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Original Research Paper

Advances in the Continuous Dual Hahn Polynomials

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Abstract.
In this paper, we investigate some advanced properties of the continuous dual Hahn polynomials which are orthogonal in a single variable. In particular, we derive various families of bilinear and bilateral generating functions from them. We also obtain recurrence relations for these polynomials with the help of three-term contiguous relations of classical hypergeometric series. Furthermore, we give some integral representations.

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1 Introduction

The continuous dual Hahn (CDH) polynomials, which generalize the Jacobi polynomials, are a family of hypergeometric orthogonal polynomials. We know from [2, 18] that the CDH polynomials reduce to the

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Jacobi polynomials with the help of some limiting procedure. So far, the CDH polynomials have been studied by many authors (see, for instance, [7, 10, 12, 3, 4]). They have also appeared in Stieltjes's (see [16]).

In the present paper, we investigate some advanced properties of these polynomials. We first derive various families of bilinear and bilateral generating functions from them. Then, we obtain recurrence relations for the CDH polynomials with the help of three-term contiguous relations of classical hypergeometric series studied by Wilson [17] and also Chu and Wang [6] (see also [15]). Furthermore, we give some integral representations in the interval $(0,1)$, $(0,\infty)$ and a triple integral representation for the CDH polynomials.

Before proceeding further, we first recall the definition of the CDH polynomials and their generating function relations.

The CDH polynomials is given by (see [11, 18])

$$S_n(x^2;\alpha,\beta,\gamma) = (\alpha+\beta)_n(\alpha+\gamma)_n \; {}_3F_2\left[-n,\alpha+ix,\alpha-ix;\alpha+\beta,\alpha+\gamma;1\right], \tag{1}$$

where ${}_3F_2$ denotes the corresponding generalized hypergeometric series. Recall that, in general, ${}_rF_s$ $(r,s\in\mathbb{N}_0=\mathbb{N}\cup\{0\})$ is defined by

$$\begin{aligned} {}_rF_s\left[\begin{array}{cc} \alpha_1,\ldots,\alpha_r; & \\ & z \\ \beta_1,\ldots,\beta_s; & \end{array}\right] &:= \sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_r)_n}{(\beta_1)_n\cdots(\beta_s)_n}\frac{z^n}{n!} \\ &= {}_rF_s\left(\alpha_1,\ldots,\alpha_r;\beta_1,\ldots,\beta_s;z\right). \end{aligned}$$

As usual, the Pochammer symbol $(\mu)_s$ is given by

$$\begin{aligned} (\mu)_s &:= \frac{\Gamma(\mu+s)}{\Gamma(\mu)} \quad (\mu\in\mathbb{C}\setminus\mathbb{Z}_0^-) \\ &= \begin{cases} 1, & \text{if } \nu=0;\ \mu\in\mathbb{C}\backslash\{0\} \\ \mu(\mu+1)...(\mu+n-1), & \text{if } s=n\in\mathbb{N};\ \mu\in\mathbb{C} \end{cases} \end{aligned}$$

provided that the Gamma quotient exists. It is assumed that $(0)_0:=1$. The CDH polynomials have the following generating function relations (see [11]):

$$(1-t)^{-\gamma+ix}\ {}_2F_1\left[\alpha+ix,\beta+ix;\alpha+\beta;t\right]=\sum_{n=0}^{\infty}\frac{S_n(x^2;\alpha,\beta,\gamma)}{(\alpha+\beta)_n n!}t^n, \tag{2}$$

$$(1-t)^{-\beta+ix}\ {}_2F_1\left[\alpha+ix,\gamma+ix;\alpha+\gamma;t\right]=\sum_{n=0}^{\infty}\frac{S_n(x^2;\alpha,\beta,\gamma)}{(\alpha+\gamma)_n n!}t^n, \tag{3}$$

$$(1-t)^{-\alpha+ix}\ {}_2F_1\left[\beta+ix,\gamma+ix;\beta+\gamma;t\right]=\sum_{n=0}^{\infty}\frac{S_n(x^2;\alpha,\beta,\gamma)}{(\beta+\gamma)_n n!}t^n, \tag{4}$$

$$e^t\ {}_2F_2\left[\alpha+ix,\alpha-ix;\alpha+\beta,\alpha+\gamma;-t\right]=\sum_{n=0}^{\infty}\frac{S_n(x^2;\alpha,\beta,\gamma)}{(\alpha+\beta)_n(\alpha+\gamma)_n n!}t^n. \tag{5}$$

2 Mixed Generating Functions

Here, we obtain several families of generating functions for the CDH polynomials $S_n(x^2;\alpha,\beta,\gamma)$ given by (1). We should note such an investigation has been considered for other polynomials, such as Gottlieb and Cesaro polynomials, in the recent papers [9, 13].

Throughout this section, let $p,s\in\mathbb{N}$; $\mu,\psi\in\mathbb{C}$; $\alpha_k\in\mathbb{C}\setminus\{0\}$ and

$$\Omega_\mu:\mathbb{C}^s\longrightarrow\mathbb{C}\setminus\{0\}$$

be a bounded function. We also need the following function:

$$\Lambda_{\mu,\psi}\left[\xi_1,...,\xi_s;\tau\right]:=\sum_{k=0}^{\infty}\alpha_k\Omega_{\mu+\psi k}(\xi_1,...,\xi_s)\tau^k. \tag{6}$$

So, we get the following theorems.

