Discrete ADM: A tools for solving a class of classic and fractional difference problems

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Abstract

Discrete fractional calculus (DFC) is continuously spreading in the neural networks, chaotic maps, engineering practice, and image encryption, which is appropriately assumed for discrete-time modelling in continuum problems. For solving problems including difference operators (classic and fractional), we employ a discrete version of the Adomian decomposition method (ADM). This method help to find the solutions of linear and nonlinear classic and fractional difference problems (CDPs and FDPs). Examples are given to clarify and confirm the obtained results and some of particular cases of CDPs and FDPs are highlighted.

Keywords: Discrete calculus, Classic difference operator, Fractional difference operator, Disceret Adomian Decomposition Method

1. Introduction

Fractional calculus (FC) is a branch of mathematics that permits the derivative and integral order to be a fraction, this is regarded as a kind of some extension to the classic derivative in which the derivative and integral order is restricted to integers. Practical results have proved time and again that it is worth the effort to model real world phenomenon using fractional integral and differential equations compared to the integer calculus. There is a general consensus that this observation is wholly attributed to fractional calculus's ability to take into account the hereditary and memory influence in predicting the future, a characteristic that the classic derivative does not possess. For a detailed discussion of fractional calculus, particularly as introductory texts to the subject, we refer the reader to [1, 2, 3]. More applications of FC in applied mathematics, science, economics, engineering and other disciplines can be found in [4, 5, 6, 7, 8, 9].
To gain maximum benefits from a good mathematical model, it is of paramount importance that the methods of its solution be computationally efficient, consistent and highly accurate. There are no methods that are exclusively reserved for fractional calculus models. Any technique that is applicable to an integer order differential and integral equation will work perfectly in the fractional calculus setting.

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However, there is no doubt that the accommodation of the fractional order feature in fractional calculus increases the labour required to solve fractional differential and integral equations. Thus, in solving these kind equations, engaging a method of solution that is both computationally inexpensive and accurate is ideal, although it is a challenging exercise to strike the balance. Common methods that have been applied successfully to solve fractional calculus models include, homotopy analysis method [10], Adams-Bashforth method [8], homotopy perturbation method [11], meshless method [12, 13], Adomian decomposition method [14, 15], operational matrix method and [16, 17, 18, 19, 20, 21].

During the last twenty years, the theory of special functions and discrete fractional calculus (DFC) have been gotten by to attract increasing attention from the physical and mathematical communities. Specifically, the strict correlation between these two models has been acting as the driving force for the most recent developments and generalizations in the literature on these subjects. In 1974, Daiz et al. [22] introduced the idea of DFC and composed it with an infinite sum. Later on, in 1988, Gray et al. [23] extended this concept and implemented it to the finite sum. This concept is known as the nabla difference operator in the literature. Atici and Eloe [24] proposed the theory of fractional difference equations, although the practical implementation is presented in [25].

The aim of this research paper is to present a new version of ADM, that is discrete ADM to solve CDPs and FDPs. Also we confirm this method is very good comparing the results CDPs and FDPs. The outline of our study is as follows. Preliminaries and notations of discrete fractional calculus are recalled in Section 2. Section 3, we construct a new version of ADM (discrete ADM). Our findings with some graphs are illustrated in section 4. Section 5 contains final concluding remarks.

2. Preliminaries and notations

In this section, we recall some basics concepts of discrete fractional calculus (DF–calculus), which will be necessary in proceeding to obtain our discrete results.

Definition 2.1 (See [26]). *Get for $a \in \mathbb{R}$,*

$$\mathcal{N}_a := \{a, a+1, a+2, \cdots\},$$

or for $a, b \in \mathbb{R}$ and $b > a$,

$$\mathcal{N}_a^b := \{a, a+1, a+2, \cdots, b\}.$$

The forward difference operator $\triangle$ and $\triangle^2$ are written as (1) if $w : \mathcal{N}_a^b \to \mathbb{R}$:

$$\begin{aligned} &\triangle h(t) = h(t+1) - h(t), \quad t \in \mathcal{N}_a^{b-1}, \\ &\triangle^2 h(t) = h(t+2) - 2h(t+1) + h(t). \end{aligned} \tag{1}$$

Theorem 2.1 (See [26]). *Assume $s_1,\ s_2,\ s_3$ are constants. Then the following hold:*

$$\begin{aligned} &\int (t - s_1)^{\underline{s_2}}\ \triangle t = \frac{1}{s_2 + 1}(t - s_1)^{\underline{s_2+1}} + C, \quad s_2 \neq -1, \\ &\int s_1^t\ \triangle t = \frac{1}{s_1 - 1} s_1^t + C, \quad s_1 \neq 1. \end{aligned}$$

Definition 2.2 (See [26]). *The falling function, $t^{\underline{s}}$, is given as follows:*

i) *for* $s \in \mathbb{N}$,

$$t^{\underline{s}} := t(t-1)(t-2)\cdots(t-s+1), \quad t^{\underline{0}} = 1,$$

ii) *for* $s \in \mathbb{R}$,

$$t^{\underline{s}} := \frac{\Gamma(t+1)}{\Gamma(t-s+1)}, \quad t \in \mathbb{R} - \left\{\mathbb{Z}^- \cup \{0\}\right\}, \quad 0^{\underline{0}} = 0.$$

Lemma 2.1. *Let* $0 < \varsigma < 1$, *then*

$$\sum_{r=1-\varsigma}^{t-\varsigma} (t-r-1)^{\underline{\varsigma-1}} = \frac{\Gamma(t+\varsigma)}{\varsigma\Gamma(t)}.$$

Proof. First, we can write

$$\begin{aligned}\sum_{r=1-\varsigma}^{t-\varsigma} (t-r-1)^{\underline{\varsigma-1}} &= \sum_{r=1-\varsigma}^{t-\varsigma} \frac{\Gamma(t-r)}{\Gamma(t-r-\varsigma+1)} \\ &= \sum_{r=1-\varsigma}^{t-\varsigma-1} \frac{\Gamma(t-r)}{\Gamma(t-r-\varsigma+1)} + \Gamma(\varsigma).\end{aligned}$$

Let $t > r$, $t, r \in \mathbb{R}$, $r > -1$, $t > -1$, then [27]

$$\frac{\Gamma(t+1)}{\Gamma(r+1)\Gamma(t-r+1)} = \frac{\Gamma(t+2)}{\Gamma(r+2)\Gamma(t-r+1)} - \frac{\Gamma(t+1)}{\Gamma(r+2)\Gamma(t-r)},$$

that is

$$\frac{\Gamma(t+1)}{\Gamma(t-r+1)} = \frac{1}{r+1}\left[\frac{\Gamma(t+2)}{\Gamma(t-r+1)} - \frac{\Gamma(t+1)}{\Gamma(t-r)}\right].$$

Then

$$\begin{aligned}\sum_{r=1-\varsigma}^{t-\varsigma} (t-r-1)^{\underline{\varsigma-1}} &= \sum_{r=1-\varsigma}^{t-\varsigma-1} \frac{1}{\varsigma}\left[\frac{\Gamma(t-r+1)}{\Gamma(t-r-\varsigma+1)} - \frac{\Gamma(t-r)}{\Gamma(t-r-\varsigma)}\right] + \Gamma(\varsigma) \\ &= \frac{1}{\varsigma}\left[\frac{\Gamma(t+\varsigma)}{\Gamma(t)} - \frac{\Gamma(t+\varsigma)}{\Gamma(1)}\right] + \Gamma(\varsigma) \\ &= \frac{\Gamma(t+\varsigma)}{\varsigma\Gamma(t)}.\end{aligned}$$

