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Original Research Paper

Discrete ADM: A tool for solving a class of classic and fractional difference problems

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Abstract. Discrete fractional calculus (DFC) is a contemporary branch of fractional calculus with a discrete form. DFC is continuously spreading in neural networks, chaotic maps, engineering practice, and image encryption, which is appropriately assumed for discrete-time modeling in continuum problems. In this study, we solve a few problems with classic and fractional difference operators using a discrete variant of the Adomian decomposition method (ADM). This method helps to find the solutions of linear and nonlinear classic and fractional difference problems (CDPs and FDPs). A few examples are given to clarify and confirm the obtained results and some of particular cases of CDPs and FDPs are highlighted.

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1 Introduction

Fractional calculus (FC) is a field of mathematics that allows the order of derivative and integral operators to be arbitrary. It is seen as a form of extension of the classic derivative, which restricts the derivative and integral order to integers. Practical results have proved time and again that it is worth the effort to model real world phenomena using fractional integral and differential equations compared to integer calculus. There is a general consensus that this observation is wholly attributed to fractional calculus's ability to take into account the hereditary and memory influence in predicting the future, a characteristic that the classic derivative does not possess. For a detailed discussion of fractional calculus, particularly as introductory texts to the subject, we refer the reader to [1, 2, 3]. More applications of FC in applied mathematics, science, economics, engineering and other disciplines can be found in [4, 5, 6, 7, 8, 9].

To gain maximum benefits from a good mathematical model, it is of paramount importance that the methods of its solution be computationally efficient, consistent and highly accurate. There are no methods that are exclusively reserved for fractional calculus models. Any technique that is applicable to an integer order differential and integral equation will work perfectly in the fractional calculus setting.

However, there is no doubt that the accommodation of the fractional order feature in fractional calculus increases the labour required to solve fractional differential and integral equations. Thus, in solving these kind equations, engaging a method of solution that is both computationally inexpensive and accurate is ideal, although it is a challenging exercise to strike the balance. Common methods that have been applied successfully to solve fractional calculus models include, homotopy analysis method [10], Adams-Bashforth method [8], homotopy perturbation method [11], meshless method [12, 13], Adomian decomposition method [14, 15], optimal homotopy asymptotic method [16], operational matrix method [17, 18, 19].

During the last two decades, the theory of special functions and discrete fractional calculus (DFC) have been gotten by to attract increasing attention from the physical and mathematical communities. Specifically, the strict correlation between these two models has been acting as the

driving force for the most recent developments and generalizations in the literature on these subjects. In 1974, Daiz et al. [20] introduced the idea of DFC and composed it with an infinite sum. Later on, in 1988, Gray et al. [21] extended this concept and implemented it to the finite sum. This concept is known as the nabla difference operator in the literature. Atici and Eloe [22] proposed the theory of fractional difference equations, although the practical implementation is presented in [23]. In [24], the Ulam-Hyers-Rassias stability of two structures of discrete fractional three-point boundary value problems is discussed. In 2022, Alzabut et al. have considered a discrete time fractional order host-immune-tumor cells interaction model [25].
This research paper aims to use a new form of ADM, that is discrete ADM to solve a class of CDPs and FDPs.
The outline of our study is as follows. Preliminaries and notations of discrete fractional calculus are recalled in Section 2. Section 3, we construct a new version of ADM (discrete ADM). Our findings with some graphs are illustrated in section 4. Section 5 contains final concluding remarks.

2 Preliminaries and notations

In this section, we recall some basics concepts of discrete fractional calculus (DF–calculus), which will be necessary in proceeding to obtain our discrete results.

Definition 2.1 (See [26])**.** Get for $a \in \mathbb{R}$,

$$\mathcal{N}_a := \{a, a+1, a+2, \cdots\},$$

or for $a, b \in \mathbb{R}$ and $b > a$,

$$\mathcal{N}_a^b := \{a, a+1, a+2, \cdots, b\}.$$

The forward difference operator $\triangle$ and $\triangle^2$ are written as (1) if $w : \mathcal{N}_a^b \to \mathbb{R}$:

$$\begin{aligned} &\triangle h(t) = h(t+1) - h(t), \quad t \in \mathcal{N}_a^{b-1}, \\ &\triangle^2 h(t) = h(t+2) - 2h(t+1) + h(t). \end{aligned} \tag{1}$$

Theorem 2.2 (See [26]). *Assume $s_1,\ s_2,\ s_3$ are constants. Then the following hold:*

$$\int (t-s_1)^{\underline{s_2}}\ \triangle t = \frac{1}{s_2+1}(t-s_1)^{\underline{s_2+1}} + C, \quad s_2 \neq -1,$$
$$\int s_1^t\ \triangle t = \frac{1}{s_1-1}s_1^t + C, \quad s_1 \neq 1.$$

Definition 2.3 (See [26]). The falling function, $t^{\underline{s}}$, is given as follows:

i) for $s \in \mathbb{N}$,

$$t^{\underline{s}} := t(t-1)(t-2)\cdots(t-s+1), \quad t^{\underline{0}} = 1,$$

ii) for $s \in \mathbb{R}$,

$$t^{\underline{s}} := \frac{\Gamma(t+1)}{\Gamma(t-s+1)}, \quad t \in \mathbb{R} - \left\{\mathbb{Z}^- \cup \{0\}\right\}, \quad 0^{\underline{0}} = 0.$$

Lemma 2.4. *Let $0 < \varsigma < 1$, then*

$$\sum_{r=1-\varsigma}^{t-\varsigma} (t-r-1)^{\underline{\varsigma-1}} = \frac{\Gamma(t+\varsigma)}{\varsigma\Gamma(t)}.$$

Proof. *First, we can write*

$$\sum_{r=1-\varsigma}^{t-\varsigma} (t-r-1)^{\underline{\varsigma-1}} = \sum_{r=1-\varsigma}^{t-\varsigma} \frac{\Gamma(t-r)}{\Gamma(t-r-\varsigma+1)}$$
$$= \sum_{r=1-\varsigma}^{t-\varsigma-1} \frac{\Gamma(t-r)}{\Gamma(t-r-\varsigma+1)} + \Gamma(\varsigma).$$

Let $t > r$, $t, r \in \mathbb{R}$, $r > -1$, $t > -1$, then [27]

$$\frac{\Gamma(t+1)}{\Gamma(r+1)\Gamma(t-r+1)} = \frac{\Gamma(t+2)}{\Gamma(r+2)\Gamma(t-r+1)} - \frac{\Gamma(t+1)}{\Gamma(r+2)\Gamma(t-r)},$$

that is

$$\frac{\Gamma(t+1)}{\Gamma(t-r+1)} = \frac{1}{r+1}\left[\frac{\Gamma(t+2)}{\Gamma(t-r+1)} - \frac{\Gamma(t+1)}{\Gamma(t-r)}\right].$$