Theorem 2.1. *Let*

$$\begin{aligned}&U_{n,p}^{\mu,\psi}(x;\alpha,\beta,\gamma;\xi_1,...,\xi_s;z)\\&:=\sum_{k=0}^{[n/p]}\alpha_k\frac{S_{n-pk}(x^2;\alpha,\beta,\gamma)}{(\alpha+\beta)_{n-pk}(n-pk)!}\Omega_{\mu+\psi k}(\xi_1,...,\xi_s)z^k.\end{aligned} \tag{7}$$

Then, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} U_{n,p}^{\mu,\psi}\left(x;\alpha,\beta,\gamma;\xi_1,...,\xi_s;\frac{\eta}{t^p}\right)t^n \\
&= (1-t)^{-\gamma+ix}\ {}_2F_1\left[\alpha+ix,\beta+ix;\alpha+\beta;t\right] \\
&\times\Lambda_{\mu,\psi}\left[\xi_1,...,\xi_s;\eta\right],
\end{aligned} \tag{8}$$

where $\Lambda_{\mu,\psi}$ *is given by* (6).

Proof. Let A denote the left side of (8). Then, using (7), we immediately get

$$A=\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]}\alpha_k\frac{S_{n-pk}(x^2;\alpha,\beta,\gamma)}{(\alpha+\beta)_{n-pk}(n-pk)!}\Omega_{\mu+\psi k}(\xi_1,...,\xi_s)\eta^k t^{n-pk}.$$

Replacing n by $n+pk$ and considering (2), we observe that

$$\begin{aligned}
A &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\alpha_k\frac{S_n(x^2;\alpha,\beta,\gamma)}{(\alpha+\beta)_n n!}\Omega_{\mu+\psi k}(\xi_1,...,\xi_s)\eta^k t^n \\
&= \sum_{n=0}^{\infty}\frac{S_n(x^2;\alpha,\beta,\gamma)}{(\alpha+\beta)_n n!}t^n\sum_{k=0}^{\infty}\alpha_k\Omega_{\mu+\psi k}(\xi_1,...,\xi_s)\eta^k \\
&= (1-t)^{-\gamma+ix}\ {}_2F_1\left[\alpha+ix,\beta+ix;\alpha+\beta;t\right]\Lambda_{\mu,\psi}\left[\xi_1,...,\xi_s;\eta\right],
\end{aligned}$$

which is the right member of (8). □

Theorem 2.2. *Let*

$$\begin{aligned}
&V_{n,p}^{\mu,\psi}(x;\alpha,\beta,\gamma;\xi_1,...,\xi_s;z) \\
&:= \sum_{k=0}^{[n/p]}\alpha_k\frac{S_{n-pk}(x^2;\alpha,\beta,\gamma)}{(\alpha+\gamma)_{n-pk}(n-pk)!}\Omega_{\mu+\psi k}(\xi_1,...,\xi_s)z^k.
\end{aligned}$$

Then, we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} V_{n,p}^{\mu,\psi}\left(x;\alpha,\beta,\gamma;\xi_1,...,\xi_s;\frac{\eta}{t^p}\right)t^n \\
&= (1-t)^{-\beta+ix}\ {}_2F_1\left[\alpha+ix,\gamma+ix;\alpha+\gamma;t\right] \\
&\times\Lambda_{\mu,\psi}\left[\xi_1,...,\xi_s;\eta\right],
\end{aligned}$$

where $\Lambda_{\mu,\psi}$ *is given by* (6).

Proof. Applying a similar method in the proof of Theorem 2.1 and also using relation (3), we easily obtain the desired result. □

Theorem 2.3. *Let*

$$\begin{aligned}&W_{n,p}^{\mu,\psi}(x;\alpha,\beta,\gamma;\xi_1,...,\xi_s;z)\\&:=\sum_{k=0}^{[n/p]}\alpha_k\frac{S_{n-pk}(x^2;\alpha,\beta,\gamma)}{(\beta+\gamma)_{n-pk}(n-pk)!}\Omega_{\mu+\psi k}(\xi_1,...,\xi_s)z^k.\end{aligned}$$

Then, we have

$$\begin{aligned}&\sum_{n=0}^{\infty}W_{n,p}^{\mu,\psi}\left(x;\alpha,\beta,\gamma;\xi_1,...,\xi_s;\frac{\eta}{t^p}\right)t^n\\&=(1-t)^{-\alpha+ix}\ {}_2F_1\left[\beta+ix,\gamma+ix;\beta+\gamma;t\right]\\&\times\Lambda_{\mu,\psi}\left[\xi_1,...,\xi_s;\eta\right],\end{aligned}$$

where $\Lambda_{\mu,\psi}$ *is given by* (6).

