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Original Research Paper

A Study on Various Subclasses of Uniformly Harmonic Starlike Mappings by Pascal Distribution Series

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Abstract. In this paper, we investigate the subclasses of harmonic univalent functions by implementing specific convolution operators such as the Pascal distribution series. We also, examine the inclusion relations of these functions. Moreover, we investigate several mapping properties involving these subclasses.

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1 Introduction

Let $\mathcal{A}$ denote the class of functions h of the form

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

that are analytic in the open unit disk $U = \{z \in C : |z| < 1\}$, and satisfy the normalizasyon condition $h(0) = h^{'}(0) - 1 = 0$. Let $\mathcal{H}$ be the family of all harmonic functions of the form

$$f = h + \bar{g}, \tag{1}$$

where

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = \sum_{k=1}^{\infty} b_k z^k, \quad (z \in U) \tag{2}$$

are in the class $\mathcal{A}$ and then $f(z)$ is given by,

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k + \overline{\sum_{k=1}^{\infty} b_k z^k}, \quad (z \in U). \tag{3}$$

Denote by the $S_{\mathcal{H}}$ the subclass of $\mathcal{H}$ that are univalent and sense-preserving in U. One shows easily that the sense-preserving property implies that $|b_1| < 1$. Note that $\frac{f - \overline{b_1 f}}{1 - |b_1|^2} \in S_{\mathcal{H}}$ whenever $f \in S_{\mathcal{H}}$. We also let the subclass $S_{\mathcal{H}}^0$ or $S_{\mathcal{H}}$,

$$S_{\mathcal{H}}^0 = \{f = h + \bar{g} \in S_{\mathcal{H}} : g^{'}(0) = b_1 = 0\}.$$

The classes $S_{\mathcal{H}}^0$ and $S_{\mathcal{H}}$ were first studied in ([13]). Also, let $S_{\mathcal{H}}^{*,0}$, $C_{\mathcal{H}}^0$ and $K_{\mathcal{H}}^0$, denote the subclasses of $S_{\mathcal{H}}^0$ of harmonic functions which are, respectively, starlike, close-to-convex and convex in U. For definitions and properties of these classes, one may refer to ([1, 2, 3, 13, 14]).

Let $\mathcal{T}_H$ be the family of all harmonic functions of the form $f = h + \bar{g}$, where

$$h(z) = z - \sum_{k=2}^{\infty} |a_k| z^k, \quad g(z) = \sum_{k=1}^{\infty} |b_k| z^k, \quad (z \in U). \tag{4}$$

This class was studied by Silverman [27].

For $0 \leq \alpha < 1$, $0 \leq \mathrm{r} < 1$ and $0 \leq \theta \leq 2\pi$, let

$$N_{\mathcal{H}}(\alpha) = \{f \in \mathcal{H} : \mathrm{Re}(\frac{f^{'}(z)}{z^{'}}) \geq \alpha, z = re^{i\theta}\},$$

where

$$f^{'}(z) = \frac{\partial}{\partial\theta} f(re^{i\theta}) = i(zh^{'}(z) - \overline{zg^{'}(z)}), \quad z^{'} = \frac{\partial}{\partial\theta}(re^{i\theta})$$

and define

$$\mathcal{T}N_{\mathcal{H}}(\alpha) = N_{\mathcal{H}}(\alpha) \cap \mathcal{T}_{\mathcal{H}}.$$

These classes are studied by Ahuja and Jahangiri [3]. One can find different areas which young researchers can find some connections with the field of this work. For more contributions see [4, 5, 6, 7, 8, 17, 19, 9, 28].

1.1 Definition [2]

$\mathcal{A}$ function $f = h + \bar{g}$ is said to be γ-uniformly harmonic starlike functions in U if satisfied the following condition:

$$\mathrm{Re}\left(\frac{zf^{'}(z)}{z^{'}[(1-\eta)z + \eta(h(z) + \overline{g(z)})]} - \delta\right) \geq \gamma\left|\frac{zf^{'}(z)}{z^{'}[(1-\eta)z + \eta(h(z) + \overline{g(z)})]} - 1\right|,$$

for $0 \leq \eta \leq 1$, $0 \leq \delta < 1$, $0 \leq \gamma < \infty$.