Then

$$\begin{aligned}\sum_{r=1-\varsigma}^{t-\varsigma}(t-r-1)^{\underline{\varsigma-1}} &= \sum_{r=1-\varsigma}^{t-\varsigma-1}\frac{1}{\varsigma}\left[\frac{\Gamma(t-r+1)}{\Gamma(t-r-\varsigma+1)}-\frac{\Gamma(t-r)}{\Gamma(t-r-\varsigma)}\right]+\Gamma(\varsigma)\\ &= \frac{1}{\varsigma}\left[\frac{\Gamma(t+\varsigma)}{\Gamma(t)}-\frac{\Gamma(t+\varsigma)}{\Gamma(1)}\right]+\Gamma(\varsigma)\\ &= \frac{\Gamma(t+\varsigma)}{\varsigma\Gamma(t)}.\end{aligned}$$

□

Definition 2.5 (See [26, 28])**.** The fractional sum of order ς is defined setting $\varsigma > 0$ and $w : \mathbb{N}_a \to \mathbb{R}$ as,

$$\triangle_a^{-\varsigma}\, h(t) = \frac{1}{\Gamma(\varsigma)}\sum_{r=a}^{t-\varsigma}(t-\sigma(r))^{\underline{\varsigma-1}}\, h(r), \quad t \in \mathcal{N}_{a+\varsigma},$$

where $\sigma(r) = r + 1$. Set $h(t) = t^{\underline{\gamma}}$, then

$$\triangle_a^{-\varsigma}\, h(t) = \frac{\Gamma(\gamma+1)}{\Gamma(\varsigma+\gamma+1)}t^{\underline{\varsigma+\gamma}}, \quad \gamma \in \mathbb{R}^+.$$

Definition 2.6 (See [26, 28])**.** The Caputo delta difference is given $0 < \varsigma < 1$ and $h : \mathbb{N}_a \to \mathbb{R}$ as,

$${}^{C}\triangle_a^{\varsigma}h(t) = {}^{C}\triangle_a^{-(1-\varsigma)}\,\triangle\, h(t) = \frac{1}{\Gamma(1-\varsigma)}\sum_{r=a}^{t+\varsigma-1}(t-\sigma(r))^{\underline{-\varsigma}}\,\triangle\, h(r), \quad t \in \mathcal{N}_{a-\varsigma+1},$$

where $\sigma(r) = r + 1$.

3 Discrete ADM for CDPs and FDPs

In this section, we apply the discrete ADM to for CDPs and as well as FDPs. Using this method, we can easily handle nonlinear problems with the large order of non linearity [29]. Let us discuss a brief outline of discrete ADM. Consider a general nonlinear difference equation in the form

$$\triangle^{\varsigma}\, h + L(h) + N(h) = g, \qquad s-1 < \varsigma \le s, \tag{2}$$

where L and N are the linear and nonlinear difference operators respectively. Also, g is the source term. Applying the operator $\triangle^{-\varsigma}$ (inverse of $\triangle^{\varsigma}$) on both sides of equation (2) and using the given initial conditions gives us,

$$h = \sum_{r=0}^{s-1} a_r \frac{t^{\underline{r}}}{r!} + \triangle^{-\varsigma} \left(g - L(h) - N(h) \right).$$

where $a_r,\ r = 0, \cdots, s-1$ are constants of integration and can be found form the boundary or initial conditions. In the Adomian decomposition method, we assume the solution h can be expanded into an infinite series as

$$h = \sum_{i=0}^{\infty} h_i. \tag{3}$$

Also, the nonlinear term $N(h)$ will be written as

$$N(h) = \sum_{i=0}^{\infty} A_i, \tag{4}$$

where A_i are called Adomain polynomials. By specified A_i, the next component of can be determined:

$$h_{i+1} = \triangle^{-\varsigma} \sum_{i=0}^{\infty} A_i,$$

Finally, after some iterations and getting sufficient accuracy, the solution of the equation can be expressed by equation (3). In equation (4), the Adomian polynomials can be computed by several techniques. Here we use the following recursive formula [14]:

$$A_i = \frac{1}{i!} \left[\frac{d^i}{d\lambda^i} N \left(\sum_{i=0}^{\infty} \lambda^i h_i \right) \right]_{\lambda=0}, \qquad i \geq 0.$$

Since the method does not resort to linearization or assumption of weak nonlinearity, the solution generated is in general more realistic than those achieved by simplifying the model of the physical problem.

4 Test problems

This section includes two subsections, Test CDPs and Test FDPs. In these subsections, a few problems are solved and tested to illustrate ability and reliability of ADM technique.

4.1 Test CDPs

Example 4.1. Consider the following CDP

$$\begin{cases} \triangle h(t) - h(t) = 0, \\ h(0) = a. \end{cases}$$

Since $\varsigma = 1$, we apply the operator $\triangle^{-1}$ on both sides of the above equation and using the given initial condition leads us,

$$\begin{cases} h_0(t) = a, \\ h_{n+1}(t) = \triangle^{-1} h_n(t), \end{cases}$$

therefore

$$\begin{aligned} & h_1(t) = \triangle^{-1} h_0(t) = at^{\underline{1}}, \quad h_2(t) = \triangle^{-1} h_1(t) = a\tfrac{t^{\underline{2}}}{2!}, \\ & h_3(t) = \triangle^{-1} h_2(t) = a\tfrac{t^{\underline{3}}}{3!}, \quad h_4(t) = \triangle^{-1} h_3(t) = a\tfrac{t^{\underline{4}}}{4!}, \\ & \vdots \end{aligned}$$

then, yields

$$\begin{aligned} h(t) &= \sum_{i=0}^{\infty} h_i(t) = h_0(t) + h_1(t) + h_2(t) + h_3(t) + h_4(t) + \cdots \\ &= a + a\frac{t^{\underline{1}}}{1!} + a\frac{t^{\underline{2}}}{2!} + a\frac{t^{\underline{3}}}{3!} + a\frac{t^{\underline{4}}}{4!} + \cdots = a\left(1 + \frac{t^{\underline{1}}}{1!} + \frac{t^{\underline{2}}}{2!} + \frac{t^{\underline{3}}}{3!} + \frac{t^{\underline{4}}}{4!} + \cdots\right) \\ &= a\sum_{i=0}^{\infty} \frac{t^{\underline{i}}}{i!} = a.2^t. \end{aligned}$$

