Journal of Mathematical Extension
Vol. 1, N0. 2, (2007), 69-80

On the Universal Interpolating Sequences on $H^2(\beta)$

B. Yousefi
Shiraz University

L. Bagheri, and T. Rastegar
Islamic Azad University-Shiraz Branch
Shiraz Payame-Noor University

Abstract. In this paper we investigate the relation between universal interpolating sequence and the approximate point spectrum of the adjoint multiplication operator acting on the Hilbert spaces of formal power series.

AMS Subject Classification: Primary 47B37; Secondary 47 A25.
Keywords and Phrases: Hilbert spaces of formal power series, universal interpolating sequence, spectrum, point spectrum, approximate point spectrum, closed graph theorem, Riesz representation theorem.

1. Introduction

Let $\{\beta(n)\}$ be a sequence of positive numbers with $\beta(0) = 1$. We consider the space of sequences $f = \{\hat{f}(n)\}_{n=0}^{\infty}$ such that

$$\|f\|^2 = \|f\|_\beta^2 = \sum_{n=0}^{\infty} |\hat{f}(n)|^2 \beta(n)^2 < \infty.$$

The notation

$$f(z)=\sum_{n=0}^{\infty}\hat{f}(n)z^n$$

shall be used whether or not the series converges for any value of z. These are called formal power series. Let $H^2(\beta)$ denote the space of such formal power series. These are Hilbert spaces with the inner product

$$<f,g>=\sum_{n=0}^{\infty}\hat{f}(n)\overline{\hat{g}(n)}\beta(n)^2,$$

where

$$f(z)=\sum_{n=0}^{\infty}\hat{f}(n)z^n\in H^2(\beta)$$

and

$$g(z)=\sum_{n=0}^{\infty}\hat{g}(n)z^n\in H^2(\beta).$$

Let $\hat{f}_k(n)=\delta_k(n)$. So $f_k(z)=z^k$ and then $\{f_k\}_k$ is a basis such that $\|f_k\|=\beta(k)$. Now consider M_z, the operator of multiplication by z on $H^2(\beta)$ defined by

$$(M_zf)(z)=\sum_{n=0}^{\infty}\hat{f}(n)z^{n+1},$$

where

$$f(z)=\sum_{n=0}^{\infty}\hat{f}(n)z^n\in H^2(\beta).$$

In other words,

$$(M_z\hat{f})(n)=\begin{cases}\hat{f}(n-1) & n\geqslant 1\\ 0 & n=0\end{cases}.$$

Clearly M_z shifts the basis $\{f_k\}_k$. The operator M_z is bounded if and only if $\{\beta(k+1)/\beta(k)\}_k$ is bounded and in this case we have

$$\|M_z^n\| = \sup_k \frac{\beta(n+k)}{\beta(k)} \quad , \quad n = 0, 1, 2, \dots .$$

Also note that if

$$\sup_n \sum_{i=1}^{n} \left(\frac{\beta(n)}{\beta(i)\beta(n-i)}\right)^2 < \infty,$$

then clearly by the Holder inequality one can see that $H^2(\beta)$ is an algebra. For a good source in formal series, we refer the reader to [4, 9, 11, 12, 14, 15, 18, 19, 21].

We say that a complex number λ is a bounded point evaluation on $H^2(\beta)$ if the functional $e_\lambda : H^2(\beta) \longrightarrow \mathbb{C}$ defined by $e_\lambda(f) = f(\lambda)$ is bounded. If the point evaluation is continuous at λ, then the Riesz representation theorem implies that there is a unique function $k_\lambda \in H^2(\beta)$ such that

$$e_\lambda(f) = f(\lambda) =< f, k_\lambda >, \qquad f \in H^2(\beta).$$

The function k_λ is called the reproducing kernel for the point λ.

Let E be a Banach space. The set of bounded linear operators on E is denoted by $B(E)$. If $A \in B(E)$, the point spectrum of A, $\sigma_p(A)$, is defined by

$$\sigma_p(A) = \{\lambda \in \mathbb{C} : \ \ker(A - \lambda) \neq (0)\}.$$

Also, the approximate Point spectrum of A, $\sigma_{ap}(A)$, is defined by

$$\begin{aligned}\sigma_{ap}(A) = \{\lambda \in \mathbb{C} \quad &: \quad \text{there is a sequence } \{x_n\} \text{ in } E \text{ such that} \\ \|x_n\| \quad &= \quad 1 \text{ for all } n \text{ and } \|(A-\lambda)x_n\| \to 0\}.\end{aligned}$$

Note that $\sigma_p(A) \subset \sigma_{ap}(A)$.

Let X be a separable reflexive Banach space whose elements are analytic functions on a complex domain Ω. A complex valued function φ on Ω for which $\varphi f \in X$ for every $f \in X$ is called a multiplier of X and the collection of all these multipliers is denoted by $M(X)$.