The family of this functions is denoted by $\mathfrak{G}_{\mathcal{H}}(\gamma,\delta,\eta)$. Also, define $\mathfrak{V}\mathfrak{G}_{\mathcal{H}}(\gamma,\delta,\eta)=\mathfrak{G}_{\mathcal{H}}(\gamma,\delta,\eta)\cap\mathcal{T}_{\mathcal{H}}$.

The above defined class includes several simpler subclasses. We point out here some of these special cases as follows:

(a) Putting $\gamma = 0$ and $\eta = 0$, we obtain $N_{\mathcal{H}}(\delta)$, which was studied by Ahuja and Jahangiri[3];

(b) Putting $\gamma = 0$ and $\eta = 1$, we obtain $S^{*}_{\mathcal{H}}(\delta)$, which was studied by Jahangiri[16];

(c) Putting $\eta = 1$ and $\delta = 1$, we obtain $G^{*}_{\mathcal{H}}(\gamma)$, which was studied by Rosy et al.[24];

(d) Putting $\gamma = 1$, $\delta = 0$, $\eta = 1$ and $g(z) \equiv 0$, we obtain US^*, which was studied by Rønning[25];

(e) Putting $\eta = 1$ we obtain $HUS^*(\gamma, \delta)$, which was studied by Porwal and Srivastava[23].

Lemma 1.1. *[13] If $f \in K^0_{\mathcal{H}}$ and $f = h + \bar{g}$ where h and g are given by (2) with $b_1 = 0$, then*

$$|a_k| \leq \frac{k+1}{2} \quad \textit{and} \quad |b_k| \leq \frac{k-1}{2} \qquad (k \geq 1).$$

Lemma 1.2. *[1] If $f \in C^0_{\mathcal{H}}$ or $S^{*,0}_{\mathcal{H}}$ and $f = h + \bar{g}$ where h and g are given by (2) with $b_1 = 0$, then*

$$|a_k| \leq \frac{(2k+1)(k+1)}{6} \quad \textit{and} \quad |b_k| \leq \frac{(2k-1)(k-1)}{6} \qquad (k \geq 1).$$

Lemma 1.3. *[3] If $f \in \mathcal{T}N_{\mathcal{H}}(\alpha)$ and $f = h + \bar{g}$ where h and g are given by (4), then*

$$|a_k| \leq \frac{1-\alpha}{k} \quad \textit{and} \quad |b_k| \leq \frac{1-\alpha}{k} \qquad (k \geq 1, 0 \leq \alpha < 1).$$

Lemma 1.4. *[2] Let $0 \leq \eta \leq 1$, $0 \leq \delta < 1$ and $0 \leq \gamma < \infty$. Also, let $f = h + \bar{g}$, where h and g are given by (2). If the following condition*

$$\sum_{k=2}^{\infty} \frac{k(\gamma+1) - \eta(\gamma+\delta)}{1-\delta} |a_k| + \sum_{k=1}^{\infty} \frac{k(\gamma+1) + \eta(\gamma+\delta)}{1-\delta} |b_k| \leq 1 \qquad (5)$$

is hold, then f is sense-preserving and harmonic mapping in U and $f \in \mathfrak{G}_{\mathcal{H}}(\gamma, \delta, \eta)$.

Lemma 1.5. *[2] Let $0 \leq \eta \leq 1$, $0 \leq \delta < 1$ and $0 \leq \gamma < \infty$. Also let, $f = h + \bar{g}$, where h and g are given by (4). A function $f \in \mathfrak{V}\mathfrak{G}_{\mathcal{H}}(\gamma, \delta, \eta)$ if and only if the condition (5) holds. Moreover, if $f \in \mathfrak{V}\mathfrak{G}_{\mathcal{H}}(\gamma, \delta, \eta)$, then*

$$|a_k| \leq \frac{1-\delta}{k(\gamma+1)-\eta(\gamma+\delta)} \qquad (k \geq 2),$$

$$|b_k| \leq \frac{1-\delta}{k(\gamma+1)+\eta(\gamma+\delta)} \qquad (k \geq 1).$$

In this paper, motivated by the earlier works studied by Porwal and Srivastava [23], we consider subclasses of harmonic univalent functions $f \in \mathcal{H}$ given by (2) and made an attempt to study inclusion relations making use of Pascal distribution series.