Example 4.2. Consider the following CDP

$$\begin{cases} \triangle h(t) - h(t) = 2^{2t+1}, \\ h(0) = 2. \end{cases}$$

By applying ADM on the above equation, yields

$$\begin{cases} h_0(t) = \frac{2}{3}.4^t + \frac{4}{3}, \\ h_{n+1}(t) = \triangle^{-1}\ h_n(t), \end{cases}$$

therefore

$$\begin{aligned}
h_1(t) &= \triangle^{-1}\ h_0(t) = \frac{2}{3^2}4^t + \frac{4}{3}t^{\underline{1}} - \frac{2}{3^2}, \\
h_2(t) &= \triangle^{-1}\ h_1(t) = \frac{2}{3^3}4^t + \frac{4}{3}\frac{t^{\underline{2}}}{2!} - \frac{2}{3^2}t^{\underline{1}} - \frac{2}{3^3}, \\
h_3(t) &= \triangle^{-1}\ h_2(t) = \frac{2}{3^4}4^t + \frac{4}{3}\frac{t^{\underline{3}}}{3!} + -\frac{2}{3^2}\frac{t^{\underline{2}}}{2!} - \frac{2}{3^3}t^{\underline{1}} - \frac{2}{3^4}, \\
h_4(t) &= \triangle^{-1}\ h_3(t) = \frac{2}{3^5}4^t + \frac{4}{3}\frac{t^{\underline{4}}}{4!} - \frac{2}{3^2}\frac{t^{\underline{3}}}{3!} - \frac{2}{3^3}\frac{t^{\underline{2}}}{2!} - \frac{2}{3^4}t^{\underline{1}} - \frac{2}{3^5}, \\
&\vdots
\end{aligned}$$

then, yields

$$\begin{aligned}
h(t) &= \sum_{i=0}^{\infty} h_i(t) = h_0(t) + h_1(t) + h_2(t) + h_3(t) + h_4(t) + \cdots \\
&= \left(\frac{2}{3}.4^t + \frac{4}{3}\right) + \left(\frac{2}{3^2}4^t + \frac{4}{3}t^{\underline{1}} - \frac{2}{3^2}\right) + \left(\frac{2}{3^3}4^t + \frac{4}{3}\frac{t^{\underline{2}}}{2!} - \frac{2}{3^2}t^{\underline{1}} - \frac{2}{3^3}\right) \\
&\quad + \left(\frac{2}{3^4}4^t + \frac{4}{3}\frac{t^{\underline{3}}}{3!} - \frac{2}{3^2}\frac{t^{\underline{2}}}{2!} - \frac{2}{3^3}t^{\underline{1}} - \frac{2}{3^4}\right) + \left(\frac{2}{3^5}4^t + \frac{4}{3}\frac{t^{\underline{4}}}{4!} - \frac{2}{3^2}\frac{t^{\underline{3}}}{3!} - \frac{2}{3^3}\frac{t^{\underline{2}}}{2!} - \frac{2}{3^4}t^{\underline{1}} - \frac{2}{3^5}\right) \\
&= \frac{2}{3}.4^t\underbrace{\left(1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots\right)}_{\frac{3}{2}} + \frac{4}{3}\underbrace{\left(1 + \frac{t^{\underline{1}}}{1!} + \frac{t^{\underline{2}}}{2!} + \frac{t^{\underline{3}}}{3!} + \frac{t^{\underline{4}}}{4!} + \cdots\right)}_{2^t} \\
&\quad - \frac{2}{3^2}\underbrace{\left(1 + \frac{t^{\underline{1}}}{1!} + \frac{t^{\underline{2}}}{2!} + \frac{t^{\underline{3}}}{3!} + \frac{t^{\underline{4}}}{4!} + \cdots\right)}_{2^t} - \frac{2}{3^3}\underbrace{\left(1 + \frac{t^{\underline{1}}}{1!} + \frac{t^{\underline{2}}}{2!} + \frac{t^{\underline{3}}}{3!} + \frac{t^{\underline{4}}}{4!} + \cdots\right)}_{2^t} \\
&\quad - \frac{2}{3^4}\underbrace{\left(1 + \frac{t^{\underline{1}}}{1!} + \frac{t^{\underline{2}}}{2!} + \frac{t^{\underline{3}}}{3!} + \frac{t^{\underline{4}}}{4!} + \cdots\right)}_{2^t} - \frac{2}{3^5}\underbrace{\left(1 + \frac{t^{\underline{1}}}{1!} + \frac{t^{\underline{2}}}{2!} + \frac{t^{\underline{3}}}{3!} + \frac{t^{\underline{4}}}{4!} + \cdots\right)}_{2^t} + \cdots \\
&= 4^t + \frac{4}{3}.2^t - \frac{2}{3^2}.2^t\left(1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots\right) = 4^t + 2^t.
\end{aligned}$$

Example 4.3. Consider the following CDP

$$\begin{cases} \triangle^2 h(t) - h(t) = 0, \\ h(0) = 1, \quad \triangle\, h(0) = 1. \end{cases}$$

We apply the operator $\triangle^{-2}$ on both sides of the above equation, after that, by using the given initial conditions, we have

$$\begin{cases} h_0(t) = 1 + t^{\underline{1}}, \\ h_{n+1}(t) = \triangle^{-2}\, h_n(t), \end{cases}$$

therefore

$$\begin{array}{ll} h_1(t) = \triangle^{-2}\, h_0(t) = \frac{t^{\underline{2}}}{2!} + \frac{t^{\underline{3}}}{3!}, & h_2(t) = \triangle^{-2}\, h_1(t) = \frac{t^{\underline{4}}}{4!} + \frac{t^{\underline{5}}}{5!}, \\ h_3(t) = \triangle^{-2}\, h_2(t) = \frac{t^{\underline{6}}}{6!} + \frac{t^{\underline{7}}}{7!}, & h_4(t) = \triangle^{-2}\, h_3(t) = \frac{t^{\underline{8}}}{8!} + \frac{t^{\underline{9}}}{9!}, \\ \vdots & \end{array}$$

then, yields

$$\begin{aligned} h(t) &= \sum_{i=0}^{\infty} h_i(t) = h_0(t) + h_1(t) + h_2(t) + h_3(t) + h_4(t) + \cdots \\ &= \left(1 + t^{\underline{1}}\right) + \left(\frac{t^{\underline{2}}}{2!} + \frac{t^{\underline{3}}}{3!}\right) + \left(\frac{t^{\underline{4}}}{4!} + \frac{t^{\underline{5}}}{5!}\right) + \left(\frac{t^{\underline{6}}}{6!} + \frac{t^{\underline{7}}}{7!}\right) + \left(\frac{t^{\underline{8}}}{8!} + \frac{t^{\underline{9}}}{9!}\right) + \cdots \\ &= 2^t. \end{aligned}$$

Example 4.4. Consider the following CDP

$$\begin{cases} \triangle^2 h(t) - h(t) = 3^{t+1}, \\ h(0) = 1, \quad \triangle\, h(0) = 2. \end{cases}$$