2. Main Theorem

Let X be a separable reflexive Banach space whose elements are analytic functions on a complex domain Ω. From [16] we note that a sequence $\{w_n\}_{n=1}^{\infty}$ of points of Ω is said an interpolating sequence for X if there exists a positive weight sequence $\{k_n\}_{n=1}^{\infty}$ such that the sequence $\{f(w_n)k_n\}_{n=1}^{\infty}$ is in ℓ^∞ for all f in X and conversely every sequence in ℓ^∞ can be written in that form. Also a sequence $\{w_n\}_{n=1}^{\infty}$ of points of Ω is said an interpolating sequence for $M(X)$ if for each bounded sequence $\{a_n\}_{n=1}^{\infty} \subset \mathbb{C}$, there exists $\varphi \in M(X)$ such that $\phi(w_n) = a_n$ for all $n \in \mathbb{N}$.

In [3] Carleson proved a necessary and sufficient condition for a sequence to be interpolating for H^∞. In [8] Carleson's result was generalized to the Hardy space H^p. Also, Berndtsson, Chang, and Lin in [2] studied the analogue of Carleson's condition for the polydisk. The multiplier space is a proper subspace of bounded analytic functions on a plane domain Ω, so that it is harder for a sequence to be interpolating for $M(X)$ than for $H^\infty(\Omega)$. Not every interpolating sequence for $H^\infty(\Omega)$ is interpolating for $M(X)$, as can be seen from the study by Sundberg and Wolff of interpolating sequences for spaces that properly lie between $M(X)$ and H^∞ ([10]). Interpolating sequences for the set of multipliers of the Dirichlet space has been studied in [1] by Axler. When H is a Hilbert space of analytic functions on a plane domain, the interpolating sequence for $M(H)$ was studied in [6] and it's extension for $M(X)$ where X is a Banach space of analytic functions on a special plane domain was studied in [16]. Also the interpolating sequence for a Banach space of analytic functions, on a special plane domain, was studied in [13].

From now on we suppose that $\lim_n \frac{\beta(n+1)}{\beta(n)} = 1$ or $\liminf_n \beta(n)^{\frac{1}{n}} = 1$. Then $H^2(\beta)$ consists of functions analytic on the open unit disc $\mathbb{D}$ and each point of the open unit disk $\mathbb{D}$ is a bounded point evaluation on $H^2(\beta)$ ([11]).

Now following the interpolation theory for the Hardy space H^2 in [8] and for certain Banach spaces of analytic functions in [13] we give the following definition.

Definition. *Suppose that* $\{w_n\}_{n\in N}$ *is a sequence of distinct points in* $\mathbb{D}$ *and consider the linear transformation* $T : H^2(\beta) \to \ell^2$ *defined by*

$$Tf = \{\frac{f(w_n)}{||e_{w_n}||}\}_{n\in N}.$$

The sequence $\{w_n\}_{n\in N}$ *is called a universal interpolating sequence for* $H^2(\beta)$ *if* T *maps* $H^2(\beta)$ *onto* ℓ^2.

For some sources on the interpolating topics one can see [5, 7, 8, 13, 16, 17, 20].

In this section we will investigate the relation between a universal interpolating sequence and the approximate point spectrum of the adjoint multiplication operator acting on $H^2(\beta)$.

Theorem. *Suppose that* $\{w_n\}_{n=1}^{\infty}$ *is a universal interpolating sequence for* $H^2(\beta)$. *If* $\mathcal{F} = \bigcap\limits_{n\in\mathbb{N}} \ker e_{w_n}$, *then*

$$(\sigma_p(M_z^*|_{\mathcal{F}^\perp}) \cap \mathbb{D})^{-} = (\sigma_{ap}(M_z^*|_{\mathcal{F}^\perp}) \cap \mathbb{D})^{-} = \{\overline{w}_n\}^{-}_{n\in N}.$$

Proof. First note that if $\mathcal{F} \in Lat(M_z)$, then $\mathcal{F}^\perp \in Lat(M_z^*)$. Now let $\{d_n\}_{n\in N}$ be the canonical basis for ℓ^2. If $f \in H^2(\beta)$ and $n \in \mathbb{N}$, then we get

$$\begin{aligned} < f, T^* d_n > &= < Tf, d_n > = < \{\frac{f(w_k)}{||e_{w_k}||}\}, d_n > \\ &= \frac{f(w_n)}{||e_{w_n}||} = < f, \frac{e_{w_n}}{||e_{w_n}||} > . \end{aligned}$$