□

Definition 2.3 (See [28])**.** *The fractional sum of order* ς *is defined setting* $\varsigma > 0$ *and* $w : \mathbb{N}_a \to \mathbb{R}$ *as,*

$$\triangle_a^{-\varsigma} h(t) = \frac{1}{\Gamma(\varsigma)} \sum_{r=a}^{t-\varsigma} (t-\sigma(r))^{\underline{\varsigma-1}}\, h(r), \quad t \in \mathcal{N}_{a+\varsigma},$$

where $\sigma(r) = r+1$. *Set* $h(t) = t^{\underline{\gamma}}$, *then*

$$\triangle_a^{-\varsigma} h(t) = \frac{\Gamma(\gamma+1)}{\Gamma(\varsigma+\gamma+1)} t^{\underline{\varsigma+\gamma}}, \quad \gamma \in \mathbb{R}^+.$$

Definition 2.4 (See [28]). *The Caputo delta difference is given* $0 < \varsigma < 1$ *and* $h : \mathbb{N}_a \to \mathbb{R}$ *as,*

$$^{C}\triangle_a^{\varsigma} h(t) = {}^{C}\triangle_a^{-(1-\varsigma)} \triangle h(t) = \frac{1}{\Gamma(1-\varsigma)} \sum_{r=a}^{t+\varsigma-1} (t-\sigma(r))^{\underline{-\varsigma}} \triangle h(r), \quad t \in \mathcal{N}_{a-\varsigma+1},$$

where $\sigma(r) = r + 1$.

3. Discrete ADM in CDPs and FDPs

This section introduces the powerful approximate method of ADM. Helping this method, one can easily handle nonlinear problems with the large order of nonlinearity [29]. Let us discuss a brief outline of discrete ADM. For this, we consider a general nonlinear equation in the form

$$\triangle^{\varsigma} h + L(h) + N(h) = g, \qquad s-1 < \varsigma \leq s, \tag{2}$$

where L and N present the linear and nonlinear difference operators respectively. Also, g is the source term. Applying the operator $\triangle^{-\varsigma}$, an inverse of $\triangle^{\varsigma}$ on both sides of equation (2) and using the given conditions gives us,

$$h = \sum_{r=0}^{s-1} a_r \frac{t^{\underline{r}}}{r!} + \triangle^{-\varsigma} \left(g - L(h) - N(h)\right).$$

where $a_r,\ r = 0, \cdots, s-1$ are constants of integration and can be found form the boundary or initial conditions. The Adomian method assumes the solution h can be expanded into an infinite series as

$$h = \sum_{i=0}^{\infty} h_i. \tag{3}$$

Also, the nonlinear term $N\ h$ will be written as

$$N(h) = \sum_{i=0}^{\infty} A_i, \tag{4}$$

where A_i are the special Adomain polynomials. By specified A_i, the next component of can be determined:

$$h_{i+1} = \triangle^{-\varsigma} \sum_{i=0}^{\infty} A_i,$$

Finally, after some iterations and getting sufficient accuracy, the solution of the equation can be expressed by equation (3). In equation (4), the Adomian polynomials can be generated by several means. Here we used the following recursive formulation [14]:

$$A_i = \frac{1}{i!} \left[\frac{d^i}{d\lambda^i} N \left(\sum_{i=0}^{\infty} \lambda^i h_i \right) \right]_{\lambda=0}, \qquad i \geq 0.$$

Since the method does not resort to linearization or assumption of weak nonlinearity, the solution generated is in general more realistic than those achieved by simplifying the model of the physical problem.

4. Test problems

This section includes two subsections, Test CDPs and Test FDPs. In these subsections are solved and tested several problems to illustrate ability and reliability of ADM technique.

4.1. Test CDPs

Example 4.1. *Consider the following CDP*

$$\begin{cases} \triangle h(t) - h(t) = 0, \\ h(0) = a. \end{cases}$$

Since $\varsigma = 1$, we apply the operator $\triangle^{-1}$ on both sides of the above equation and using the given condition gives us,

$$\begin{cases} h_0(t) = a, \\ h_{n+1}(t) = \triangle^{-1} h_n(t), \end{cases}$$

therefore

$$\begin{aligned} &h_1(t) = \triangle^{-1} h_0(t) = at^{\underline{1}}, \quad h_2(t) = \triangle^{-1} h_1(t) = a\tfrac{t^{\underline{2}}}{2!}, \\ &h_3(t) = \triangle^{-1} h_2(t) = a\tfrac{t^{\underline{3}}}{3!}, \quad h_4(t) = \triangle^{-1} h_3(t) = a\tfrac{t^{\underline{4}}}{4!}, \\ &\vdots \end{aligned}$$

then, yields

$$\begin{aligned} h(t) &= \sum_{i=0}^{\infty} h_i(t) = h_0(t) + h_1(t) + h_2(t) + h_3(t) + h_4(t) + \cdots \\ &= a + a\frac{t^{\underline{1}}}{1!} + a\frac{t^{\underline{2}}}{2!} + a\frac{t^{\underline{3}}}{3!} + a\frac{t^{\underline{4}}}{4!} + \cdots = a\left(1 + \frac{t^{\underline{1}}}{1!} + \frac{t^{\underline{2}}}{2!} + \frac{t^{\underline{3}}}{3!} + \frac{t^{\underline{4}}}{4!} + \cdots\right) \\ &= a\sum_{i=0}^{\infty} \frac{t^{\underline{i}}}{i!} = a.2^t. \end{aligned}$$

Example 4.2. *Consider the following CDP*

$$\begin{cases} \triangle h(t) - h(t) = 2^{2t+1}, \\ h(0) = 2. \end{cases}$$

By applying ADM on the above equation, yields

$$\begin{cases} h_0(t) = \frac{2}{3}.4^t + \frac{4}{3}, \\ h_{n+1}(t) = \triangle^{-1} h_n(t), \end{cases}$$

therefore

$$\begin{aligned}
h_1(t) &=\triangle^{-1} h_0(t) = \frac{2}{3^2}4^t + \frac{4}{3}t^{\underline{1}} - \frac{2}{3^2},\\
h_2(t) &=\triangle^{-1} h_1(t) = \frac{2}{3^3}4^t + \frac{4}{3}\frac{t^{\underline{2}}}{2!} - \frac{2}{3^2}t^{\underline{1}} - \frac{2}{3^3},\\
h_3(t) &=\triangle^{-1} h_2(t) = \frac{2}{3^4}4^t + \frac{4}{3}\frac{t^{\underline{3}}}{3!} + -\frac{2}{3^2}\frac{t^{\underline{2}}}{2!} - \frac{2}{3^3}t^{\underline{1}} - \frac{2}{3^4},\\
h_4(t) &=\triangle^{-1} h_3(t) = \frac{2}{3^5}4^t + \frac{4}{3}\frac{t^{\underline{4}}}{4!} - \frac{2}{3^2}\frac{t^{\underline{3}}}{3!} - \frac{2}{3^3}\frac{t^{\underline{2}}}{2!} - \frac{2}{3^4}t^{\underline{1}} - \frac{2}{3^5},\\
&\vdots
\end{aligned}$$