By applying ADM and the similar process, gets

$$\begin{cases} h_0(t) = \frac{3}{4}.3^t + \frac{1}{2} t^{\underline{1}} + \frac{1}{4}, \\ h_{n+1}(t) = \triangle^{-2}\, h_n(t), \end{cases}$$

therefore

$$
\begin{aligned}
h_1(t) =& \triangle^{-2}\, h_0(t) = \frac{3}{4^2}3^t + \frac{1}{2}\frac{t^{\underline{3}}}{3!} + \frac{1}{4}\frac{t^{\underline{2}}}{2!} - \frac{3}{2^3}t^{\underline{1}} - \frac{3}{4^2},\\
h_2(t) =& \triangle^{-2}\, h_1(t) = \frac{3}{4^3}3^t + \frac{1}{2}\frac{t^{\underline{5}}}{5!} + \frac{1}{4}\frac{t^{\underline{4}}}{4!} - \frac{3}{2^3}\frac{t^{\underline{3}}}{3!} - \frac{3}{4^2}\frac{t^{\underline{2}}}{2!} - \frac{3}{2^5}t^{\underline{1}} - \frac{3}{4^3},\\
h_3(t) =& \triangle^{-2}\, h_2(t) = \frac{3}{4^4}3^t + \frac{1}{2}\frac{t^{\underline{7}}}{7!} + \frac{1}{4}\frac{t^{\underline{6}}}{6!} - \frac{3}{2^3}\frac{t^{\underline{5}}}{5!} - \frac{3}{4^2}\frac{t^{\underline{4}}}{4!} - \frac{3}{2^5}\frac{t^{\underline{3}}}{3!} - \frac{3}{4^3}\frac{t^{\underline{2}}}{2!} - \frac{3}{2^7}t^{\underline{1}} - \frac{3}{4^4},\\
\vdots&
\end{aligned}
$$

then, yields

$$
\begin{aligned}
h(t) =& \sum_{i=0}^{\infty} h_i(t) = h_0(t) + h_1(t) + h_2(t) + h_3(t) + \cdots\\
=& \left(\frac{3}{4}.3^t + \frac{1}{2}t^{\underline{1}} + \frac{1}{4}\right) + \left(\frac{3}{4^2}3^t + \frac{1}{2}\frac{t^{\underline{3}}}{3!} + \frac{1}{4}\frac{t^{\underline{2}}}{2!} - \frac{3}{2^3}t^{\underline{1}} - \frac{3}{4^2}\right)\\
&+ \left(\frac{3}{4^3}3^t + \frac{1}{2}\frac{t^{\underline{5}}}{5!} + \frac{1}{4}\frac{t^{\underline{4}}}{4!} - \frac{3}{2^3}\frac{t^{\underline{3}}}{3!} - \frac{3}{4^2}\frac{t^{\underline{2}}}{2!} - \frac{3}{2^5}t^{\underline{1}} - \frac{3}{4^3}\right)\\
&+ \left(\frac{3}{4^4}3^t + \frac{1}{2}\frac{t^{\underline{7}}}{7!} + \frac{1}{4}\frac{t^{\underline{6}}}{6!} - \frac{3}{2^3}\frac{t^{\underline{5}}}{5!} - \frac{3}{4^2}\frac{t^{\underline{4}}}{4!} - \frac{3}{2^5}\frac{t^{\underline{3}}}{3!} - \frac{3}{4^3}\frac{t^{\underline{2}}}{2!} - \frac{3}{2^7}t^{\underline{1}} - \frac{3}{4^4}\right) + \cdots,
\end{aligned}
$$

then,

$$
\begin{aligned}
h(t) =& \frac{3}{4}.3^t \underbrace{\left(1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots\right)}_{\frac{4}{3}} + \frac{1}{4}\left(1 + \frac{t^{\underline{2}}}{2!} + \frac{t^{\underline{4}}}{4!} + \cdots\right)\\
&+ \frac{1}{2}\left(\frac{t^{\underline{1}}}{1!} + \frac{t^{\underline{3}}}{3!} + \frac{t^{\underline{5}}}{5!} + \cdots\right) - \frac{3}{2^3}\left(\frac{t^{\underline{1}}}{1!} + \frac{t^{\underline{3}}}{3!} + \frac{t^{\underline{5}}}{5!} + \cdots\right)\\
&- \frac{3}{4^2}\left(1 + \frac{t^{\underline{2}}}{2!} + \frac{t^{\underline{4}}}{4!} + \cdots\right) - \frac{3}{2^5}\left(\frac{t^{\underline{1}}}{1!} + \frac{t^{\underline{3}}}{3!} + \frac{t^{\underline{5}}}{5!} + \cdots\right) + \cdots\\
=& 3^t.
\end{aligned}
$$

Example 4.5. Given the following CDE,

$$
\begin{cases}
\triangle_t\, h(x,t) = \frac{1}{2}\, \triangle_x^2\, h(x,t) + h(x,t),\\
h(x,0) = x,
\end{cases}
$$

its exact solution is $h(x,t) = x.2^t$. To access the solution, we must apply operator $\triangle_t^{-1}$ on the above equation, this yields,

$$h(x,t) = h(x,0) + \triangle_t^{-1}\left(\frac{1}{2}\triangle_x^2 h(x,t) + h(x,t)\right). \tag{5}$$

Therefore the equation (5) can be rewritten as

$$h(x,t) = h(x,0) + \triangle_t^{-1}\left(\frac{1}{2}h(x+2,t) - h(x+1,t) + \frac{3}{2}h(x,t)\right).$$

Now, let $h(x,t) = h_{x,t} = \sum_{n=0}^{\infty} h_{x_n,t}$ and by substituting in the above equation, we get

$$\sum_{n=0}^{\infty} h_{x_n,t} = h_{x_0,t} + \triangle_t^{-1}\left(\frac{1}{2}\sum_{n=0}^{\infty} h_{x_{n+2},t} - \sum_{n=0}^{\infty} h_{x_{n+1},t} + \frac{3}{2}\sum_{n=0}^{\infty} h_{x_n,t}\right).$$

Therefore, we infer that the first term and the recursive formula series are as,

$$\begin{cases} h_{x_0,t} = x, \\ h_{x_{n+1},t} = \triangle_t^{-1}\left(\frac{1}{2}\sum_{n=0}^{\infty} h_{x_{n+2},t} - \sum_{n=0}^{\infty} h_{x_{n+1},t} + \frac{3}{2}\sum_{n=0}^{\infty} h_{x_n,t}\right), \end{cases}$$

then, we get

$$h_{x_1,t} = \triangle_t^{-1}\left(\frac{1}{2}h_{x_0+2,t} - h_{x_0+1,t} + \frac{3}{2}h_{x_0,t}\right) = \triangle_t^{-1}\left(\frac{1}{2}(x+2) - (x+1) + \frac{3}{2}x\right) = x\frac{t^{\underline{1}}}{1!},$$

$$h_{x_2,t} = \triangle_t^{-1}\left(\frac{1}{2}h_{x_1+2,t} - h_{x_1+1,t} + \frac{3}{2}h_{x_1,t}\right) = \triangle_t^{-1}\left(\frac{1}{2}(x+2) - (x+1) + \frac{3}{2}x\right) = x\frac{t^{\underline{2}}}{2!},$$