Proof. Applying a similar method in the proof of Theorem 2.1 and also using relation (4), the proof follows immediately. □

Theorem 2.4. *Let*

$$\begin{aligned}&\theta_{n,p}^{\mu,\psi}(x;\alpha,\beta,\gamma;\xi_1,...,\xi_s;z)\\&:=\sum_{k=0}^{[n/p]}\alpha_k\frac{S_{n-pk}(x^2;\alpha,\beta,\gamma)}{(\alpha+\beta)_{n-pk}(\alpha+\gamma)_{n-pk}(n-pk)!}\Omega_{\mu+\psi k}(\xi_1,...,\xi_s)z^k.\end{aligned}$$

Then, we have

$$\begin{aligned}&\sum_{n=0}^{\infty}\theta_{n,p}^{\mu,\psi}\left(x;\alpha,\beta,\gamma;\xi_1,...,\xi_s;\frac{\eta}{t^p}\right)t^n\\&=e^t\ {}_2F_2\left[\alpha+ix,\alpha-ix;\alpha+\beta,\alpha+\gamma;-t\right]\\&\times\Lambda_{\mu,\psi}\left[\xi_1,...,\xi_s;\eta\right],\end{aligned}$$

where $\Lambda_{\mu,\psi}$ *is given by* (6).

Proof. Again using the idea as in the proof of Theorem 2.1 and considering relation (5), the proof is easily obtained. □

Since the multivariable function $\Omega_{\mu+\psi k}$ is quite general, we may derive some applications of our results by choosing appropriate functions $\Omega_{\mu+\psi k}(\xi_1, ..., \xi_s)$, $k \in \mathbb{N}_0$, $s \in \mathbb{N}$. We now introduce some of them.

Example 2.5. Setting $s = 1$, $\alpha_k = \frac{1}{(\alpha+\beta)_k(\alpha+\gamma)_k k!}$, $\mu = 0$, $\psi = 1$ and taking the CDH polynomials instead of the function $\Omega_{\mu+\psi k}$ in Theorem 2.1, and also using (5), we get the following class of bilinear generating functions for the CDH polynomials:

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} \frac{S_{n-pk}(x^2;\alpha,\beta,\gamma)S_k(y^2;\alpha,\beta,\gamma)}{(\alpha+\beta)_k(\alpha+\gamma)_k(\alpha+\beta)_{n-pk}(n-pk)!k!}\eta^k t^{n-pk}\\ &= e^{\eta}(1-t)^{-\gamma+ix}{}_2F_1\left[\alpha+ix,\beta+ix;\alpha+\beta;t\right]\\ &\times {}_2F_2\left[\alpha+iy,\alpha-iy;\alpha+\beta,\alpha+\gamma;-\eta\right].\end{aligned}$$

Example 2.6. Recall that the Lagrange polynomials $g_n^{(\alpha,\beta)}(y,z)$ are generated by (see [8])

$$(1-yt)^{-\alpha}(1-zt)^{-\beta} = \sum_{n=0}^{\infty} g_n^{(\alpha,\beta)}(y,z)\,t^n, \tag{9}$$

where $|t| < \min\left\{|y|^{-1}, |z|^{-1}\right\}$. If we take $s = 2$, $\alpha_k = 1$, $\mu = 0$, $\psi = 1$ and replace the function $\Omega_{\mu+\psi k}$ in Theorem 2.2 with the Lagrange polynomials, then, using the relation (9) and Theorem 2.2, we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} \frac{S_{n-pk}(x^2;\alpha,\beta,\gamma)g_n^{(\alpha,\beta)}(y,z)}{(\alpha+\gamma)_{n-pk}(n-pk)!}\eta^k t^{n-pk}\\ &= (1-t)^{-\beta+ix}(1-y\eta)^{-\alpha}(1-z\eta)^{-\beta}\ {}_2F_1\left[\alpha+ix,\gamma+ix;\alpha+\gamma;t\right],\end{aligned}$$

which is a class of bilateral generating functions for the CDH polynomials and the Lagrange polynomials.

Before closing this section, we should note that if $\Omega_{\mu+\psi k}$ is given as a product of some simpler functions, then by using suitable coefficients α_k, our Theorems 2.1−2.4 contain many families of generating functions (in the sense of multilinear and multilateral) for the CDH polynomials (1).

3 Recurrence Relations

In 2007, by using the modified Abel Lemma Chu and Wang showed the forms A, B, C and D of contiguous relations satisfied by the ${}_3F_2\,(1)$-series (see [6]). Furthermore, in [17] Wilson obtained three-term contiguous relations for some orthogonal polynomials. In this section, using them, we derive five recurrence relations for the CDH polynomials $S_n(x^2;\alpha,\beta,\gamma)$.

We now consider the conditions $\mathrm{Re}(\beta+\mu-\alpha-\gamma-\eta)>1$ (in Part $1-3$) and $\mathrm{Re}(\mu+\eta-\alpha-\beta-\gamma)>0$ (in Part 4) for the series

$$ {}_3F_2\left[\begin{matrix}\alpha,\gamma,\eta;\\ \beta,\mu;\end{matrix}\ 1\right]. $$

Furthermore, we need the conditions $\mathrm{Re}(\beta+\gamma)+n-1>0$ (in Part $1-3$) and $\mathrm{Re}(\beta+\gamma)+n>0$ (in Part 4) for the parameters of all recurrence relations for the CDH polynomials.