then, yields

$$\begin{aligned}
h(t) &= \sum_{i=0}^{\infty} h_i(t) = h_0(t) + h_1(t) + h_2(t) + h_3(t) + h_4(t) + \cdots\\
&= \left(\frac{2}{3}.4^t + \frac{4}{3}\right) + \left(\frac{2}{3^2}4^t + \frac{4}{3}t^{\underline{1}} - \frac{2}{3^2}\right) + \left(\frac{2}{3^3}4^t + \frac{4}{3}\frac{t^{\underline{2}}}{2!} - \frac{2}{3^2}t^{\underline{1}} - \frac{2}{3^3}\right)\\
&\quad + \left(\frac{2}{3^4}4^t + \frac{4}{3}\frac{t^{\underline{3}}}{3!} - \frac{2}{3^2}\frac{t^{\underline{2}}}{2!} - \frac{2}{3^3}t^{\underline{1}} - \frac{2}{3^4}\right) + \left(\frac{2}{3^5}4^t + \frac{4}{3}\frac{t^{\underline{4}}}{4!} - \frac{2}{3^2}\frac{t^{\underline{3}}}{3!} - \frac{2}{3^3}\frac{t^{\underline{2}}}{2!} - \frac{2}{3^4}t^{\underline{1}} - \frac{2}{3^5}\right)\\
&= \frac{2}{3}.4^t\underbrace{\left(1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots\right)}_{\frac{3}{2}} + \frac{4}{3}\underbrace{\left(1 + \frac{t^{\underline{1}}}{1!} + \frac{t^{\underline{2}}}{2!} + \frac{t^{\underline{3}}}{3!} + \frac{t^{\underline{4}}}{4!} + \cdots\right)}_{2^t}\\
&\quad - \frac{2}{3^2}\underbrace{\left(1 + \frac{t^{\underline{1}}}{1!} + \frac{t^{\underline{2}}}{2!} + \frac{t^{\underline{3}}}{3!} + \frac{t^{\underline{4}}}{4!} + \cdots\right)}_{2^t} - \frac{2}{3^3}\underbrace{\left(1 + \frac{t^{\underline{1}}}{1!} + \frac{t^{\underline{2}}}{2!} + \frac{t^{\underline{3}}}{3!} + \frac{t^{\underline{4}}}{4!} + \cdots\right)}_{2^t}\\
&\quad - \frac{2}{3^4}\underbrace{\left(1 + \frac{t^{\underline{1}}}{1!} + \frac{t^{\underline{2}}}{2!} + \frac{t^{\underline{3}}}{3!} + \frac{t^{\underline{4}}}{4!} + \cdots\right)}_{2^t} - \frac{2}{3^5}\underbrace{\left(1 + \frac{t^{\underline{1}}}{1!} + \frac{t^{\underline{2}}}{2!} + \frac{t^{\underline{3}}}{3!} + \frac{t^{\underline{4}}}{4!} + \cdots\right)}_{2^t} + \cdots\\
&= 4^t + \frac{4}{3}.2^t - \frac{2}{3^2}.2^t\left(1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots\right) = 4^t + 2^t.
\end{aligned}$$

Example 4.3. *Consider the following CDP*

$$\begin{cases}\triangle^2 h(t) - h(t) = 0,\\ h(0) = 1, \quad \triangle\, h(0) = 1.\end{cases}$$

By applying the operator $\triangle^{-2}$ *on both sides of the above equation and using the given condition gives us,*

$$\begin{cases}h_0(t) = 1 + t^{\underline{1}},\\ h_{n+1}(t) =\triangle^{-2} h_n(t),\end{cases}$$

therefore

$$\begin{aligned}
&h_1(t) = \triangle^{-2}\, h_0(t) = \tfrac{t^{\underline{2}}}{2!} + \tfrac{t^{\underline{3}}}{3!}, \quad h_2(t) = \triangle^{-2}\, h_1(t) = \tfrac{t^{\underline{4}}}{4!} + \tfrac{t^{\underline{5}}}{5!},\\
&h_3(t) = \triangle^{-2}\, h_2(t) = \tfrac{t^{\underline{6}}}{6!} + \tfrac{t^{\underline{7}}}{7!}, \quad h_4(t) = \triangle^{-2}\, h_3(t) = \tfrac{t^{\underline{8}}}{8!} + \tfrac{t^{\underline{9}}}{9!},\\
&\vdots
\end{aligned}$$

then, yields

$$\begin{aligned}
h(t) &= \sum_{i=0}^{\infty} h_i(t) = h_0(t) + h_1(t) + h_2(t) + h_3(t) + h_4(t) + \cdots\\
&= \left(1 + t^{\underline{1}}\right) + \left(\frac{t^{\underline{2}}}{2!} + \frac{t^{\underline{3}}}{3!}\right) + \left(\frac{t^{\underline{4}}}{4!} + \frac{t^{\underline{5}}}{5!}\right) + \left(\frac{t^{\underline{6}}}{6!} + \frac{t^{\underline{7}}}{7!}\right) + \left(\frac{t^{\underline{8}}}{8!} + \frac{t^{\underline{9}}}{9!}\right) + \cdots\\
&= 2^t.
\end{aligned}$$

Example 4.4. *Consider the following CDP*

$$\begin{cases} \triangle^2 h(t) - h(t) = 3^{t+1},\\ h(0) = 1, \quad \triangle\, h(0) = 2. \end{cases}$$

By applying ADM and the similar process, gets

$$\begin{cases} h_0(t) = \frac{3}{4}.3^t + \frac{1}{2}t^{\underline{1}} + \frac{1}{4},\\ h_{n+1}(t) = \triangle^{-2}\, h_n(t), \end{cases}$$

therefore

$$\begin{aligned}
&h_1(t) = \triangle^{-2}\, h_0(t) = \frac{3}{4^2}3^t + \frac{1}{2}\frac{t^{\underline{3}}}{3!} + \frac{1}{4}\frac{t^{\underline{2}}}{2!} - \frac{3}{2^3}t^{\underline{1}} - \frac{3}{4^2},\\
&h_2(t) = \triangle^{-2}\, h_1(t) = \frac{3}{4^3}3^t + \frac{1}{2}\frac{t^{\underline{5}}}{5!} + \frac{1}{4}\frac{t^{\underline{4}}}{4!} - \frac{3}{2^3}\frac{t^{\underline{3}}}{3!} - \frac{3}{4^2}\frac{t^{\underline{2}}}{2!} - \frac{3}{2^5}t^{\underline{1}} - \frac{3}{4^3},\\
&h_3(t) = \triangle^{-2}\, h_2(t) = \frac{3}{4^4}3^t + \frac{1}{2}\frac{t^{\underline{7}}}{7!} + \frac{1}{4}\frac{t^{\underline{6}}}{6!} - \frac{3}{2^3}\frac{t^{\underline{5}}}{5!} - \frac{3}{4^2}\frac{t^{\underline{4}}}{4!} - \frac{3}{2^5}\frac{t^{\underline{3}}}{3!} - \frac{3}{4^3}\frac{t^{\underline{2}}}{2!} - \frac{3}{2^7}t^{\underline{1}} - \frac{3}{4^4},\\
&\vdots
\end{aligned}$$