$$h_{x_3,t} = \triangle_t^{-1}\left(\frac{1}{2}h_{x_2+2,t} - h_{x_2+1,t} + \frac{3}{2}h_{x_1,t}\right) = \triangle_t^{-1}\left(\frac{1}{2}(x+2) - (x+1) + \frac{3}{2}x\right) = x\frac{t^{\underline{3}}}{3!},$$

$$h_{x_4,t} = \triangle_t^{-1}\left(\frac{1}{2}h_{x_3+2,t} - h_{x_3+1,t} + \frac{3}{2}h_{x_3,t}\right) = \triangle_t^{-1}\left(\frac{1}{2}(x+2) - (x+1) + \frac{3}{2}x\right) = x\frac{t^{\underline{4}}}{4!},$$
$$\vdots$$

then, we can write

$$\begin{aligned} h(x,t) = h_{x,t} &= \sum_{n=0}^{\infty} h_{x_n,t} = h_{x_0,t} + h_{x_1,t} + h_{x_2,t} + h_{x_3,t} + h_{x_4,t} + \cdots \\ &= x + x\frac{t^{\underline{1}}}{1!} + x\frac{t^{\underline{2}}}{2!} + x\frac{t^{\underline{3}}}{3!} + x\frac{t^{\underline{4}}}{4!} + \cdots = x\left(1 + \frac{t^{\underline{1}}}{1!} + \frac{t^{\underline{2}}}{2!} + \frac{t^{\underline{3}}}{3!} + \frac{t^{\underline{4}}}{4!} + \cdots\right) = x.2^t. \end{aligned}$$

4.2 Test FDPs

Example 4.6. Consider the following FDP

$$\begin{cases} \triangle^{\varsigma} h(t) = h(t+\varsigma-1), \quad 0 < \varsigma \leq 1, \\ y(0) = a. \end{cases}$$

By applying operator $\triangle^{-\varsigma}$ on the above equation, this yields,

$$\begin{cases} h_0(t) = a, \\ h_{n+1}(t) = \triangle^{-\varsigma} h_n(t+\varsigma-1), \end{cases}$$

therefore

$$\begin{aligned} &h_1(t) = \triangle^{-\varsigma} h_0(t+\varsigma-1) = a\tfrac{(t+\varsigma-1)^{\underline{\varsigma}}}{\Gamma(\varsigma+1)}, \quad h_2(t) = \triangle^{-\varsigma} h_1(t+\varsigma-1) = a\tfrac{(t+2\varsigma-2)^{\underline{2\varsigma}}}{\Gamma(2\varsigma+1)}, \\ &h_3(t) = \triangle^{-\varsigma} h_2(t+\varsigma-1) = a\tfrac{(t+3\varsigma-3)^{\underline{3\varsigma}}}{\Gamma(3\varsigma+1)}, \quad h_4(t) = \triangle^{-\varsigma} h_3(t+\varsigma-1) = a\tfrac{(t+4\varsigma-4)^{\underline{4\varsigma}}}{\Gamma(4\varsigma+1)}, \\ &\vdots \end{aligned}$$

then, yields

$$\begin{aligned} h(t) &= \sum_{i=0}^{\infty} h_i(t) = h_0(t) + h_1(t) + h_2(t) + h_3(t) + \cdots \\ &= a + a\frac{(t+\varsigma-1)^{\underline{\varsigma}}}{\Gamma(\varsigma+1)} + a\frac{(t+2\varsigma-2)^{\underline{2\varsigma}}}{\Gamma(2\varsigma+1)} + a\frac{(t+3\varsigma-3)^{\underline{3\varsigma}}}{\Gamma(3\varsigma+1)} + a\frac{(t+4\varsigma-4)^{\underline{4\varsigma}}}{\Gamma(4\varsigma+1)} + \cdots \\ &= a\sum_{i=0}^{\infty} \frac{(t+i(\varsigma-1))^{\underline{i\varsigma}}}{\Gamma(i\varsigma+1)}. \end{aligned}$$

The approximation solutions $h(t)$ considering the first five and ten terms for different ς are shown in Figure 1. We can see the different behaviors of the discrete FDE with different fractional parameters. It is clear when ς is close to 1 the approximation solution tends to the exact solution.

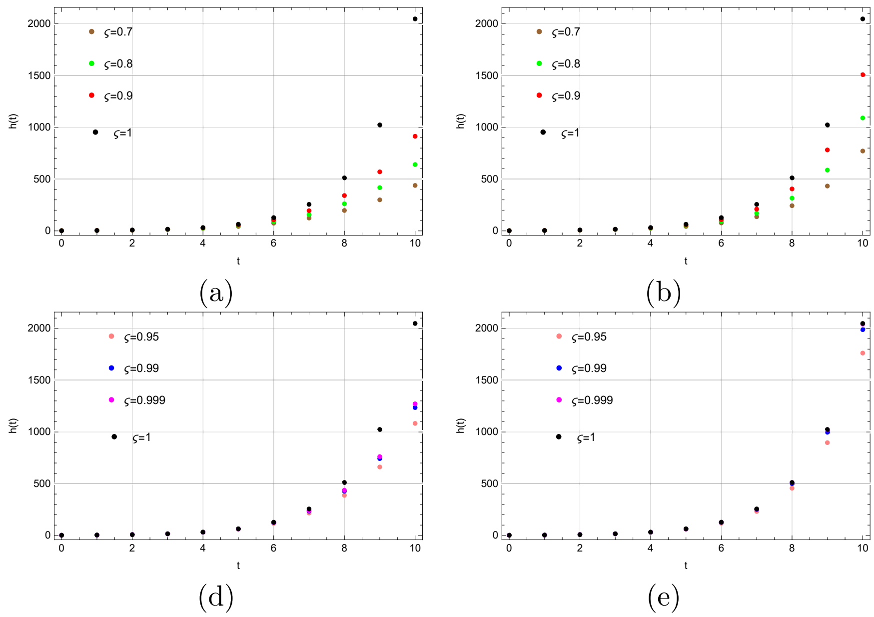

Figure 1: (Example 4.6) The approximation solution $h(x,t)$ with setting $a=2$ (a) The first five terms (b) The first ten terms (c) The first five terms (d) The first ten terms.