Part 1. For five complex numbers α,β,γ,μ and η, there holds the three-term relation [6]

$$ \begin{aligned} &{}_3F_2\left[\begin{matrix}\alpha,\gamma,\eta;\\ \beta,\mu;\end{matrix}\ 1\right]=A\ {}_3F_2\left[\begin{matrix}\alpha+1,\gamma,\eta;\\ \beta+1,\mu;\end{matrix}\ 1\right]\\ &+B\ {}_3F_2\left[\begin{matrix}\alpha+2,\gamma+1,\eta+1;\\ \beta+2,\mu+1;\end{matrix}\ 1\right], \end{aligned} \tag{10} $$

where the coefficients A and B are defined by

$$ \begin{aligned} A&=\frac{(1+\alpha-\mu)\beta+\gamma\eta}{(1+\alpha-\mu)\beta},\\ B&=\frac{(1+\alpha+\gamma+\eta-\beta-\mu)(1+\alpha)\gamma\eta}{(1+\alpha-\mu)(1+\beta)\beta\mu}. \end{aligned} $$

Then, one can get the following theorem.

Theorem 3.1. *The CDH polynomials have the following recurrence relation for $n\geq 2$:*

$$ \begin{aligned} &S_n(x^2;\alpha,\beta,\gamma)\\ &=\left[(\alpha+\gamma+n-1)(\alpha+\beta)-(\alpha^2+x^2)\right]S_{n-1}(x^2;\alpha,\beta+1,\gamma)\\ &+\left[(\beta+\gamma+n-1)(1-n)(\alpha^2+x^2)\right]S_{n-2}(x^2;\alpha+1,\beta+1,\gamma). \end{aligned} $$

Proof. If we take $\alpha \to -n$, $\beta \to \alpha+\beta$, $\gamma \to \alpha+ix$, $\mu \to \alpha+\gamma$ and $\eta \to \alpha-ix$ in (10), we have

$$
{}_3F_2\left[\begin{matrix} -n, \alpha+ix, \alpha-ix; \\ \alpha+\beta, \alpha+\gamma; \end{matrix}\ 1\right] = A\ {}_3F_2\left[\begin{matrix} -n+1, \alpha+ix, \alpha-ix; \\ \alpha+\beta+1, \alpha+\gamma; \end{matrix}\ 1\right] + B\ {}_3F_2\left[\begin{matrix} -n+2, \alpha+ix+1, \alpha-ix+1; \\ \alpha+\beta+2, \alpha+\gamma+1; \end{matrix}\ 1\right],
$$

where

$$
A = \frac{(1-n-\alpha-\gamma)(\alpha+\beta)+(\alpha^2+x^2)}{(1-n-\alpha-\gamma)(\alpha+\beta)},
$$
$$
B = \frac{(1-n-\beta-\gamma)(1-n)(\alpha^2+x^2)}{(1-n-\alpha-\gamma)(1+\alpha+\beta)(\alpha+\beta)(\alpha+\gamma)}.
$$

Using the definition of the CDH polynomials given by (1), we obtain

$$
\begin{aligned}
&\frac{S_n(x^2;\alpha,\beta,\gamma)}{(\alpha+\beta)_n(\alpha+\gamma)_n} \\
&= \left[\frac{(1-n-\alpha-\gamma)(\alpha+\beta)+(\alpha^2+x^2)}{(1-n-\alpha-\gamma)(\alpha+\beta)}\right] \frac{S_{n-1}(x^2;\alpha,\beta+1,\gamma)}{(\alpha+\beta+1)_{n-1}(\alpha+\gamma)_{n-1}} \\
&+ \left[\frac{(1-n-\beta-\gamma)(1-n)(\alpha^2+x^2)}{(1-n-\alpha-\gamma)(1+\alpha+\beta)(\alpha+\beta)(\alpha+\gamma)}\right] \\
&\times \frac{S_{n-2}(x^2;\alpha+1,\beta+1,\gamma)}{(\alpha+\beta+2)_{n-2}(\alpha+\gamma+1)_{n-2}},
\end{aligned}
$$

and then

$$
\begin{aligned}
&S_n(x^2;\alpha,\beta,\gamma) \\
&= \left[\frac{(1-n-\alpha-\gamma)(\alpha+\beta)+(\alpha^2+x^2)}{(1-n-\alpha-\gamma)}\right] \frac{(\alpha+\gamma)_n}{(\alpha+\gamma)_{n-1}} S_{n-1}(x^2;\alpha,\beta+1,\gamma) \\
&+ \left[\frac{(1-n-\beta-\gamma)(1-n)(\alpha^2+x^2)}{(1-n-\alpha-\gamma)}\right] \frac{(\alpha+\gamma)_n}{(\alpha+\gamma)_{n-1}} \\
&\times S_{n-2}(x^2;\alpha+1,\beta+1,\gamma).
\end{aligned}
$$