This implies that for $a = \{a_n\}_{n\in N} \in \ell^2$ we have

$$T^*a = \sum_{n\in N} a_n \frac{e_{w_n}}{||e_{w_n}||}.$$

If $f \in H^2(\beta)$, then

$$\begin{aligned} |< f, \sum_{n\in N} a_n \frac{e_{w_n}}{||e_{w_n}||} >| &\leqslant \sum_{n\in N} |a_n| \frac{|f(w_n)|}{||e_{w_n}||} \\ &\leqslant ||a||_2 ||Tf|| < \infty. \end{aligned}$$

Note that for all f in $\mathcal{F}$ we have $e_{w_n}(f) = f(w_n) = 0$. Thus $e_{w_n} \in \mathcal{F}^\perp$ for all $n \in \mathbb{N}$. Also if $f \in H^2(\beta)$, then we have

$$\begin{aligned} < f, M_z^* e_{w_n} > &= < M_z f, e_{w_n} >= w_n f(w_n) \\ &= w_n < f, e_{\overline{w}_n} >=< f, w_n e_{w_n} > . \end{aligned}$$

Therefore

$$M_z^* e_{w_n} = \overline{w}_n e_{w_n}.$$

But

$$(M_z - w_n)^* = M_z^* - \overline{w}_n,$$

so indeed

$$\begin{aligned} \{\overline{w}_n\}_{n\in N} &\subseteq \sigma_p(M_z^*|_{\mathcal{F}^\perp}) \cap \mathbb{D} \\ &\subseteq \sigma_{ap}(M_z^*|_{\mathcal{F}^\perp}) \cap \mathbb{D}. \end{aligned}$$

Thus to complete the proof it is sufficient to show that

$$(\sigma_{ap}(M_z^*|_{\mathcal{F}^\perp}) \cap \mathbb{D}\bar{)} \subseteq \{\overline{w}_n\bar{\}}_{n\in N}.$$

Let $\overline{w} \in \mathbb{D} \backslash \overline{\{\overline{w}_n\}}_{n\in N}$. So there exists $r > 0$ such that $|w - w_n| > r$ for all $n \in \mathbb{N}$. Now if

$$\overline{w} \in \sigma_{ap}(M_z^*|_{\mathcal{F}^\perp}),$$

then there exists a sequence $\{h_n\}_{n\in N}$ of unit vectors in $\mathcal{F}^\perp$ such that $||(M_z - w)^* h_n|| \to 0$. Since

$$(M_z - w) = \ker \ e_w,$$

the subspace $(M_z - w)$ is closed and so $(M_z - w)^*$ is also closed. But

$$\ker(M_z - w)^* = (\,(M_z - w))^\perp = [k_w].$$

Thus $(M_z - w)^*$ is injective on $[k_w]^\perp$ and hence $(M_z - w)^*$ is bounded below on $[k_w]^\perp$. So $k_w \in \mathcal{F}^\perp$. Clearly we have $T^* = \mathcal{F}^\perp$, hence there exists a nonzero sequence $a = \{a_n\}_{n\in N} \in \ell^2$ such that $T^* a = k_w$. Now we have

$$\begin{aligned}
||(M_z - w)^* k_w|| &= ||(M_z - w)^* T^* a|| \\
&= ||\sum_{n\in N} \frac{a_n}{||k_{w_n}||}(M_z^* - \overline{w})k_{w_n}|| \\
&= ||\sum_{n\in N} a_n(\overline{w}_n - \overline{w})\frac{k_{w_n}}{||k_{w_n}||}|| \\
&= ||T^* d||,
\end{aligned}$$

where

$$d = \{a_n(\ \overline{w_n - w}\ )\}_{n\in N} \in \ell^2$$

and $||d||_2 \geqslant r||a||_2$. On the other hand since $\{w_n\}_{n\in N}$ is a universal interpolating sequence, the operator $T : H^2(\beta) \to \ell^2$ is onto and so $T^* : \ell^2 \to H^2(\beta)$ is bounded below. So there exists $\alpha > 0$ such that $||T^*b|| \geqslant \alpha||b||_2$ for all $b \in \ell^2$. Also, since $(M_z - w)^*k_w = 0$, we obtain

$$0 = ||T^*d|| \geqslant \alpha||d||_2 \geqslant \alpha\ r||a||_2 > 0$$

which is a contradiction. Hence

$$\overline{w} \notin \sigma_{ap}(M_z^*|_{\mathcal{F}^\perp})$$

and we have proved that

$$\mathbb{D}\backslash\overline{\{\overline{w}_n\}}_{n\in N} \subseteq \mathbb{C}\backslash\ \sigma_{ap}(M_z^*|_{\mathcal{F}^\perp}).$$

Now clearly we get

$$\overline{(\sigma_{ap}(M_z^*|_{\mathcal{F}^\perp}) \cap \mathbb{D})} \subseteq \overline{\{\overline{w}_n\}}_{n\in N}.$$

This completes the proof. $\square$

Remark. *The above theorem has been extended for the Banach spaces of analytic functions on a plane domain ([22]).*

Acknowledgment

The second and third author, thank the Research Council of Islamic Azad University-Shiraz Branch.