then, yields

$$\begin{aligned}
h(t) &= \sum_{i=0}^{\infty} h_i(t) = h_0(t) + h_1(t) + h_2(t) + h_3(t) + \cdots\\
&= \left(\frac{3}{4}.3^t + \frac{1}{2}t^{\underline{1}} + \frac{1}{4}\right) + \left(\frac{3}{4^2}3^t + \frac{1}{2}\frac{t^{\underline{3}}}{3!} + \frac{1}{4}\frac{t^{\underline{2}}}{2!} - \frac{3}{2^3}t^{\underline{1}} - \frac{3}{4^2}\right)\\
&\quad + \left(\frac{3}{4^3}3^t + \frac{1}{2}\frac{t^{\underline{5}}}{5!} + \frac{1}{4}\frac{t^{\underline{4}}}{4!} - \frac{3}{2^3}\frac{t^{\underline{3}}}{3!} - \frac{3}{4^2}\frac{t^{\underline{2}}}{2!} - \frac{3}{2^5}t^{\underline{1}} - \frac{3}{4^3}\right)\\
&\quad + \left(\frac{3}{4^4}3^t + \frac{1}{2}\frac{t^{\underline{7}}}{7!} + \frac{1}{4}\frac{t^{\underline{6}}}{6!} - \frac{3}{2^3}\frac{t^{\underline{5}}}{5!} - \frac{3}{4^2}\frac{t^{\underline{4}}}{4!} - \frac{3}{2^5}\frac{t^{\underline{3}}}{3!} - \frac{3}{4^3}\frac{t^{\underline{2}}}{2!} - \frac{3}{2^7}t^{\underline{1}} - \frac{3}{4^4}\right) + \cdots,
\end{aligned}$$

then,

$$h(t) = \frac{3}{4}.3^t \underbrace{\left(1+\frac{1}{4}+\frac{1}{4^2}+\frac{1}{4^3}+\cdots\right)}_{\frac{4}{3}} + \frac{1}{4}\left(1+\frac{t^{\underline{2}}}{2!}+\frac{t^{\underline{4}}}{4!}+\cdots\right)$$
$$+\frac{1}{2}\left(\frac{t^{\underline{1}}}{1!}+\frac{t^{\underline{3}}}{3!}+\frac{t^{\underline{5}}}{5!}+\cdots\right) - \frac{3}{2^3}\left(\frac{t^{\underline{1}}}{1!}+\frac{t^{\underline{3}}}{3!}+\frac{t^{\underline{5}}}{5!}+\cdots\right)$$
$$-\frac{3}{4^2}\left(1+\frac{t^{\underline{2}}}{2!}+\frac{t^{\underline{4}}}{4!}+\cdots\right) - \frac{3}{2^5}\left(\frac{t^{\underline{1}}}{1!}+\frac{t^{\underline{3}}}{3!}+\frac{t^{\underline{5}}}{5!}+\cdots\right)+\cdots$$
$$= 3^t.$$

Example 4.5. *Given the following CDE,*

$$\begin{cases} \triangle_t\ h(x,t) = \frac{1}{2}\ \triangle_x^2\ h(x,t) + h(x,t), \\ h(x,0) = x, \end{cases}$$

its exact solution is $h(x,t) = x.2^t$ *To access the solution, we must apply operator* $\triangle_t^{-1}$ *on the above equation, this yields,*

$$h(x,t) = h(x,0) + \triangle_t^{-1}\left(\frac{1}{2}\ \triangle_x^2\ h(x,t) + h(x,t)\right). \tag{5}$$

Therefore the equation (5) *can be rewritten as*

$$h(x,t) = h(x,0) + \triangle_t^{-1}\left(\frac{1}{2}h(x+2,t) - h(x+1,t) + \frac{3}{2}h(x,t)\right).$$

Now, let $h(x,t) = h_{x,t} = \sum_{n=0}^{\infty} h_{x_n,t}$ *and by substituting in the above equation, we get*

$$\sum_{n=0}^{\infty} h_{x_n,t} = h_{x_0,t} + \triangle_t^{-1}\left(\frac{1}{2}\sum_{n=0}^{\infty} h_{x_{n+2},t} - \sum_{n=0}^{\infty} h_{x_{n+1},t} + \frac{3}{2}\sum_{n=0}^{\infty} h_{x_n,t}\right).$$

Therefore, we infer that the first term and the recursive formula series are as,

$$\begin{cases} h_{x_0,t} = x, \\ h_{x_{n+1},t} = \triangle_t^{-1}\left(\frac{1}{2}\sum_{n=0}^{\infty} h_{x_{n+2},t} - \sum_{n=0}^{\infty} h_{x_{n+1},t} + \frac{3}{2}\sum_{n=0}^{\infty} h_{x_n,t}\right), \end{cases}$$

then, we get

$$h_{x_1,t} = \triangle_t^{-1}\left(\frac{1}{2}h_{x_0+2,t} - h_{x_0+1,t} + \frac{3}{2}h_{x_0,t}\right) = \triangle_t^{-1}\left(\frac{1}{2}(x+2) - (x+1) + \frac{3}{2}x\right) = x\frac{t^{\underline{1}}}{1!},$$

$$h_{x_2,t} = \triangle_t^{-1}\left(\frac{1}{2}h_{x_1+2,t} - h_{x_1+1,t} + \frac{3}{2}h_{x_1,t}\right) = \triangle_t^{-1}\left(\frac{1}{2}(x+2) - (x+1) + \frac{3}{2}x\right) = x\frac{t^{\underline{2}}}{2!},$$

$$h_{x_3,t} = \triangle_t^{-1}\left(\frac{1}{2}h_{x_2+2,t} - h_{x_2+1,t} + \frac{3}{2}h_{x_1,t}\right) = \triangle_t^{-1}\left(\frac{1}{2}(x+2) - (x+1) + \frac{3}{2}x\right) = x\frac{t^{\underline{3}}}{3!},$$

$$h_{x_4,t} = \triangle_t^{-1}\left(\frac{1}{2}h_{x_3+2,t} - h_{x_3+1,t} + \frac{3}{2}h_{x_3,t}\right) = \triangle_t^{-1}\left(\frac{1}{2}(x+2) - (x+1) + \frac{3}{2}x\right) = x\frac{t^{\underline{4}}}{4!},$$

$$\vdots$$

then, we can write

$$\begin{aligned} h(x,t) = h_{x,t} &= \sum_{n=0}^{\infty} h_{x_n,t} = h_{x_0,t} + h_{x_1,t} + h_{x_2,t} + h_{x_3,t} + h_{x_4,t} + \cdots \\ &= x + x\frac{t^{\underline{1}}}{1!} + x\frac{t^{\underline{2}}}{2!} + x\frac{t^{\underline{3}}}{3!} + x\frac{t^{\underline{4}}}{4!} + \cdots = x\left(1 + \frac{t^{\underline{1}}}{1!} + \frac{t^{\underline{2}}}{2!} + \frac{t^{\underline{3}}}{3!} + \frac{t^{\underline{4}}}{4!} + \cdots\right) = x.2^t. \end{aligned}$$

4.2. Test FDPs

Example 4.6. *Consider the following FDP*

$$\begin{cases} \triangle^{\varsigma}h(t) = h(t+\varsigma-1), \quad 0 < \varsigma \leq 1, \\ y(0) = a. \end{cases}$$