So, the desired result easily follows from some simple calculations. □

Part 2. For five complex numbers $\alpha, \beta, \gamma, \mu$ and η, there holds the three-term relation [6]

$$ {}_3F_2\left[\begin{array}{c}\alpha, \gamma, \eta;\\ \beta, \mu;\end{array} 1\right] = C\,{}_3F_2\left[\begin{array}{c}\alpha-1, \gamma, \eta;\\ \beta, \mu-1;\end{array} 1\right] + D\,{}_3F_2\left[\begin{array}{c}\alpha, \gamma+1, \eta+1;\\ \beta+1, \mu;\end{array} 1\right], \tag{11} $$

where the coefficients C and D are defined by

$$ C = \frac{(1+\gamma+\eta-\mu)(1-\mu)}{(1+\gamma-\mu)(1+\eta-\mu)}, $$
$$ D = \frac{(1+\alpha+\gamma+\eta-\beta-\mu)\gamma\eta}{(1+\gamma-\mu)(\mu-\eta-1)\beta}. $$

Then we obtain the next result.

Theorem 3.2. *The CDH polynomials have the following recurrence relation for $n \geq 0$:*

$$ \begin{aligned} &\left[(1-\gamma)^2 + x^2\right](\alpha+\beta+n)\,S_n(x^2;\alpha,\beta,\gamma) \\ &= (\gamma-\alpha-1)S_{n+1}(x^2;\alpha,\beta,\gamma-1) \\ &+ (n+\beta+\gamma-1)(\alpha^2+x^2)S_n(x^2;\alpha+1,\beta,\gamma-1). \end{aligned} $$

Proof. Using (11) instead of (10) in the proof of Theorem 3.1, the proof is completed at once. $\square$

Part 3. For five complex numbers $\alpha, \beta, \gamma, \mu$ and η, there holds the three-term relation [6]

$$ {}_3F_2\left[\begin{array}{c}\alpha, \gamma, \eta;\\ \beta, \mu;\end{array} 1\right] = E\,{}_3F_2\left[\begin{array}{c}\alpha, \gamma-1, \eta-1;\\ \beta-1, \mu-1;\end{array} 1\right] + H\,{}_3F_2\left[\begin{array}{c}\alpha+1, \gamma, \eta;\\ \beta, \mu;\end{array} 1\right], \tag{12} $$

where the coefficients E and H are defined by

$$ E = \frac{(1-\beta)(1-\mu)}{(1+\alpha-\beta))(1+\alpha-\mu)}, $$
$$ H = \frac{\alpha(1+\alpha+\gamma+\eta-\beta-\mu)}{(1+\alpha-\beta)(1+\alpha-\mu)}. $$

Then, we get the next result.

Theorem 3.3. *The CDH polynomials have the next relation for* $n \in \mathbb{N}:$

$$S_n(x^2;\alpha,\beta,\gamma) = S_n(x^2;\alpha-1,\beta,\gamma) - n(1-n-\beta-\gamma)S_{n-1}(x^2;\alpha,\beta,\gamma).$$

Proof. Using (12) instead of (10) in the proof of Theorem 3.1, we get the proof. □

Part 4. For five complex numbers α,β,γ,μ and η, there holds the three-term relation [17]

$$\begin{aligned} &{}_3F_2\left[\begin{array}{c}\alpha,\beta,\gamma;\\ \mu-1,\eta;\end{array} 1\right] - {}_3F_2\left[\begin{array}{c}\alpha,\beta,\gamma;\\ \mu,\eta;\end{array} 1\right] \\ &= \frac{\alpha\beta\gamma}{(\mu-1)\mu\eta}\, {}_3F_2\left[\begin{array}{c}\alpha+1,\beta+1,\gamma+1;\\ \mu+1,\eta+1;\end{array} 1\right] \end{aligned}$$

Theorem 3.4. *The CDH polynomials have the next relation for* $n \in \mathbb{N}:$

$$\begin{aligned} &(\alpha+\beta+n-1)S_n(x^2;\alpha,\beta-1,\gamma) \\ &= (\alpha+\beta-1)S_n(x^2;\alpha,\beta,\gamma) - n(\alpha^2+x^2)S_{n-1}(x^2;\alpha+1,\beta,\gamma). \end{aligned}$$

Proof. If we use the relation in Part 4 instead of (10) in the proof of Theorem 3.1, the proof is completed. □

4 Integral Representations

Now, we derive various integral representations for the CDH polynomials.

Theorem 4.1. *The CDH polynomials have the next representation*:

$$\begin{aligned} S_n(x^2;\alpha,\beta,\gamma) &= \frac{\Gamma(\alpha+\beta+n)\Gamma(\gamma-ix+n)}{\Gamma(\alpha-ix)\Gamma(\beta+ix)\Gamma(\gamma-ix)} \\ &\times \int_0^1 u^{\beta+ix-1}(1-u)^{\alpha-ix-1}(1-tu)^{-\alpha-ix}du, \end{aligned}$$

where $\mathrm{Re}(\alpha+\beta) > \mathrm{Re}(\beta+ix) > 0$ *and* $|\arg(1-t)| < \pi$.