In this example, let $\varsigma = 1$, then

$$\begin{cases} h_0(t) = a, \\ h_{n+1}(t) = \Delta^{-1}\, h_n(t), \end{cases}$$

therefore

$$\begin{array}{ll} h_1(t) = a\frac{t^{1}}{1!}, & h_2(t) = a\frac{t^{2}}{2!}, \\ h_3(t) = a\frac{t^{3}}{3!}, & h_4(t) = a\frac{t^{4}}{4!}, \\ \vdots & \end{array}$$

then, yields

$$h(t) = \sum_{i=0}^{\infty} h_i(t) = a.2^t.$$

Example 4.7. Given the following FDP,

$$\begin{cases} \triangle_t^{\varsigma} \, h(x,t) = \triangle_x^2 \, h(x,t), & 0 < \varsigma \leq 1, \\ h(x,0) = 2^x, \end{cases} \tag{6}$$

its exact solution is $h(x,t) = 2^{x+t}$ when $\varsigma = 1$. To find the solution, we apply operator ${}_{1-\varsigma}\triangle_t^{-\varsigma}$ on both sides of the above equation, this yields,

$$h(x,t) = h(x,0) + {}_{1-\varsigma}\triangle_t^{-\varsigma} \left(\triangle_x^2 \, h(x,t) \right). \tag{7}$$

Therefore the equation (7) can be rewritten as

$$h(x,t) = h(x,0) + {}_{1-\varsigma}\triangle_t^{-\varsigma} \left(h(x+2,t) - 2h(x+1,t) + h(x,t) \right).$$

Now, let $h(x,t) = h_{x,t} = \sum_{n=0}^{\infty} h_{x_n,t}$ and by substituting in the above equation, we get

$$\sum_{n=0}^{\infty} h_{x_n,t} = h_{x_0,t} + {}_{1-\varsigma}\triangle_t^{-\varsigma} \left(\sum_{n=0}^{\infty} h_{x_{n+2},t} - 2 \sum_{n=0}^{\infty} w_{x_{n+1},t} + \sum_{n=0}^{\infty} w_{x_n,t} \right).$$

Therefore, we infer that the first term and the recursive formula series are as,

$$\begin{cases} h_{x_0,t} = 2^x, \\ h_{x_{n+1},t} = {}_{1-\varsigma}\triangle_t^{-\varsigma} \left(\sum_{n=0}^{\infty} h_{x_{n+2},t} - 2 \sum_{n=0}^{\infty} h_{x_{n+1},t} + \sum_{n=0}^{\infty} h_{x_n,t} \right), \end{cases}$$

then, we get

$$h_{x_1,t} = {}_{1-\varsigma}\triangle_t^{-\varsigma} \left(h_{x_0+2,t} - 2h_{x_0+1,t} + h_{x_0,t} \right) = {}_{1-\varsigma}\triangle_t^{-\varsigma} \left(2^{x+2} - 2.2^{x+1} + 2^x \right) = 2^x \frac{(t+\varsigma-1)^{\underline{\varsigma}}}{\Gamma(\varsigma+1)},$$

$$h_{x_2,t} = {}_{1-\varsigma}\Delta_t^{-\varsigma}\left(h_{x_1+2,t} - 2h_{x_1+1,t} + h_{x_1,t}\right) = 2^x\left({}_{1-\varsigma}\Delta_t^{-\varsigma}\frac{(t+\varsigma-1)^{\underline{\varsigma}}}{\Gamma(\varsigma+1)}\right) = 2^x\frac{(t+2\varsigma-2)^{\underline{2\varsigma}}}{\Gamma(2\varsigma+1)},$$

$$h_{x_3,t} = {}_{1-\varsigma}\Delta_t^{-\varsigma}\left(h_{x_2+2,t} - 2h_{x_2+1,t} + h_{x_2,t}\right) = 2^x\left({}_{1-\varsigma}\Delta_t^{-\varsigma}\frac{(t+2\varsigma-2)^{\underline{2\varsigma}}}{\Gamma(2\varsigma+1)}\right) = 2^x\frac{(t+3\varsigma-3)^{\underline{3\varsigma}}}{\Gamma(3\varsigma+1)},$$

$$h_{x_4,t} = {}_{1-\varsigma}\Delta_t^{-\varsigma}\left(h_{x_3+2,t} - 2h_{x_3+1,t} + h_{x_3,t}\right) = 2^x\left({}_{1-\varsigma}\Delta_t^{-\varsigma}\frac{(t+3\varsigma-3)^{\underline{3\varsigma}}}{\Gamma(3\varsigma+1)}\right) = 2^x\frac{(t+4\varsigma-4)^{\underline{4\varsigma}}}{\Gamma(4\varsigma+1)},$$

$$\vdots$$

then, we can write

$$\begin{aligned}
h(x,t) = h_{x,t} &= \sum_{n=0}^{\infty} h_{x_n,t} = h_{x_0,t} + h_{x_1,t} + h_{x_2,t} + h_{x_3,t} + h_{x_4,t} + \cdots \\
&= 2^x + 2^x\frac{(t+\varsigma-1)^{\underline{\varsigma}}}{\Gamma(\varsigma+1)} + 2^x\frac{(t+2\varsigma-2)^{\underline{2\varsigma}}}{\Gamma(2\varsigma+1)} + 2^x\frac{(t+3\varsigma-3)^{\underline{3\varsigma}}}{\Gamma(3\varsigma+1)} + 2^x\frac{(t+4\varsigma-4)^{\underline{4\varsigma}}}{\Gamma(4\varsigma+1)} + \cdots \\
&= 2^x\left(1 + \frac{(t+\varsigma-1)^{\underline{\varsigma}}}{\Gamma(\varsigma+1)} + \frac{(t+2\varsigma-2)^{\underline{2\varsigma}}}{\Gamma(2\varsigma+1)} + \frac{(t+3\varsigma-3)^{\underline{3\varsigma}}}{\Gamma(3\varsigma+1)} + \frac{(t+4\varsigma-4)^{\underline{4\varsigma}}}{\Gamma(4\varsigma+1)} + \cdots\right) \\
&= 2^x\sum_{n=0}^{\infty}\frac{(t+n(\varsigma-1))^{\underline{n\varsigma}}}{\Gamma(n\varsigma+1)}.
\end{aligned}$$

When $\varsigma = 1$, then

$$h(x,t) = 2^{x+t}.$$

Figures 2 and 3 show the approximation solutions $h(x,t)$ considering the first five and ten terms for different values of ς. We can see the different behaviors of the discrete FDE with different fractional parameters. It is clear when ς tends to 1 the approximation solution tends to the exact solution.

Example 4.8. Given the following FDE,

$$\begin{cases} \Delta_t^{\varsigma}\, h(x,t) = \frac{1}{2}\,\Delta_x^2\, h(x,t) + w(x,t), & 0 < \varsigma \le 1, \\ h(x,0) = x, \end{cases}$$

