1 + ad_\eta$ where $\eta \in X^*$ and $D_1 : A^\sharp \to e \cdot X^* \cdot e$ is a derivation. Now $D_1|_A : A \to e \cdot X^* \cdot e$ is a derivation and A is amenable modulo I so there is $\psi \in e \cdot X^* \cdot e$ such that $D_1|_A = ad_\psi$ on $A \setminus I$. Then $D(a) = ad_\psi(a) + ad_\eta(a)\,(a \in A \setminus I)$. Thus $D_1 e = e \cdot De \cdot e = (De - De \cdot e) \cdot e = 0$. It follows that $De = D_1 e + ad_\eta(e)$ so $De = ad_\eta(e)$. Hence D is inner on $A^\sharp \setminus I$.
Conversely, let X be an A-bimodule such that $I \cdot X = X \cdot I = 0$ and $D : A \to X^*$ be a derivation. Set $\tilde{D} : A^\sharp \to X^*$ by $\tilde{D}(a, \lambda) = D(a)\,((a, \lambda) \in A^\sharp)$ and $\tilde{D}(e) = 0$. Then X can be made into Banach $A^\sharp$-bimodule by defining

$$(a, \lambda) \cdot x = a \cdot x + \lambda x \quad , \quad x \cdot (a, \lambda) = x \cdot a + \lambda x \qquad (x \in X,\ (a, \lambda) \in A^\sharp).$$

We have $e \cdot Da = Da = Da \cdot e\,(a \in A)$ and

$$\begin{aligned}
\tilde{D}((a_1, \lambda_1)(a_2, \lambda_2)) &= \tilde{D}(a_1 a_2 + \lambda_2 a_1 + \lambda_1 a_2, \lambda_1 \lambda_2) \\
&= D(a_1 a_2) + \lambda_2 e \cdot D(a_1) + \lambda_1 e \cdot D(a_2) \\
&= D(a_1) \cdot a_2 + a_1 \cdot D(a_2) + \lambda_2 e \cdot D(a_1) + \lambda_1 e \cdot D(a_2) \\
&= D(a_1) \cdot (a_2 + \lambda_2 e) + (a_1 + \lambda_1 e) \cdot D(a_2) \\
&= \tilde{D}(a_1, \lambda_1) \cdot (a_2, \lambda_2) + (a_1, \lambda_1) \cdot \tilde{D}(a_2, \lambda_2)
\end{aligned}$$

where $a_1, a_2 \in A, \lambda_1, \lambda_2 \in \mathbb{C}$. Since $A^\sharp$ is amenable modulo I, there exists $\psi \in X^*$ such that $\tilde{D} = \delta_\psi$ on $A^\sharp \setminus I$. So

$$\begin{aligned}
\langle \tilde{D}(a, \lambda)\,,\, x \rangle &= \langle (a, \lambda) \cdot \psi - \psi \cdot (a, \lambda)\,,\, x \rangle \\
&= \langle (a, \lambda) \cdot \psi\,,\, x \rangle - \langle \psi \cdot (a, \lambda)\,,\, x \rangle \\
&= \langle \psi\,,\, x \cdot a + \lambda x \rangle - \langle \psi\,,\, a \cdot x + \lambda x \rangle \\
&= \langle \psi\,,\, x \cdot a \rangle + \langle \psi\,,\, \lambda x \rangle - \langle \psi\,,\, a \cdot x \rangle - \langle \psi\,,\, \lambda x \rangle \\
&= \langle a \cdot \psi - \psi \cdot a\,,\, x \rangle
\end{aligned}$$

Then D is inner on $A \setminus I$. $\square$

Theorem 3.3. *A is contractible modulo I if and only if $A^\sharp$ is contractible modulo I.*

Proof. In the proof of Theorem 3.2, replace X by X^*. $\square$

Replacing X instead of X^* in the Lemma 2.3 and 2.4 [10], we have the following lemmas.

Lemma 3.4. *Let A be a unital Banach algebra with identity e, X an A-bimodule and $D : A \to X$ be a derivation. Then there is a derivation $D_1 : A \to e \cdot X \cdot e$ and $\eta \in X$ such that*

1. *$\|\eta\| \leqslant 2C_X\|D\|$;*
2. *$D = D_1 + ad_\eta$ on $A \setminus I$.*

Lemma 3.5. *Let A be an unital Banach algebra contractible modulo I, X be an A-bimodule such that $I \cdot X = X \cdot I = 0$ and $D : A \to X$ a derivation. Then there are $\phi \in e \cdot X \cdot e$, and $\eta \in X$, such that*

1. *$\|\eta\| \leqslant 2C_X\|D\|$;*
2. *$D = ad_\phi + ad_\eta$ on $A \setminus I$.*

Theorem 3.6. *Let I be a closed ideal of A and A be amenable modulo I. If X is a commutative Banach A-bimodule such that $I \cdot X = 0$, then every derivation $D : A \to X$ vanishes on $A \setminus I$.*

Proof. Let $D : A \to X$ be a derivation and τ be the canonical mapping of X into X^{**}. Clearly that $\tau \circ D : A \to X^{**}$ is a derivation and $I \cdot X^* = X^* \cdot I = 0$. Since A is amenable modulo I, there is a $f \in X^{**}$ such that $\tau \circ D = \delta_f$ on $A \setminus I$. Now X^{**} is a commutative Banach A-bimodule and $\tau \circ D = 0$ on $A \setminus I$. Thus $D = 0$ on $A \setminus I$. $\square$

Let A be an algebra and I be a linear subspace of A such that $I.A = A.I = 0$. Then I is an ideal in A and it shown that these ideals are the annihilator ideals of A [4].

Corollary 3.7. *Let I be an annihilator ideal of A and A be a commutative amenable modulo I. Then every derivation $D : A \to A$ vanishes on $A \setminus I$.*

Theorem 3.8. *Let I be a closed ideal of A. If A is amenable modulo I and J is a closed ideal of A such that $J \subset I$, then $\frac{A}{J}$ is amenable modulo $\frac{I}{J}$.*

Proof. Suppose that X is an $\frac{A}{J}$-bimodule such that $\frac{I}{J} \cdot X = X \cdot \frac{I}{J} = 0$ and $D : \frac{A}{J} \to X^*$ is a derivation. Let $\pi : A \to \frac{A}{J}$ be natural quotient map. Now X can be made into Banach A-bimodule by defining

$$a \cdot x := \pi(a) \cdot x \ , \ x \cdot a := x \cdot \pi(a) \quad (a \in A \, , \, x \in X)$$

It is clear that $I \cdot X = X \cdot I = 0$ and $D \circ \pi : A \to X^*$ is a bounded derivation. Then there is $\phi \in X^*$ such that $D \circ \pi = \delta_\phi$ on $A \setminus I$. Let $\tilde{a} = a + J \in \frac{A}{J} \setminus \frac{I}{J}\, (a \in A \setminus I)$, we have

$$D(\tilde{a}) = D \circ \pi(a) = \delta_\phi(a) = a \cdot \phi - \phi \cdot a = \tilde{a} \cdot \phi - \phi \cdot \tilde{a}$$

Thus $D = \delta_\phi$ on $\frac{A}{J} \setminus \frac{I}{J}$. $\square$

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