2 Applications to Pascal Distribution Series

The Pascal distribution series is a current subject of study in Geometric Function Theory (see,[10, 11, 12, 15, 30]). Taking into account the consequences on relations between various subclasses of analytic and harmonic univalent functions by using hypergeometric functions (see [2, 3, 13, 18, 20, 21, 22, 26]), we establish several relations between the classes $\mathfrak{G}^0_{\mathcal{H}}(\gamma, \delta, \eta)$, $K^0_{\mathcal{H}}$ and $S^{*,0}_{\mathcal{H}}$ by applying the convolutaion operator $P^{r,s}_{p,q}$ associated with Pascal distribution series are built.

Let us consider a non-negative discrete random variable $\mathcal{X}$ with a Pascal probability generating function

$$P(\mathcal{X}=k) = \binom{k+r-1}{r-1} p^k (1-p)^r, \qquad k \in \{0, 1, 2, 3, ...\},$$

where p, r are named as the parameters.

Currently, a power series whose coefficients are probabilities of the Pascal distribution is introduced

$$P^r_p(z) = z + \sum_{k=2}^{\infty} \binom{k+r-2}{r-1} p^{k-1} (1-p)^r z^k, \qquad (r \geq 1, 0 \leq p \leq 1, z \in U). \tag{6}$$

Note that using the ratio test yields the following conclusion:
The radius of convergence of the power series given above is infinite. In conclusion, the formulas used are as follows:

$$\sum_{k=0}^{\infty}\binom{k+r-1}{r-1}p^k=\frac{1}{(1-p)^r},\quad \sum_{k=0}^{\infty}\binom{k+r-2}{r-2}p^k=\frac{1}{(1-p)^{r-1}},$$

$$\sum_{k=0}^{\infty}\binom{k+r}{r}p^k=\frac{1}{(1-p)^{r+1}},\quad \sum_{k=0}^{\infty}\binom{k+r+1}{r+1}p^k=\frac{1}{(1-p)^{r+2}},\, |p|<1.$$

Further, throughout this paper unless otherwise stated, let $r \geq 1$ and $0 \leq p < 1$.

Now, for $r, s \geq 1$ and $0 \leq p, q < 1$, the operator is being introduced

$$P_{p,q}^{r,s}(f)(z) = P_p^r(z) * h(z) + \overline{P_q^s(z) * g(z)} = H(z) + \overline{G(z)},$$

where

$$H(z) = z + \sum_{k=2}^{\infty}\binom{k+r-2}{r-1}p^{k-1}(1-p)^r a_k z^k$$

$$G(z) = b_1 z + \sum_{k=2}^{\infty}\binom{k+s-2}{s-1}p^{k-1}(1-p)^s b_k z^k \tag{7}$$

and "$*$" represents the convolution (or Hadamard product) of power series.

To be able to build relations between harmonic convex functions and Goodman-Rønning-type harmonic univalent functions, the following conclusion is needed.

Lemma 2.1. *[30] Let $r, s \geq 1$ and $0 \leq p, q < 1$. Also, let $f = h + \bar{g} \in \mathcal{H}$ is given by (3). If the inequalities*

$$\sum_{k=2}^{\infty}|a_k| + \sum_{k=1}^{\infty}|b_k| \leq 1, \quad (|b_1| < 1) \tag{8}$$

and

$$(1-p)^r + (1-q)^s \geq 1 + |b_1| + \frac{rp}{1-p} + \frac{sq}{1-q} \tag{9}$$

are hold, then $P_{p,q}^{r,s}(f) \in \mathcal{SH}^*$.