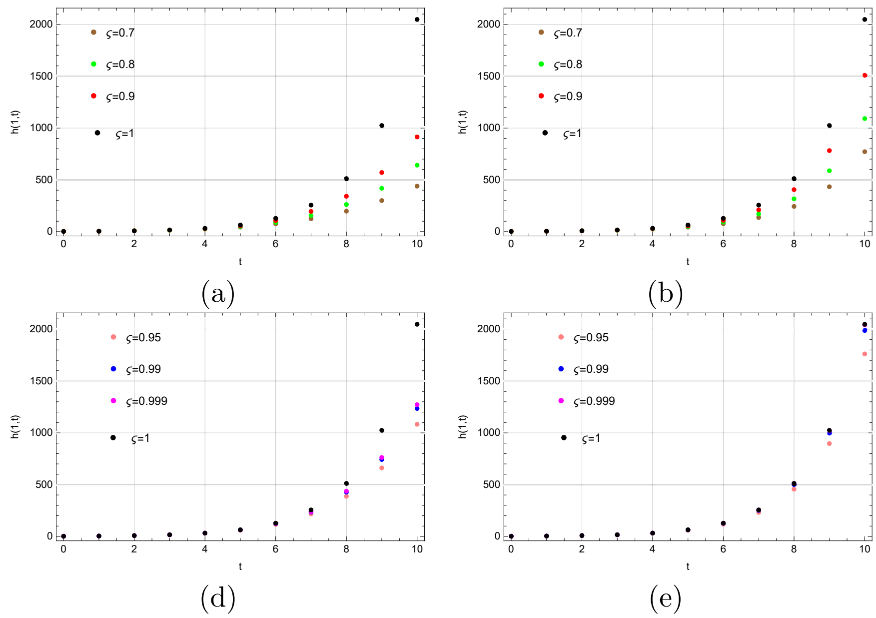

Figure 2: (Example 4.7) The approximation solution $h(x,t)$ when $x=1$ and considering (a) The first five terms (b) The first ten terms (c) The first five terms (d) The first ten terms.

its exact solution is $h(x,t) = x\, 2^t$ when $\varsigma = 1$. To access the solution, we must apply operator ${}_{1-\varsigma}\Delta_t^{-\varsigma}$ on the above equation, this yields,

$$h(x,t) = h(x,0) + {}_{1-\varsigma}\Delta_t^{-\varsigma}\left(\frac{1}{2}\,\Delta_x^2\, h(x,t) + h(x,t)\right). \tag{8}$$

Therefore the equation (8) can be rewritten as

$$h(x,t) = hw(x,0) + {}_{1-\varsigma}\Delta_t^{-\varsigma}\left(\frac{1}{2}h(x+2,t) - h(x+1,t) + \frac{3}{2}h(x,t)\right).$$

Now, let $h(x,t) = h_{x,t} = \displaystyle\sum_{n=0}^{\infty} h_{x_n,t}$ and by substituting in the above equation, we get

$$\sum_{n=0}^{\infty} hw_{x_n,t} = h_{x_0,t} + {}_{1-\varsigma}\Delta_t^{-\varsigma}\left(\frac{1}{2}\sum_{n=0}^{\infty} h_{x_{n+2},t} - \sum_{n=0}^{\infty} h_{x_{n+1},t} + \frac{3}{2}\sum_{n=0}^{\infty} h_{x_n,t}\right).$$

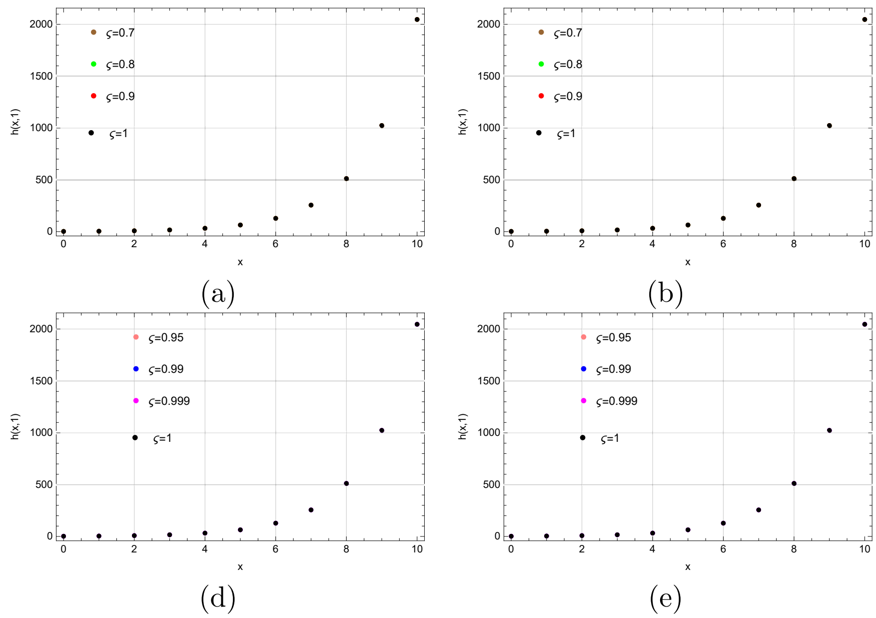

Figure 3: (Example 4.7) The approximation solution $h(x,t)$ when $t=1$ and considering (a) The first five terms (b) The first ten terms (c) The first five sentences (d) The first ten terms.

Therefore, we infer that the first term and the recursive formula series are as,

$$\begin{cases} h_{x_0,t} = x, \\ h_{x_{n+1},t} = {}_{1-\varsigma}\Delta_t^{-\varsigma}\left(\frac{1}{2}\sum_{n=0}^{\infty} h_{x_{n+2},t} - \sum_{n=0}^{\infty} h_{x_{n+1},t} + \frac{3}{2}\sum_{n=0}^{\infty} h_{x_n,t}\right), \end{cases}$$

then, we get

$$\begin{aligned}
h_{x_1,t} &= {}_{1-\varsigma}\Delta_t^{-\varsigma}\left(\frac{1}{2}h_{x_0+2,t} - h_{x_0+1,t} + \frac{3}{2}h_{x_0,t}\right) = {}_{1-\varsigma}\Delta_t^{-\varsigma}\left(\frac{1}{2}(x+2) - (x+1) + \frac{3}{2}x\right)\\
&= x\frac{(t+\varsigma-1)^{\underline{\varsigma}}}{\Gamma(\varsigma+1)},\\
h_{x_2,t} &= {}_{1-\varsigma}\Delta_t^{-\varsigma}\left(\frac{1}{2}h_{x_1+2,t} - h_{x_1+1,t} + \frac{3}{2}h_{x_1,t}\right) = x\left({}_{1-\varsigma}\Delta_t^{-\varsigma}\frac{(t+\varsigma-1)^{\underline{\varsigma}}}{\Gamma(\varsigma+1)}\right)\\
&= x\frac{(t+2\varsigma-2)^{\underline{2\varsigma}}}{\Gamma(2\varsigma+1)},\\
h_{x_3,t} &= {}_{1-\varsigma}\Delta_t^{-\varsigma}\left(\frac{1}{2}h_{x_2+2,t} - h_{x_2+1,t} + \frac{3}{2}h_{x_2,t}\right) = x\left({}_{1-\varsigma}\Delta_t^{-\varsigma}\frac{(t+2\varsigma-2)^{\underline{2\varsigma}}}{\Gamma(2\varsigma+1)}\right)\\
&= x\frac{(t+3\varsigma-3)^{\underline{3\varsigma}}}{\Gamma(3\varsigma+1)},\\
h_{x_4,t} &= {}_{1-\varsigma}\Delta_t^{-\varsigma}\left(\frac{1}{2}h_{x_3+2,t} - h_{x_3+1,t} + \frac{3}{2}h_{x_3,t}\right) = x\left({}_{1-\varsigma}\Delta_t^{-\varsigma}\frac{(t+3\varsigma-3)^{\underline{3\varsigma}}}{\Gamma(3\varsigma+1)}\right)\\
&= x\frac{(t+4\varsigma-4)^{\underline{4\varsigma}}}{\Gamma(4\varsigma+1)},\\
&\vdots
\end{aligned}$$