By applying operator $\triangle^{-\varsigma}$ on the above equation, this yields,

$$\begin{cases} h_0(t) = a, \\ h_{n+1}(t) = \triangle^{-\varsigma}h_n(t+\varsigma-1), \end{cases}$$

therefore

$$\begin{aligned} &h_1(t) = \triangle^{-\varsigma}h_0(t+\varsigma-1) = a\tfrac{(t+\varsigma-1)^{\underline{\varsigma}}}{\Gamma(\varsigma+1)}, \quad h_2(t) = \triangle^{-\varsigma}h_1(t+\varsigma-1) = a\tfrac{(t+2\varsigma-2)^{\underline{2\varsigma}}}{\Gamma(2\varsigma+1)}, \\ &h_3(t) = \triangle^{-\varsigma}h_2(t+\varsigma-1) = a\tfrac{(t+3\varsigma-3)^{\underline{3\varsigma}}}{\Gamma(3\varsigma+1)}, \quad h_4(t) = \triangle^{-\varsigma}h_3(t+\varsigma-1) = a\tfrac{(t+4\varsigma-4)^{\underline{4\varsigma}}}{\Gamma(4\varsigma+1)}, \\ &\vdots \end{aligned}$$

then, yields

$$\begin{aligned} h(t) &= \sum_{i=0}^{\infty} h_i(t) = h_0(t) + h_1(t) + h_2(t) + h_3(t) + \cdots \\ &= a + a\frac{(t+\varsigma-1)^{\underline{\varsigma}}}{\Gamma(\varsigma+1)} + a\frac{(t+2\varsigma-2)^{\underline{2\varsigma}}}{\Gamma(2\varsigma+1)} + a\frac{(t+3\varsigma-3)^{\underline{3\varsigma}}}{\Gamma(3\varsigma+1)} + a\frac{(t+4\varsigma-4)^{\underline{4\varsigma}}}{\Gamma(4\varsigma+1)} + \cdots \\ &= a\sum_{i=0}^{\infty} \frac{(t+i(\varsigma-1))^{\underline{i\varsigma}}}{\Gamma(i\varsigma+1)}. \end{aligned}$$

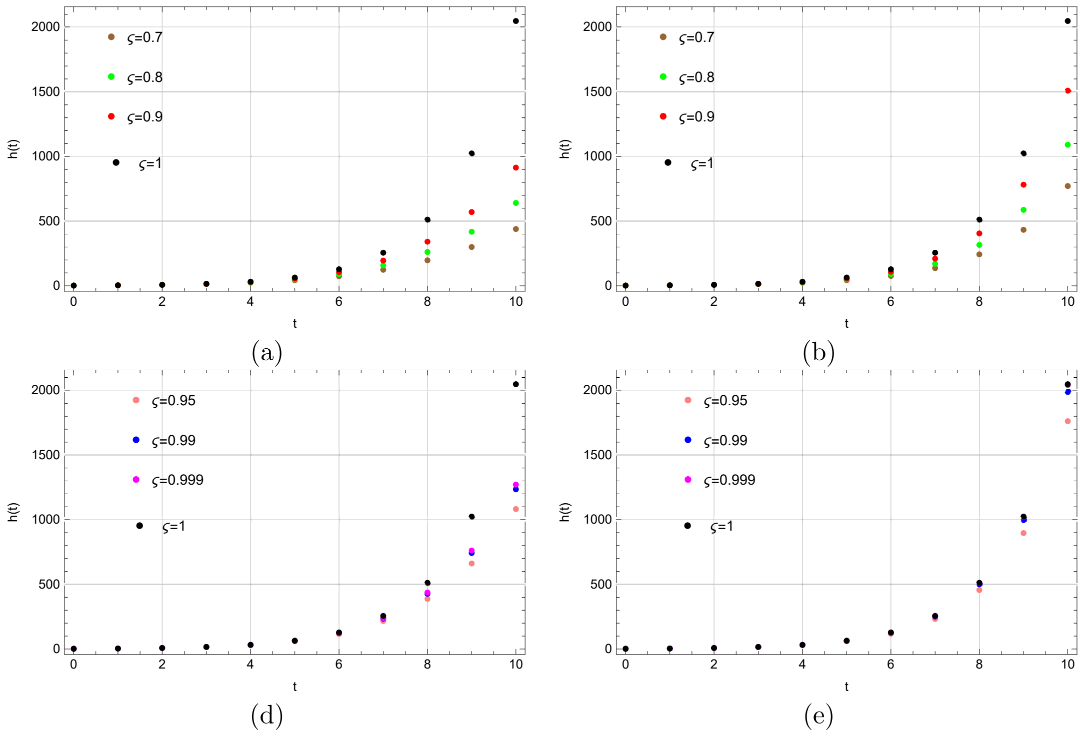

Figure 1: (Example 4.6) The approximation solution $h(x,t)$ with setting $a = 2$ (a) The first five sentences (b) The first ten sentences (c) The first five sentences (d) The first ten sentences.

The approximation solutions $h(t)$ considering the first five and ten sentences for different ς are shown in Figure 1. We can see the different behaviors of the discrete FDE with different fractional parameters. It is clear when ς is close to 1 the approximation solution tends to the exact solution.

In this example, let $\varsigma = 1$, then

$$\begin{cases} h_0(t) = a, \\ h_{n+1}(t) = \triangle^{-1} h_n(t), \end{cases}$$

therefore

$$\begin{array}{l} h_1(t) = a\frac{t^{\underline{1}}}{1!}, \quad h_2(t) = a\frac{t^{\underline{2}}}{2!}, \\ h_3(t) = a\frac{t^{\underline{3}}}{3!}, \quad h_4(t) = a\frac{t^{\underline{4}}}{4!}, \\ \vdots \end{array}$$

then, yields

$$h(t) = \sum_{i=0}^{\infty} h_i(t) = a.2^t.$$

Example 4.7. *Given the following FDP,*

$$\begin{cases} \triangle_t^{\varsigma} h(x,t) = \triangle_x^2 h(x,t), \quad 0 < \varsigma \leq 1, \\ h(x,0) = 2^x, \end{cases} \tag{6}$$

its exact solution is $h(x,t) = 2^{x+t}$ *when* $\varsigma = 1$. *To access the solution, we must apply operator* ${}_{1-\varsigma}\triangle_t^{-\varsigma}$ *on the above equation, this yields,*

$$h(x,t) = h(x,0) + {}_{1-\varsigma}\triangle_t^{-\varsigma}\left(\triangle_x^2 h(x,t)\right). \tag{7}$$

Therefore the equation (7) *can be rewritten as*

$$h(x,t) = h(x,0) + {}_{1-\varsigma}\triangle_t^{-\varsigma}\left(h(x+2,t) - 2h(x+1,t) + h(x,t)\right).$$

Now, let $h(x,t) = h_{x,t} = \sum_{n=0}^{\infty} h_{x_n,t}$ *and by substituting in the above equation, we get*

$$\sum_{n=0}^{\infty} h_{x_n,t} = h_{x_0,t} + {}_{1-\varsigma}\triangle_t^{-\varsigma}\left(\sum_{n=0}^{\infty} h_{x_{n+2},t} - 2\sum_{n=0}^{\infty} w_{x_{n+1},t} + \sum_{n=0}^{\infty} w_{x_n,t}\right).$$