3 Main Results

In this part, we will obtain the inclusion relations of harmonic classes $\mathfrak{G}^0_{\mathcal{H}}(\gamma,\delta,\eta)$ with the classes $K^0_{\mathcal{H}}$ and $S^*_{\mathcal{H}}$ respectively.

Theorem 3.1. *Let $r, s \geq 1$ and $0 \leq p, q < 1$. If the inequality*

$$\frac{(\gamma+1)}{2}\frac{p^2r(r+1)}{(1-p)^2}+\frac{[2(\gamma+1)-\frac{\eta}{2}(\gamma+\delta)]pr}{1-p}+\frac{(\gamma+1)}{2}\frac{q^2s(s+1)}{(1-q)^2}$$
$$+((\gamma+1)-\eta(\gamma+\delta))[1-(1-p)^r]+\frac{[(\gamma+1)+\frac{\eta}{2}(\gamma+\delta)]qs}{1-q}$$
$$\leq 1-\delta \tag{10}$$

is hold then

$$P^{r,s}_{p,q}(K^0_{\mathcal{H}}) \subset \mathfrak{G}^0_{\mathcal{H}}(\gamma,\delta,\eta).$$

Proof. Let $f = h + \bar{g} \in K^0_{\mathcal{H}}$ where h and g are of the form (2) with $b_1 = 0$. We need to show that $P^{r,s}_{p,q}(f)$=$H + \overline{G} \in \mathfrak{G}^0_{\mathcal{H}}(\gamma,\delta,\eta)$, where H and G defined by (7) with $b_1 = 0$ are analytic functions in $\mathcal{U}$. In view of Lemma 1.4, we need to prove that

$$\Psi_1 = \sum_{k=2}^{\infty}[k(\gamma+1)-\eta(\gamma+\delta)]\left|\binom{k+r-2}{r-1}(1-p)^r p^{k-1}a_k\right|$$
$$+\sum_{k=2}^{\infty}[k(\gamma+1)+\eta(\gamma+\delta)]\left|\binom{k+s-2}{s-1}(1-q)^s q^{k-1}b_k\right|$$
$$\leq 1-\delta.$$

Considering Lemma 1.1, we get

$$\begin{aligned}
\Psi_1 &\leq \sum_{k=2}^{\infty}\left[\frac{k(\gamma+1)}{2}-\frac{\eta}{2}(\gamma+\delta)\right](k+1)\binom{k+r-2}{r-1}(1-p)^r p^{k-1}\\
&+\sum_{k=2}^{\infty}\left[\frac{k(\gamma+1)}{2}+\frac{\eta}{2}(\gamma+\delta)\right](k-1)\binom{k+s-2}{s-1}(1-q)^s q^{k-1}\\
&=\sum_{k=2}^{\infty}\left[\frac{(\gamma+1)}{2}(k-1)(k-2)+(2(\gamma+1)-\frac{\eta}{2}(\gamma+\delta))(k-1)\right.\\
&\left.+((\gamma+1)-\eta(\gamma+\delta))\right]\times\binom{k+r-2}{r-1}(1-p)^r p^{k-1}\\
&+\sum_{k=2}^{\infty}\left[\frac{(\gamma+1)}{2}(k-1)(k-2)+((\gamma+1)+\frac{\eta}{2}(\gamma+\delta))(k-1)\right]\\
&\times\binom{k+s-2}{s-1}(1-q)^s q^{k-1}\\
&=\frac{\gamma+1}{2}p^2 r(r+1)(1-p)^r\sum_{k=0}^{\infty}\binom{k+r+1}{r+1}p^k\\
&+\left[2(\gamma+1)-\frac{\eta}{2}(\gamma+\delta)\right]pr(1-p)^r\sum_{k=0}^{\infty}\binom{k+r}{r}p^k\\
&+\left[(\gamma+1)-\eta(\gamma+\delta)\right](1-p)^r\left[\sum_{k=0}^{\infty}\binom{k+r-1}{r-1}p^k-1\right]\\
&+\frac{\gamma+1}{2}q^2 s(s+1)(1-q)^s\sum_{k=0}^{\infty}\binom{k+s+1}{s+1}q^k\\
&+\left[(\gamma+1)+\frac{\eta}{2}(\gamma+\delta)\right]qs(1-q)^s\sum_{k=0}^{\infty}\binom{k+s}{s}q^k\\
&=\frac{(\gamma+1)}{2}\frac{p^2 r(r+1)}{(1-p)^2}+\frac{[2(\gamma+1)-\frac{\eta}{2}(\gamma+\delta)]pr}{1-p}+\frac{(\gamma+1)}{2}\frac{q^2 s(s+1)}{(1-q)^2}\\
&+[(\gamma+1)-\eta(\gamma+\delta)][1-(1-p)^r]+\frac{[(\gamma+1)+\frac{\eta}{2}(\gamma+\delta)]qs}{1-q}\\
&\leq 1-\delta.
\end{aligned}$$