References

[1] S. Axler, Interpolation by multipliers of the Dirichlet space, *Quart. J. Math. Oxford*, 43 (2) (1992), 409-419.

[2] B. Berndtsson, S. Y. A. Chang, and K. C. Lin, Interpolating sequences in the polydisc, *Trans. Amer. Math. Soc.*, 302 (1987), 161-169.

[3] L. Carleson, An interpolation problem for bounded analytic functions, *Amer. J. Math.*, 80 (1958), 921-930.

[4] J. Doroodgar and B. Yousefi, Weighted sequence spaces and cyclicity, *Journal of Mathematical Extension*, 1 (1) (2006), 11-20.

[5] K.C. Chan and A.L. Shields, Zero sets, interpolating sequences, and cyclic vectors for Dirichlet spaces, *Michigan Math. J.*, 39 (1992), 289-307.

[6] K. Mosaleheh and K. Seddighi, Interpolation by multipliers on certain spaces of analytic functions, *Internat. J. Math. and Math. Sci.*, 23 (8) (2000), 547-554.

[7] S. Richter, Invariant subspaces in Banach spaces of analytic functions, *Trans. Amer. Math. Soc.*, 304 (1987), 585-616.

[8] H. S. Shapiro and A. L. Shields, On some interpolation problems for analytic functions, *Amer. J. Math.*, 83 (1961), 513-532.

[9] A.L. Shields, Weighted shift operators and analytic function theory, *Math. Survey, AMS Providence*, 13 (1974), 49-128.

[10] C. Sundberg and T. H. Wolff, Interpolating sequences for QA_n, *Trans. Amer. Math. Soc.*, 276 (1983), 551-581.

[11] B. Yousefi, On the space $\ell^p(\beta)$, *Rendiconti del Circolo Matematico di Palermo*, 49 (2000), 115-120.

[12] B. Yousefi, Unicellularity of the multiplication operator on Banach spaces of formal power series, *Studia Mathematica*, 147 (3) (2001), 201-209.

[13] B. Yousefi, Interpolating sequence on certain Banach spaces of analytic functions, *Bull. Austral. Math. Soc.*, 65 (2002), 177-182.

[14] B. Yousefi and S. Jahedi, Composition operators on Banach spaces of formal power series, *Bollettino Della Unione Matematica Italiano*, 6-B (8) (2003),481-487.

[15] B. Yousefi, Strictly cyclic algebra of operators acting on Banach spaces $H^p(\beta)$, *Czechoslovak Mathematical Journal*, 54 (129) (2004),261-26

[16] B. Yousefi and K. Mosaleheh, Interpolation sequences for multipliers of Banach spaces of analytic functions, *Italian Journal of Pure and Applied Mathematics*, 17 (2005), 207-212.

[17] B. Yousefi and B. Tabatabaie, Universal interpolating sequence on some function spaces, *Czechoslovak Mathematical Journal*, 55 (130) (2005), 773-780.

[18] B. Yousefi, On the eighteenth question of Allen Shields, *International Journal of Mathematics*, 16 (1) (2005), 37-42.

[19] B. Yousefi and A. I. Kashkuli, Cyclicity and unicellularity of the differentiation operator on Banach spaces of formal power series, *Mathematical Proceedings of the Royal Irish Academy*, 105A (1) (2005), 1-7.

[20] B. Yousefi and J. Doroodgar, Interpolation and spectral properties on $H^p(\beta)$, *International Journal of Applied Mathematics*, 19 (2) (2006), 165-169.

[21] B. Yousefi, Z. Kamali, and L. Bagheri, Isometries of weighted Hardy spaces among composition operators, *Korean Annals of Math.*, 23 (2) (2006), 119-125.

[22] B. Yousefi, Universal interpolating sequence on spaces of analytic functions, *Exp. Math. Journal*, submitted.

Bahmann Yousefi
Department of Mathematics
College of Sciences
Shiraz University
Shiraz 71454, Iran
E-mail: yousefi@susc.ac.ir

Leila Bagheri
Department of Mathematics
Islamic Azad University-Shiraz Branch
Shiraz, Iran

Shiraz Payame-Noor University
Shiraz, Iran
Email: sky_sun_1980@yahoo.com

Tahereh Rastegar
Department of Mathematics
Islamic Azad University-Shiraz Branch
Shiraz, Iran
Shiraz Payame-Noor University
Shiraz, Iran
Email: mt_rastegar@yahoo.com