Therefore, we infer that the first term and the recursive formula series are as,

$$\begin{cases} h_{x_0,t} = 2^x, \\ h_{x_{n+1},t} = {}_{1-\varsigma}\triangle_t^{-\varsigma}\left(\sum_{n=0}^{\infty} h_{x_{n+2},t} - 2\sum_{n=0}^{\infty} h_{x_{n+1},t} + \sum_{n=0}^{\infty} h_{x_n,t}\right), \end{cases}$$

then, we get

$$h_{x_1,t} = {}_{1-\varsigma}\triangle_t^{-\varsigma}\left(h_{x_0+2,t} - 2h_{x_0+1,t} + h_{x_0,t}\right) = {}_{1-\varsigma}\triangle_t^{-\varsigma}\left(2^{x+2} - 2.2^{x+1} + 2^x\right) = 2^x \frac{(t+\varsigma-1)^{\underline{\varsigma}}}{\Gamma(\varsigma+1)},$$

$$h_{x_2,t} = {}_{1-\varsigma}\triangle_t^{-\varsigma}\left(h_{x_1+2,t} - 2h_{x_1+1,t} + h_{x_1,t}\right) = 2^x\left({}_{1-\varsigma}\triangle_t^{-\varsigma}\frac{(t+\varsigma-1)^{\underline{\varsigma}}}{\Gamma(\varsigma+1)}\right) = 2^x \frac{(t+2\varsigma-2)^{\underline{2\varsigma}}}{\Gamma(2\varsigma+1)},$$

$$h_{x_3,t} = {}_{1-\varsigma}\triangle_t^{-\varsigma}\left(h_{x_2+2,t} - 2h_{x_2+1,t} + h_{x_2,t}\right) = 2^x\left({}_{1-\varsigma}\triangle_t^{-\varsigma}\frac{(t+2\varsigma-2)^{\underline{2\varsigma}}}{\Gamma(2\varsigma+1)}\right) = 2^x \frac{(t+3\varsigma-3)^{\underline{3\varsigma}}}{\Gamma(3\varsigma+1)},$$

$$h_{x_4,t} = {}_{1-\varsigma}\triangle_t^{-\varsigma}\left(h_{x_3+2,t} - 2h_{x_3+1,t} + h_{x_3,t}\right) = 2^x\left({}_{1-\varsigma}\triangle_t^{-\varsigma}\frac{(t+3\varsigma-3)^{\underline{3\varsigma}}}{\Gamma(3\varsigma+1)}\right) = 2^x \frac{(t+4\varsigma-4)^{\underline{4\varsigma}}}{\Gamma(4\varsigma+1)},$$

$$\vdots$$

then, we can write

$$\begin{aligned}
h(x,t) = h_{x,t} &= \sum_{n=0}^{\infty} h_{x_n,t} = h_{x_0,t} + h_{x_1,t} + h_{x_2,t} + h_{x_3,t} + h_{x_4,t} + \cdots \\
&= 2^x + 2^x \frac{(t+\varsigma-1)^{\underline{\varsigma}}}{\Gamma(\varsigma+1)} + 2^x \frac{(t+2\varsigma-2)^{\underline{2\varsigma}}}{\Gamma(2\varsigma+1)} + 2^x \frac{(t+3\varsigma-3)^{\underline{3\varsigma}}}{\Gamma(3\varsigma+1)} + 2^x \frac{(t+4\varsigma-4)^{\underline{4\varsigma}}}{\Gamma(4\varsigma+1)} + \cdots \\
&= 2^x \left(1 + \frac{(t+\varsigma-1)^{\underline{\varsigma}}}{\Gamma(\varsigma+1)} + \frac{(t+2\varsigma-2)^{\underline{2\varsigma}}}{\Gamma(2\varsigma+1)} + \frac{(t+3\varsigma-3)^{\underline{3\varsigma}}}{\Gamma(3\varsigma+1)} + \frac{(t+4\varsigma-4)^{\underline{4\varsigma}}}{\Gamma(4\varsigma+1)} + \cdots \right) \\
&= 2^x \sum_{n=0}^{\infty} \frac{(t+n(\varsigma-1))^{\underline{n\varsigma}}}{\Gamma(n\varsigma+1)}.
\end{aligned}$$

When $\varsigma = 1$*, then*

$$h(x,t) = 2^{x+t}.$$

Figures 2 and 3 show the approximation solutions $h(x,t)$ *considering the first five and ten sentences for different* ς*. We can see the different behaviors of the discrete FDE with different fractional parameters. It is clear when* ς *is close to* 1 *the approximation solution tends to the exact solution.*

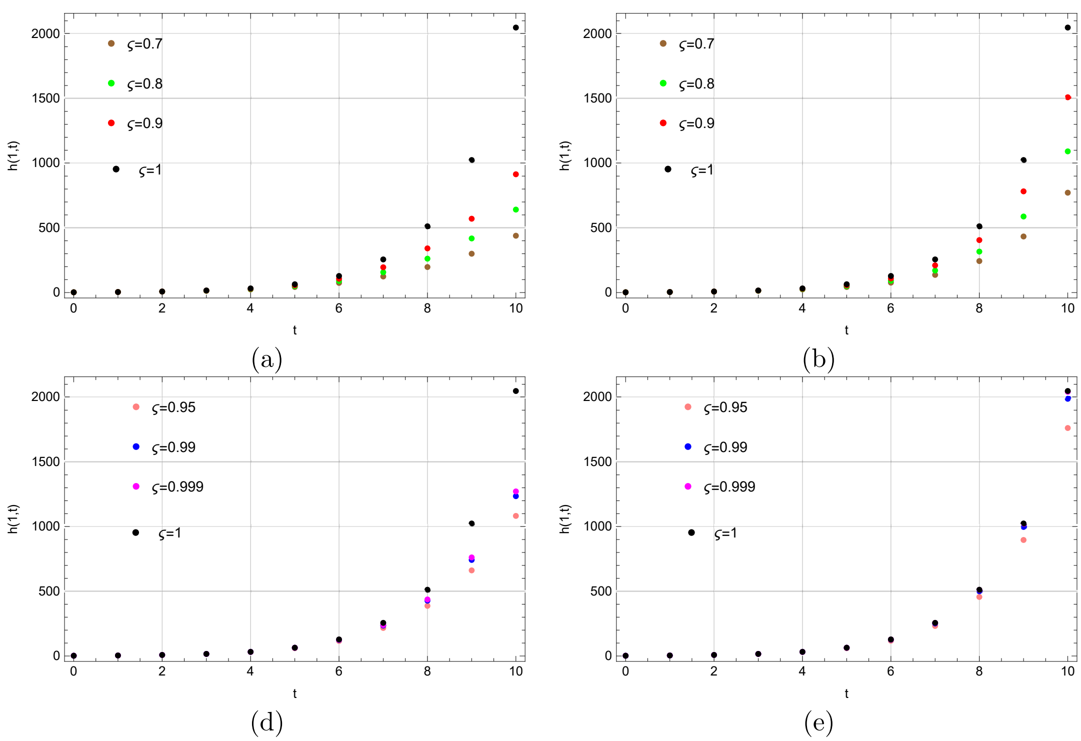

Figure 2: (Example 4.7) The approximation solution $h(x,t)$ when $x = 1$ and considering (a) The first five sentences (b) The first ten sentences (c) The first five sentences (d) The first ten sentences.