The last expression is bounded above by $(1-\delta)$ by the given condition. Thus the proof of Theorem 3.1 is complete. $\square$

Remark 3.2. Putting $\gamma = 1$ in Theorem 3.1, we improve the result obtained in ([29], Theorem 3.3).

Analogous to Theorem 3.1, we next find conditions of the class $S_{\mathcal{H}}^{*,0}$, with $\mathfrak{G}_{\mathcal{H}}^{0}(\gamma,\delta,\eta)$. However, we first need Lemma 1.2, which may be also found in [15],[14].

Theorem 3.3. *Let $r, s \geq 1$ and $0 \leq p, q < 1$. If the inequality*

$$\begin{aligned}
&\frac{(\gamma+1)r(r+1)(r+2)p^3}{(1-p)^3} + \frac{[\frac{15}{2}(\gamma+1) - \eta(\gamma+\delta)]r(r+1)p^2}{(1-p)^2} \\
&+ \frac{[12(\gamma+1) - \frac{9}{2}\eta(\gamma+\delta)]rp}{1-p} + 3[(\gamma+1) - \eta(\gamma+\delta)][1-(1-p)^r] \\
&+ \frac{(\gamma+1)s(s+1)(s+2)q^3}{(1-q)^3} + \frac{[\frac{9}{2}(\gamma+1) + \eta(\gamma+\delta)]s(s+1)q^2}{(1-q)^2} \\
&+ \frac{3[(\gamma+1) + \frac{1}{2}\eta(\gamma+\delta)]sq}{1-q} \\
&\leq 3(1-\delta)
\end{aligned}$$

is hold, then

$$P_{p,q}^{r,s}(S_{\mathcal{H}}^{*,0}) \subset \mathfrak{G}_{\mathcal{H}}^{0}(\gamma,\delta,\eta).$$

Proof. Let $f = h + \bar{g} \in S_{\mathcal{H}}^{*,0}$ where h and g are of the form (2) with $b_1 = 0$. We need to show that $P_{p,q}^{r,s}(f) = H + \overline{G} \in \mathfrak{G}_{\mathcal{H}}^{0}(\gamma,\delta,\eta)$, where H and G defined by (7) with $b_1 = 0$ are analytic functions in $\mathcal{U}$. In view of Lemma 1.4, it is enough to show that

$$\begin{aligned}
\Psi_2 = &\sum_{k=2}^{\infty}[k(\gamma+1) - \eta(\gamma+\delta)]\left|\binom{k+r-2}{r-1}(1-p)^r p^{k-1} a_k\right| \\
&+ \sum_{k=2}^{\infty}[k(\gamma+1) + \eta(\gamma+\delta)]\left|\binom{k+s-2}{s-1}(1-q)^s q^{k-1} b_k\right| \\
&\leq 1-\delta.
\end{aligned}$$