Proof. The hypergeometric function ${}_2F_1$ has the representation [14]

$$ {}_2F_1\left[\alpha,\beta;\gamma;z\right]=\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 u^{\beta-1}(1-u)^{\gamma-\beta-1}(1-zu)^{-\alpha}du, \quad (13) $$

where $\mathrm{Re}(\gamma)>\mathrm{Re}(\beta)>0$ and $|\arg(1-z)|<\pi$. If we use (13) and binomial theorem for $(1-t)^{-\gamma+ix}$ in left-hand side of (2), we have

$$ \sum_{n=0}^{\infty}\frac{(\gamma-ix)_n}{n!}t^n\frac{\Gamma(\alpha+\beta)}{\Gamma(\beta+ix)\Gamma(\alpha-ix)}\int_0^1 u^{\beta+ix-1}(1-u)^{\alpha-ix-1}(1-tu)^{-\alpha-ix}du $$
$$ =\sum_{n=0}^{\infty}\frac{S_n(x^2;\alpha,\beta,\gamma)}{(\alpha+\beta)_n n!}t^n. $$

Then, the proof easily follows from the coefficients of $\frac{t^n}{n!}$. □

Theorem 4.2. *The CDH polynomials have the next representation*:

$$ S_n(x^2;\alpha,\beta,\gamma)=\frac{\Gamma(\alpha+\beta+n)\Gamma(\gamma-ix+n)}{\Gamma(\alpha-ix)\Gamma(\beta+ix)\Gamma(\gamma-ix)} $$
$$ \int_0^\infty u^{\alpha-ix-1}(u+1)^{ix-\beta}(u-t+1)^{-\alpha-ix}du, $$

where $\mathrm{Re}(\alpha+\beta)>\mathrm{Re}(\beta+ix)>0$ *and* $|\arg(1-t)|<\pi$.

Proof. The hypergeometric function ${}_2F_1$ has the following integral representation [1]:

$$ {}_2F_1\left[\alpha,\beta;\gamma;z\right]=\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^\infty u^{-\beta+\gamma-1}(u+1)^{\alpha-\gamma}(u-z+1)^{-\alpha}du, \quad (14) $$

where $\mathrm{Re}(\gamma)>\mathrm{Re}(\beta)>0$ and $|\arg(1-z)|<\pi$. Using (14) instead of (13) in the proof of Theorem 4.1, the proof immediately follows. □

Theorem 4.3. *The CDH polynomials have the next representation*:

$$S_n(x^2;\alpha,\beta,\gamma)=\frac{\Gamma(\alpha+\beta+n)}{\Gamma(\alpha+ix)\Gamma(\alpha-ix)\Gamma(\beta+ix)\Gamma(\gamma-ix)}$$
$$\times\int_0^\infty\int_0^\infty\int_0^\infty e^{-u_1-u_2u_3-u_3}u_1^{\gamma-ix-1}u_2^{\alpha-ix-1}(u_2+1)^{-\beta+ix}u_3^{\alpha+ix-1}$$
$$\times(u_1+u_3)^n du_1du_2du_3,$$

where $\mathrm{Re}(\alpha+\beta)>\mathrm{Re}(\beta+ix)>0$ *and* $|\arg(1-t)|<\pi$.

Proof. Using the following fact

$$\alpha^{-\lambda}=\frac{1}{\Gamma(\lambda)}\int_0^\infty e^{-\alpha t}t^{\lambda-1}dt,\qquad \mathrm{Re}\,(\lambda)>0,$$

and (14) on the left side of (2), we obtain

$$\sum_{n=0}^\infty\frac{S_n(x^2;\alpha,\beta,\gamma)}{(\alpha+\beta)_n n!}t^n$$
$$=\frac{1}{\Gamma(\gamma-ix)}$$
$$\times\int_0^\infty e^{-u_1(1-t)}u_1^{\gamma-ix-1}du_1\frac{\Gamma(\alpha+\beta)}{\Gamma(\beta+ix)\Gamma(\alpha-ix)}$$
$$\times\int_0^\infty u_2^{\alpha-ix-1}(u_2+1)^{ix-\beta}(u_2-t+1)^{-\alpha-ix}du_2$$
$$=\frac{\Gamma(\alpha+\beta)}{\Gamma(\gamma-ix)\Gamma(\beta+ix)\Gamma(\alpha-ix)}$$
$$\times\int_0^\infty\int_0^\infty u_2^{\alpha-ix-1}(u_2+1)^{ix-\beta}(u_2-t+1)^{-\alpha-ix}e^{-u_1(1-t)}\ u_1^{\gamma-ix-1}du_1du_2,$$

which gives

$$\sum_{n=0}^{\infty}\frac{S_n(x^2;\alpha,\beta,\gamma)}{(\alpha+\beta)_n n!}t^n$$
$$=\frac{\Gamma(\alpha+\beta)}{\Gamma(\gamma-ix)\Gamma(\beta+ix)\Gamma(\alpha-ix)}$$
$$\times\int_0^{\infty}\int_0^{\infty}u_2^{\alpha-ix-1}(u_2+1)^{ix-\beta}\frac{1}{\Gamma(\alpha+ix)}$$
$$\times\int_0^{\infty}e^{-u_3(u_2+1-t)}u_3^{\alpha+ix-1}du_3e^{-u_1(1-t)}\ u_1^{\gamma-ix-1}du_1du_2.$$