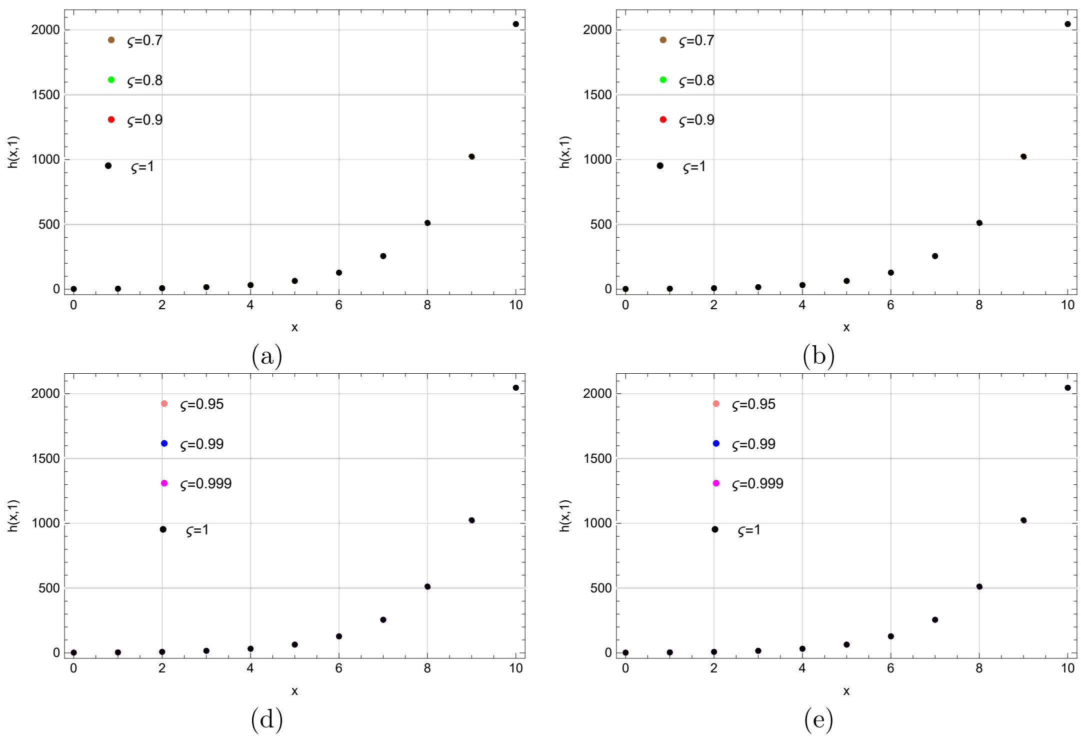

Figure 3: (Example 4.7) The approximation solution $h(x,t)$ when $t=1$ and considering (a) The first five sentences (b) The first ten sentences (c) The first five sentences (d) The first ten sentences.

Example 4.8. *Given the following FDE,*

$$\begin{cases} \triangle_t^{\varsigma}\, h(x,t) = \frac{1}{2}\, \triangle_x^2\, h(x,t) + w(x,t), \quad 0 < \varsigma \le 1, \\ h(x,0) = x, \end{cases}$$

its exact solution is $h(x,t) = x.2^t$ *when* $\varsigma = 1$. *To access the solution, we must apply operator* ${}_{1-\varsigma}\triangle_t^{-\varsigma}$ *on the above equation, this yields,*

$$h(x,t) = h(x,0) + {}_{1-\varsigma}\triangle_t^{-\varsigma} \left(\frac{1}{2}\, \triangle_x^2\, h(x,t) + h(x,t) \right). \tag{8}$$

Therefore the equation (8) *can be rewritten as*

$$h(x,t) = hw(x,0) + {}_{1-\varsigma}\triangle_t^{-\varsigma} \left(\frac{1}{2} h(x+2,t) - h(x+1,t) + \frac{3}{2} h(x,t) \right).$$

Now, let $h(x,t) = h_{x,t} = \sum_{n=0}^{\infty} h_{x_n,t}$ *and by substituting in the above equation, we get*

$$\sum_{n=0}^{\infty} hw_{x_n,t} = h_{x_0,t} + {}_{1-\varsigma}\triangle_t^{-\varsigma} \left(\frac{1}{2} \sum_{n=0}^{\infty} h_{x_{n+2},t} - \sum_{n=0}^{\infty} h_{x_{n+1},t} + \frac{3}{2} \sum_{n=0}^{\infty} h_{x_n,t} \right).$$

Therefore, we infer that the first term and the recursive formula series are as,

$$\begin{cases} h_{x_0,t} = x, \\ h_{x_{n+1},t} = {}_{1-\varsigma}\Delta_t^{-\varsigma}\left(\frac{1}{2}\sum_{n=0}^{\infty} h_{x_{n+2},t} - \sum_{n=0}^{\infty} h_{x_{n+1},t} + \frac{3}{2}\sum_{n=0}^{\infty} h_{x_n,t}\right), \end{cases}$$

then, we get

$$h_{x_1,t} = {}_{1-\varsigma}\Delta_t^{-\varsigma}\left(\frac{1}{2}h_{x_0+2,t} - h_{x_0+1,t} + \frac{3}{2}h_{x_0,t}\right) = {}_{1-\varsigma}\Delta_t^{-\varsigma}\left(\frac{1}{2}(x+2) - (x+1) + \frac{3}{2}x\right) = x\frac{(t+\varsigma-1)^{\underline{\varsigma}}}{\Gamma(\varsigma+1)},$$

$$h_{x_2,t} = {}_{1-\varsigma}\Delta_t^{-\varsigma}\left(\frac{1}{2}h_{x_1+2,t} - h_{x_1+1,t} + \frac{3}{2}h_{x_1,t}\right) = x\left({}_{1-\varsigma}\Delta_t^{-\varsigma}\frac{(t+\varsigma-1)^{\underline{\varsigma}}}{\Gamma(\varsigma+1)}\right) = x\frac{(t+2\varsigma-2)^{\underline{2\varsigma}}}{\Gamma(2\varsigma+1)},$$

$$h_{x_3,t} = {}_{1-\varsigma}\Delta_t^{-\varsigma}\left(\frac{1}{2}h_{x_2+2,t} - h_{x_2+1,t} + \frac{3}{2}h_{x_2,t}\right) = x\left({}_{1-\varsigma}\Delta_t^{-\varsigma}\frac{(t+2\varsigma-2)^{\underline{2\varsigma}}}{\Gamma(2\varsigma+1)}\right) = x\frac{(t+3\varsigma-3)^{\underline{3\varsigma}}}{\Gamma(3\varsigma+1)},$$

$$h_{x_4,t} = {}_{1-\varsigma}\Delta_t^{-\varsigma}\left(\frac{1}{2}h_{x_3+2,t} - h_{x_3+1,t} + \frac{3}{2}h_{x_3,t}\right) = x\left({}_{1-\varsigma}\Delta_t^{-\varsigma}\frac{(t+3\varsigma-3)^{\underline{3\varsigma}}}{\Gamma(3\varsigma+1)}\right) = x\frac{(t+4\varsigma-4)^{\underline{4\varsigma}}}{\Gamma(4\varsigma+1)},$$

$$\vdots$$

then, we can write

$$\begin{aligned} h(x,t) = h_{x,t} &= \sum_{n=0}^{\infty} h_{x_n,t} = h_{x_0,t} + h_{x_1,t} + h_{x_2,t} + h_{x_3,t} + h_{x_4,t} + \cdots \\ &= x + x\frac{(t+\varsigma-1)^{\underline{\varsigma}}}{\Gamma(\varsigma+1)} + x\frac{(t+2\varsigma-2)^{\underline{2\varsigma}}}{\Gamma(2\varsigma+1)} + x\frac{(t+3\varsigma-3)^{\underline{3\varsigma}}}{\Gamma(3\varsigma+1)} + x\frac{(t+4\varsigma-4)^{\underline{4\varsigma}}}{\Gamma(4\varsigma+1)} + \cdots \\ &= x\left(1 + \frac{(t+\varsigma-1)^{\underline{\varsigma}}}{\Gamma(\varsigma+1)} + \frac{(t+2\varsigma-2)^{\underline{2\varsigma}}}{\Gamma(2\varsigma+1)} + \frac{(t+3\varsigma-3)^{\underline{3\varsigma}}}{\Gamma(3\varsigma+1)} + \frac{(t+4\varsigma-4)^{\underline{4\varsigma}}}{\Gamma(4\varsigma+1)} + \cdots\right) \\ &= x\sum_{n=0}^{\infty}\frac{(t+n(\varsigma-1))^{\underline{n\varsigma}}}{\Gamma(n\varsigma+1)}. \end{aligned}$$