Considering Lemma 1.2, we have

$$\Psi_2 \leq \frac{1}{3}\left\{\sum_{k=2}^{\infty}\left[\frac{k(\gamma+1)}{2}-\frac{\eta}{2}(\gamma+\delta)\right](2k+1)(k+1)\binom{k+r-2}{r-1}(1-p)^r p^{k-1}\right.$$

$$\left.+\sum_{k=2}^{\infty}\left[\frac{k(\gamma+1)}{2}+\frac{\eta}{2}(\gamma+\delta)\right](2k-1)(k-1)\binom{k+s-2}{s-1}(1-q)^s q^{k-1}\right\}$$

$$=\frac{1}{3}\Bigg\{(\gamma+1)\sum_{k=2}^{\infty}\binom{k+r-2}{r-1}(k-1)(k-2)(k-3)(1-p)^r p^{k-1}$$

$$+\left[\frac{15}{2}(\gamma+1)-\eta(\gamma+\delta)\right]\sum_{k=2}^{\infty}\binom{k+r-2}{r-1}(k-1)(k-2)(1-p)^r p^{k-1}$$

$$+\left[12(\gamma+1)-\frac{9}{2}\eta(\gamma+\delta)\right]\sum_{k=2}^{\infty}\binom{k+r-2}{r-1}(k-1)(1-p)^r p^{k-1}$$

$$+[3(\gamma+1)-3\eta(\gamma+\delta)]\sum_{k=2}^{\infty}\binom{k+r-2}{r-1}(1-p)^r p^{k-1}$$

$$+(\gamma+1)\sum_{k=2}^{\infty}\binom{k+s-2}{s-1}(k-1)(k-2)(k-3)(1-q)^s q^{k-1}$$

$$+\left[\frac{9}{2}(\gamma+1)+\eta(\gamma+\delta)\right]\sum_{k=2}^{\infty}\binom{k+s-2}{s-1}(k-1)(k-2)(1-q)^s q^{k-1}$$

$$+\left[3(\gamma+1)+\frac{3}{2}\eta(\gamma+\delta)\right]\sum_{k=2}^{\infty}\binom{k+s-2}{s-1}(k-1)(1-q)^s q^{k-1}\Bigg\}$$

$$=\frac{1}{3}\Bigg\{(\gamma+1)r(r+1)(r+2)p^3(1-p)^r\sum_{k=0}^{\infty}\binom{k+r+2}{r+2}p^k$$

$$+\left[\frac{15}{2}(\gamma+1)-\eta(\gamma+\delta)\right]r(r+1)p^2(1-p)^r\sum_{k=0}^{\infty}\binom{k+r+1}{r+1}p^k$$

$$+\left[12(\gamma+1)-\frac{9}{2}\eta(\gamma+\delta)\right]rp(1-p)^r\sum_{k=0}^{\infty}\binom{k+r}{r}p^k$$

$$+[3(\gamma+1)-3\eta(\gamma+\delta)](1-p)^r\left[\sum_{k=0}^{\infty}\binom{k+r-1}{r-1}-1\right]$$

$$+ (\gamma+1)s(s+1)(s+2)q^3(1-q)^s \sum_{k=0}^{\infty} \binom{k+s+2}{s+2} q^k$$

$$+ \left[\frac{9}{2}(\gamma+1) + \eta(\gamma+\delta)\right] s(s+1)q^2(1-q)^s \sum_{k=0}^{\infty} \binom{k+s+1}{s+1} q^k$$

$$+ \left[3(\gamma+1) + \frac{3}{2}\eta(\gamma+\delta)\right] sq(1-q)^s \sum_{k=0}^{\infty} \binom{k+s}{s} q^k \Bigg\}$$

$$= \frac{1}{3}\Bigg\{ \frac{(\gamma+1)r(r+1)(r+2)p^3}{(1-p)^3} + \frac{\left[\frac{15}{2}(\gamma+1) - \eta(\gamma+\delta)\right] r(r+1)p^2}{(1-p)^2}$$

$$+ \frac{\left[12(\gamma+1) - \frac{9}{2}\eta(\gamma+\delta)\right] rp}{1-p} + [3(\gamma+1) - 3\eta(\gamma+\delta)]\left[1-(1-p)^r\right]$$

$$+ \frac{(\gamma+1)s(s+1)(s+2)q^3}{(1-q)^3} + \frac{\left[\frac{9}{2}(\gamma+1) + \eta(\gamma+\delta)\right] s(s+1)q^2}{(1-q)^2}$$

$$+ \frac{\left[3(\gamma+1) + \frac{3}{2}\eta(\gamma+\delta)\right] sq}{1-q} \Bigg\}$$

$$\leq 1-\delta.$$

Now $\Psi_2 \leq 1-\delta$ follows from the given condition.
□

Remark 3.4. Putting $\gamma = 1$ in Theorem 3.3, we improve the result obtained in ([29]; Theorem 3.5).