Then, we may write that

$$\sum_{n=0}^{\infty}\frac{S_n(x^2;\alpha,\beta,\gamma)}{(\alpha+\beta)_n n!}t^n$$
$$=\frac{\Gamma(\alpha+\beta)}{\Gamma(\gamma-ix)\Gamma(\beta+ix)\Gamma(\alpha-ix)\Gamma(\alpha+ix)}$$
$$\times\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}u_2^{\alpha-ix-1}u_1^{\gamma-ix-1}(u_2+1)^{ix-\beta}u_3^{\alpha+ix-1}e^{-u_1-u_2u_3-u_3}$$
$$\times e^{(u_1+u_3)t}du_1du_2du_3,$$

which implies

$$\sum_{n=0}^{\infty}\frac{S_n(x^2;\alpha,\beta,\gamma)}{(\alpha+\beta)_n n!}t^n$$
$$=\frac{\Gamma(\alpha+\beta)}{\Gamma(\gamma-ix)\Gamma(\beta+ix)\Gamma(\alpha-ix)\Gamma(\alpha+ix)}$$
$$\times\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}u_2^{\alpha-ix-1}u_1^{\gamma-ix-1}(u_2+1)^{ix-\beta}u_3^{\alpha+ix-1}e^{-u_1-u_2u_3-u_3}$$
$$\times\sum_{n=0}^{\infty}\frac{(u_1+u_3)^n t^n}{n!}du_1du_2du_3.$$

Then, the proof easily follows from the coefficients of $\frac{t^n}{n!}$. □

5 Concluding Remarks

In this paper, we derived advanced properties of the CDH polynomials. We have obtained some relations for them, such as, bilinear and bilateral generating function relations, recurrence relations and various integral representations.

In 2008, Ferreira et al. obtain asymptotic expansions between the CDH polynomials and Jacobi, Meixner–Pollaczek, Krawtchouk and Meixner polynomials. From these expansions, we may also derive many relations for the other polynomials.

Recall that the Meixner-Pollaczek polynomials are defined by [11]

$$P_n^{(\lambda)}(x;\phi) = \frac{(2\lambda)_n}{n!} e^{in\phi} \, {}_2F_1\left[-n, \lambda+ix; 2\lambda; 1-e^{-2i\phi}\right].$$

The CDH polynomials have the following asymptotic representation with respect to the Meixner-Pollaczek polynomials (see [10]):

$$\frac{S_n(x^2;\alpha,\beta,\gamma)}{(\alpha+\beta)_n \, n!} = \sum_{k=0}^{n} \gamma_k P_{n-k}^{(C)}(X;A), \tag{15}$$

where $A \neq m\pi, \quad m \in \mathbb{Z}$, is an arbitrary constant,

$$C = p_1(x)\cos A + \frac{1}{2} p_1(x)^2 - p_2(x),$$

$$X = -\frac{p_1(x)\cos(2A) + \left[p_1(x)^2 - 2p_2(x)\right]\cos A}{2\sin A}$$

with

$$\begin{aligned} p_1(x) &= \gamma + \frac{\alpha\beta - x^2}{\alpha+\beta}, \\ p_2(x) &= \frac{\gamma(\gamma+1)}{2} + \frac{\alpha\beta\gamma}{\alpha+\beta} \\ &+ \frac{\alpha\beta(1+\alpha)(1+\beta) - [\gamma + 2\alpha\beta + (1+\gamma)(1+2(\alpha+\beta))]x^2 + x^4}{2(\alpha+\beta)(1+\alpha+\beta)}. \end{aligned} \tag{16}$$

The Jacobi polynomials are defined by [14]

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}\, {}_2F_1\left[-n, n+\alpha+\beta+1; \alpha+1; \frac{1-x}{2}\right].$$

The CDH polynomials have also the following asymptotic representation with respect to the Jacobi polynomials (see [10]):

$$\frac{S_n(x^2;\alpha,\beta,\gamma)}{(\alpha+\beta)_n\, n!} = \sum_{k=0}^{n} c_k P_{n-k}^{(A,C)}(X), \tag{17}$$

where $X \neq \pm 1$ is an arbitrary constant,

$$\begin{aligned} A &= \frac{1}{X+1}\left[2p_1(x)^2 + p_1(x) + 3p_1(x)X - 4p_2(x) + X^2 - X - 2\right], \\ C &= \frac{1}{X-1}\left[2p_1(x)^2 + p_1(x) + 3p_1(x)X - 4p_2(x) + X^2 + X - 2\right], \end{aligned}$$

and $p_1(x)$, $p_2(x)$ are given in (16).