When $\varsigma = 1$*, then*

$$h(x,t) = x.2^t.$$

We set the first five and ten sentences of $h(x,t)$*, then the numerical results are plotted in Figures 4 and 5. The different behaviors of the discrete FDE with different fractional parameters observe in this Figure. Also, in this Figure, we can see the approximation solution tends to the exact solution when* ς *is close to 1.*

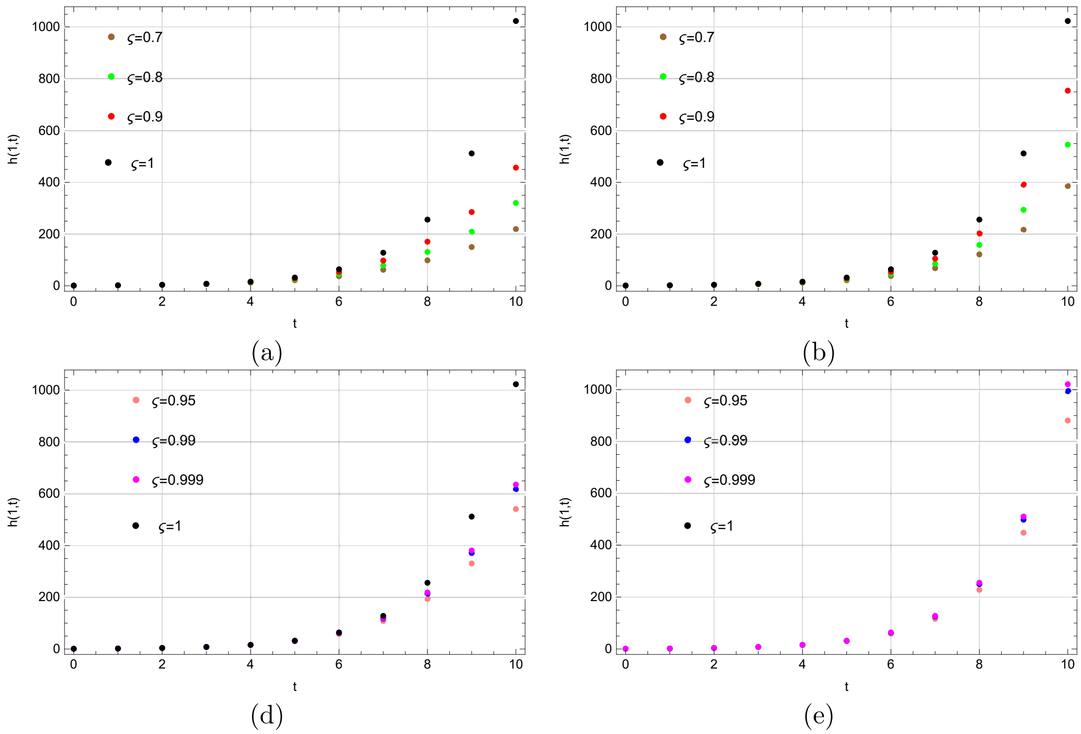

Figure 4: (Example 4.8) The approximation solution $h(x,t)$ when $x = 1$ and considering (a) The first five sentences (b) The first ten sentences (c) The first five sentences (d) The first ten sentences.

5. Conclusion

This paper is based on a new version of the Adomian decomposition method (ADM), which is called the discrete ADM. This techniqu help us to obtain a recursive formulation. Using this recursive formulation, we can achieve the solutions of linear and nonlinear classic and fractional difference problems (CDPs and FDPs). Finally, several practice problem are solved to show the accuracy of the discrete ADM approach.

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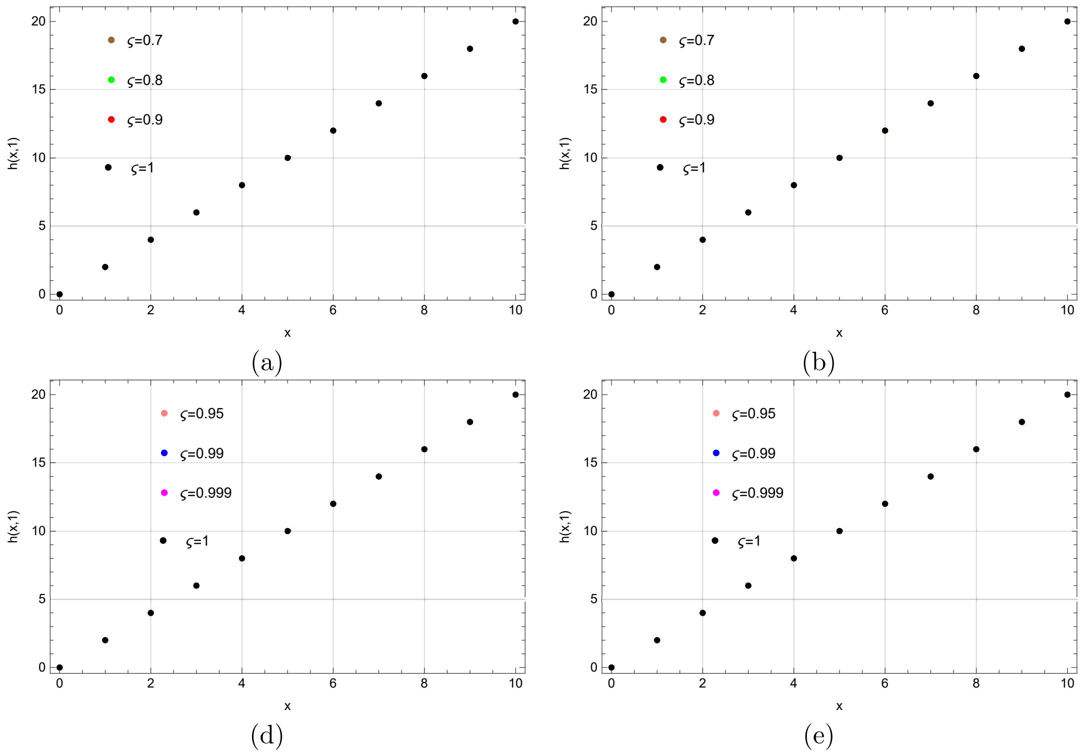

Figure 5: (Example 4.8) The approximation solution $h(x,t)$ when $t = 1$ and considering (a) The first five sentences (b) The first ten sentences (c) The first five sentences (d) The first ten sentences.

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