Theorem 3.5. *Let $r, s \geq 1$ and $0 \leq p, q < 1$. If the inequality*

$$(1-p)^r + (1-q)^s \geq 1 + \frac{[(\gamma+1) + \eta(\gamma+\delta)]|b_1|}{1-\delta} \tag{11}$$

is satisfied, then $P_{p,q}^{r,s}(\mathfrak{GV}_{\mathcal{H}}(\gamma,\delta,\eta)) \subset \mathfrak{GV}_{\mathcal{H}}(\gamma,\delta,\eta)$.

Proof. Suppose $f = h + \bar{g} \in \mathfrak{GV}_{\mathcal{H}}(\gamma,\delta,\eta)$ where h and g are provided by (4). We need to prove that the function

$$P_{p,q}^{r,s}(f)(z) = z - \sum_{k=2}^{\infty} \binom{k+r-2}{r-1} (1-p)^r p^{k-1} |a_k| z^k$$

$$+ |b_1|\overline{z} + \sum_{k=2}^{\infty} \binom{k+s-2}{s-1} (1-q)^s q^{k-1} |b_k| \overline{z}^k$$

is in $\mathfrak{GV}_{\mathcal{H}}(\gamma, \delta, \eta)$ if $\Psi_3 \leq 1-\delta$, where

$$\Psi_3 = \sum_{k=2}^{\infty} [k(\gamma+1) - \eta(\gamma+\delta)] \binom{k+r-2}{r-1} (1-p)^r p^{k-1} |a_k|$$

$$+ [(\gamma+1) + \eta(\gamma+\delta)]|b_1|$$

$$+ \sum_{k=2}^{\infty} [k(\gamma+1) + \eta(\gamma+\delta)] \binom{k+s-2}{s-1} (1-q)^s q^{k-1} |b_k|.$$

In view of Lemma 1.5, we have

$$\Psi_3 \leq (1-\delta) \left[\sum_{k=2}^{\infty} \binom{k+r-2}{r-1} (1-p)^r p^{k-1} + \sum_{k=2}^{\infty} \binom{k+s-2}{s-1} (1-q)^s q^{k-1} \right]$$

$$+ [(\gamma+1) + \eta(\gamma+\delta)]|b_1|$$

$$= (1-\delta) \Bigg[(1-p)^r \sum_{k=0}^{\infty} \binom{k+r-1}{r-1} p^k - (1-p)^r$$

$$+ (1-q)^s \sum_{k=0}^{\infty} \binom{k+s-1}{s-1} q^k - (1-q)^s \Bigg] + [(\gamma+1) + \eta(\gamma+\delta)]|b_1|$$

$$= (1-\delta) \left[2 - (1-p)^r - (1-q)^s \right] + [(\gamma+1) + \eta(\gamma+\delta)]|b_1|$$

$$\leq 1-\delta,$$

by the given condition and thus the proof of the theorem is complete.
□

Remark 3.6. Putting $\gamma = 1$ in Theorem 3.5, we improve the result obtained in ([29], Theorem 3.6).

Remark 3.7. By suitable specializing the parameter η, one can deduce the results for the subclasses $\mathfrak{NV}_{\mathcal{H}}(\alpha, \delta)$ and $f \in \mathfrak{RV}_{\mathcal{H}}(\alpha, \delta)$ which are defined, respectively, [Example 2.1 and 2.2 in [29]] and associated with the Pascal distribution series. The deatails involved may be given as a pratice for the reader willing.

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