The Meixner polynomials are defined by [5]

$$M_n(x;\beta,\gamma) = {}_2F_1\left[-n, -x; \beta; 1 - \frac{1}{\gamma}\right].$$

Another asymptotic representation with respect to the Meixner polynomials (see [10]) is given as the following:

$$\frac{S_n(x^2;\alpha,\beta,\gamma)}{(\alpha+\beta)_n\, n!} = \sum_{k=0}^{n} c_k \frac{(A)_{n-k}}{(n-k)!} M_{n-k}(X; A, C), \tag{18}$$

where $C \neq 0, 1$ is a constant,

$$\begin{aligned} X &= \frac{C^2}{1-C}\left[p_1(x)^2 + p_1(x) - 2p_2(x)\right], \\ A &= (1+C)\,p_1(x) + Cp_1(x)^2 - 2Cp_2(x), \end{aligned}$$

and $p_1(x)$, $p_2(x)$ are given in (16).

The Krawtchouk polynomials are defined by [11]

$$K_n(x;p,N) = {}_2F_1\left[-n, -x; -N; \frac{1}{p}\right].$$

Another asymptotic representation with respect to the Krawtchouk polynomials (see [10]) is given by

$$\frac{S_n(x^2;\alpha,\beta,\gamma)}{(\alpha+\beta)_n\, n!} = \sum_{k=0}^{n} \begin{pmatrix} C \\ n-k \end{pmatrix} c_k K_{n-k}\left(X;A,C\right), \tag{19}$$

where $A \neq 0,1$ is a constant,

$$X = \frac{A^2}{1-A}\left[p_1\left(x\right)^2 - p_1\left(x\right) - 2p_2\left(x\right)\right],$$
$$C = p_1\left(x\right) + \frac{X}{A},$$

and $p_1(x)$, $p_2(x)$ are given in (16).

Therefore, we can say that, in Sections 2,3,4, using asymptotic representations given respectively by (15), (17), (18) and (19) between the CDH polynomials and Meixner–Pollaczek, Jacobi, Meixner and Krawtchouk polynomials, we may also obtain new bilinear and bilateral generating functions relations, recurrence relations and integral representations for Jacobi, Meixner–Pollaczek, Krawtchouk and Meixner polynomials. But we leave the details to the readers.

References

[1] R. Askey, G. E. Andrews and R. Roy, *Special Functions* (17th Press), United Kingdom: Cambridge University (1999).

[2] R. Askey and J. A. Wilson, *Some Basic Hypergeometric Orthogonal Polynomials that Generalized Jacobi Polynomials*, Memo. Amer. Math. Soc. vol. 319, 1985.

[3] C. F. Bracciali, J. V. da Silva and A. Sri Ranga, A class of Sobolev orthogonal polynomials on the unit circle and associated continuous dual Hahn polynomials: Bounds, asymptotics and zeros, *J. Approx. Theory*, 268 (2021), paper no. 105604, 25 pp.

[4] W. Bryc, On the continuous dual Hahn process, *Stochastic Process. Appl.* 143 (2022), 185-206.

[5] T. S. Chihara, *An Introduction to Orthogonal Polynomials*, New York: Gordon and Breach Science Publishers Inc. (1978).

[6] W. Chu and X. Wang, Abel's method on summation by parts and hypergeometric contiguous relations, *Integral Transform. Spec. Funct.* 18 (11) (2007), 771-807.

[7] F. Colomo and A. G. Pronko, Square ice, alternating sign matrices, and classical orthogonal polynomials, *J. Stat. Mech. Theory Exp.* 1 (5) (2005), 33 pp.

[8] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher Transcendental Functions*, Vol. III, McGraw-Hill Book Company, New York, Toronto and London (1955).

[9] E. Erkus-Duman and J. Choi, Gottlieb polynomials and their q-extensions, *Mathematics*, 9 (13) (2021), 1499.

[10] C. Ferreira, J. López and E.P. Sinusía, Asymptotic relations between the Hahn-type polynomials and Meixner-Pollaczek, Jacobi, Meixner and Krawtchouk polynomials, *J. Comput. Appl. Math.* 217 (2008), 88-109.

[11] R. Koekoek, P. A. Lesky and R. F. Swarttouw, *Hypergeometric Orthogonal Polynomials and Their q-Analogues*, Springer (2010).

[12] E. Neuman, On Hahn polynomials and continuous dual Hahn polynomials, *J. Comput. Anal. Appl.* 8 (2006), 229-248.

[13] N. Ozmen and E. Erkus-Duman, Some families of generating functions for the generalized Cesaro polynomials, *J Comput. Anal. Appl.* 25 (4) (2018), 670-683.

[14] E. D. Rainville, *Special Functions*, New York: The Macmillan Company (1960).

[15] F. M. Sakar and A. Canbulat, A study on harmonic univalent function with (p, q)-calculus and introducing (p, q)-possion distribution series, *J. Math. Extension* 17 (3) (2023), 2654.

[16] G. Valent and W. Van Assche, The impact of Stieltjes' work on continued fractions and orthogonal polynomials: additional material, *J. Comput. Appl. Math.* 65 (1995), 419-447.

[17] J. A. Wilson, Three-term contiguous relations and some new orthogonal polynomials, *In: Padé and Rational Approximation*, eds. E. Staff and R. Varga, Academic Press, New York (1977), 227-232.

[18] J. A. Wilson, Some hypergeometric orthogonal polynomials, *SIAM J. Math. Anal.* 11 (1980), 690-701.

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