then, we can write

$$\begin{aligned}
h(x,t) = h_{x,t} &= \sum_{n=0}^{\infty} h_{x_n,t} = h_{x_0,t} + h_{x_1,t} + h_{x_2,t} + h_{x_3,t} + h_{x_4,t} + \cdots\\
&= x + x\frac{(t+\varsigma-1)^{\underline{\varsigma}}}{\Gamma(\varsigma+1)} + x\frac{(t+2\varsigma-2)^{\underline{2\varsigma}}}{\Gamma(2\varsigma+1)} + x\frac{(t+3\varsigma-3)^{\underline{3\varsigma}}}{\Gamma(3\varsigma+1)} + x\frac{(t+4\varsigma-4)^{\underline{4\varsigma}}}{\Gamma(4\varsigma+1)} + \cdots
\end{aligned}$$

$$\begin{aligned}
&= x\left(1 + \frac{(t+\varsigma-1)^{\underline{\varsigma}}}{\Gamma(\varsigma+1)} + \frac{(t+2\varsigma-2)^{\underline{2\varsigma}}}{\Gamma(2\varsigma+1)} + \frac{(t+3\varsigma-3)^{\underline{3\varsigma}}}{\Gamma(3\varsigma+1)} + \frac{(t+4\varsigma-4)^{\underline{4\varsigma}}}{\Gamma(4\varsigma+1)} + \cdots\right)\\
&= x\sum_{n=0}^{\infty}\frac{(t+n(\varsigma-1))^{\underline{n\varsigma}}}{\Gamma(n\varsigma+1)}.
\end{aligned}$$

When $\varsigma = 1$, then

$$h(x,t) = x\,2^t.$$

We set the first five and ten terms of $h(x,t)$, then the numerical results are plotted in Figures 4 and 5. The different behaviors of the discrete FDE with different fractional parameters observe in this Figure. Also, in this Figure, we can see the approximation solution tends to the exact solution when ς is close to 1.

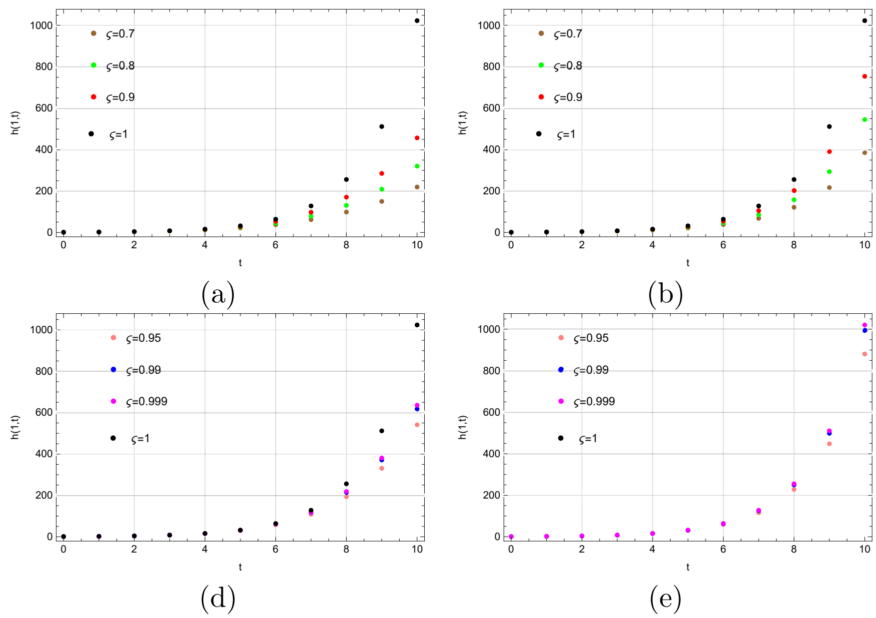

Figure 4: (Example 4.8) The approximation solution $h(x,t)$ when $x = 1$ and considering (a) The first five terms (b) The first ten terms (c) The first five terms (d) The first ten terms.

5 Conclusion

This paper was based on a new version of the Adomian decomposition method (ADM), which is called the discrete ADM. This technique helps us to obtain a recursive formulation. Using this recursive formulation, we can achieve the solutions of linear and nonlinear classic and fractional difference problems (CDPs and FDPs). Finally, several practice problems are solved to show the accuracy of the discrete ADM approach.

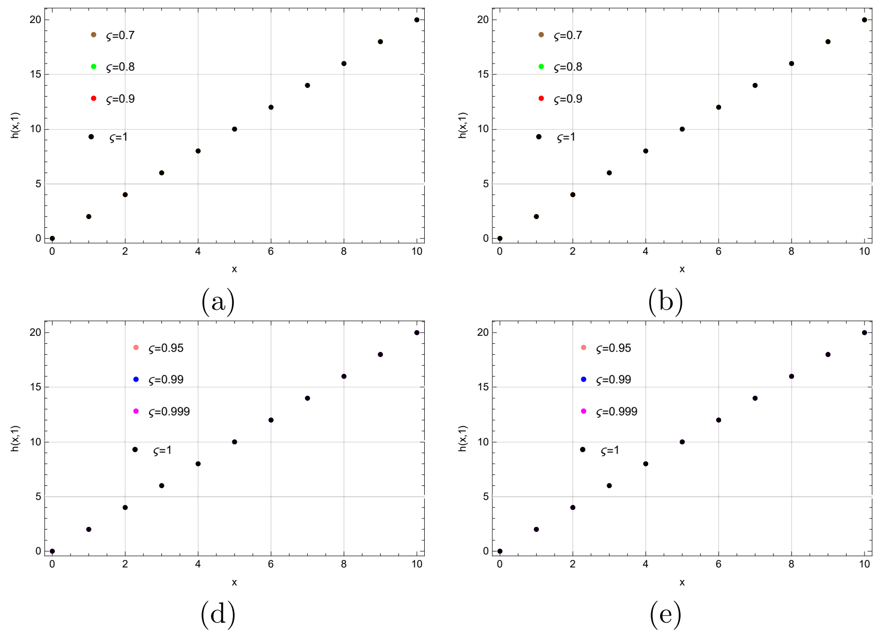

Figure 5: (Example 4.8) The approximation solution $h(x,t)$ when $t=1$ and considering (a) The first five terms (b) The first ten terms (c) The first five terms (d) The first ten terms.

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