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Irrational Rotation Algebra as a Crossed Product

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Abstract: In this paper we will consider the crossed product $C(T) \times_\alpha \mathbf{Z}$, where T is the unit circle, $\alpha(n) = \alpha_n$ is a rotation through the angle $-2\pi n\theta$ for $n \in \mathbf{Z}$, and θ is a fixed irrational number. We will apply some results about patial actions to represent this crossed product as a C^*-subalgebra of $B(L^2(T))$. Also, by a different method form the proof of Davidson, we show that this crossed product is isomorphic to the irrational rotation algebra.

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1. Introduction

Let G be a discrete group and $\theta = (\{\theta_t\}, \{U_t\})_{t\in G}$ be a partial homeomorphism [3] of a locally compact space X. Put $D_t = C_0(U_t)$ and define $\alpha_t : D_{t^{-1}} \to D_t$ by

$$\alpha_t(f)(x) := f(\theta_{t^{-1}}(x)) \ , \quad \text{for} \quad f \in D_{t^{-1}} \quad \text{and} \quad x \in U_t.$$

Then $\alpha = (\{\alpha_t\}, \{D_t\})_{t\in G}$ is a partial action of G on the C^*-algbera $C_0(X)$ in the sense of [2] and [6], which is called the partial action of G on $C_0(X)$ corresponding to θ ([4]).

Definition 1.1. **([4])** *The partial dynamical system $(C_0(X), G, \alpha)$ is topologically free if for every $t \in G\backslash\{e\}$, the set*

$$F_t := \{x \in U_{t^{-1}} : \theta_t(x) = x\}$$

has empty interior.

The concepts of reduced and full crossed products for actions are generalized by McClanahan in [6] to parital actions. It is surprising that in some situations the faithfulness of a representation of the reduced crossed product $C_0(X) \times_r G$ depends only on that of $C_0(X)$. In this relation we bring the following theorem. For the proof see [4, Theorem 2.6].

Theorem 1.2. *Suppose $(C_0(X), G, \alpha)$ is topologically free. A representation of the reduced crossed product $C_0(X) \times_r G$ is faithful if and only if it is faithful on $C_0(X)$.*

We remark that when G is an amenable group (especially when G is abelian), the reduced and full crossed products are identified with each other, and so in this case, the preceding theorem is valid for the full crossed product, with a similar proof.

2. Main Result

Fix an irrational number θ. Let R_{θ_n} be the rotation through the angle $2\pi n\theta$. That is, $R_{\theta_n} : T \to T$ is defined by $R_{\theta_n}(z) = ze^{2\pi in\theta}$ for $n \in \mathbf{Z}$. So the map $R_\theta : n \mapsto R_{\theta_n}$ is a partial action (indeed, an action) on $\mathbf{Z}$. It is clear that R_{θ_n} is a homeomorphism on the compact space T. Thus, R_θ is a partial homeomorphism.

Now, let α be the partial action of $\mathbf{Z}$ on $C_0(T) = C(T)$ corresponding to R_θ. So $\alpha_n : C(T) \to C(T)$ is defined by

$$\alpha_n(f)(z) = f(R_{\theta_{-n}}(z)) = f(ze^{-2\pi in\theta})$$

for $f \in C(T)$ and $z \in T$. Note that α is an action in this case.

To identify the crossed product $C(T) \times_\alpha \mathbf{Z}$ ([1], [6]), more explicitely, first we find a faithful representation of $C(T) \times_\alpha \mathbf{Z}$. Indeed, we represent the crossed product as a C^*-subalgebra of $B(L^2(T))$. Let $M : C(T) \to B(L^2(T))$ be given by

$$M_f(g) = fg$$

for $f \in C(T)$ and $g \in L^2(T)$. Also, define $\lambda : \mathbf{Z} \to B(L^2(T))$ by

$$\lambda_n(\xi)(z) = \xi(ze^{-2\pi in\theta})$$

for $\xi \in L^2(T)$ and $z \in T$.

Note that $\lambda_n^* = \lambda_{-n}$ and $\lambda_n = \lambda_1^n$. It is clear that M is a non-degenerate representation and λ is a unitary representation. Also it can be easily verified that

$$M(\alpha_n(f)) = \lambda_n o M_f o \lambda_n^*$$

for $f \in C(T)$. Therefore, $(M, \lambda, L^2(T))$ is a covariant representation of the C^*-dynamical system $(C(T), \mathbf{Z}, \alpha)$. By the correspondence between the covariant representations of a partial action and the representations of the associated crossed product [6], we conclude that $M \times \lambda$ is a representation of $C(T) \times_\alpha \mathbf{Z}$. Since $\mathbf{Z}$ is an abelian group, we can identify $C(T) \times_\alpha \mathbf{Z}$ with $C(T) \times_r \mathbf{Z}$. On the other hand, M is faithful on $C(T)$. So Theorem 1.2 implies that $M \times \lambda$ is a faithful representation. Note that Theorem 1.2 can be used because for every irrational θ, R_θ is topologically free. In fact, for $n \in \mathbf{Z}\backslash\{0\}, F_n = \{z \in T : ze^{-2\pi in\theta} = z\} = \emptyset$ because θ is an irrational number and so $e^{2\pi in\theta} \neq 1$.

We know that $(M \times \lambda)(f\delta_n) = M_f\lambda_n = M_f\lambda_1^n$, where $f\delta_n$ is a generator of $C(T) \times_\alpha \mathbf{Z}$. Since $\iota(z) = z$, generates the C^*-algebra $C(T)$ and 1 generates the group $\mathbf{Z}$, we have $(M \times \lambda)(C(T) \times_\alpha \mathbf{Z}) = C^*(M_z, \lambda_1)$.

We can summarize the above discussions in the following lemma.

Lemma 2.1. *Assume that M_z and λ_1 are defined as following*

$$M_z(g) = zg$$

for $g \in L^2(T)$, and

$$\lambda_1(\xi)(z) = \xi(ze^{-2\pi i\theta})$$

for $\xi \in L^2(T)$ and $z \in T$. Then

$$C(T) \times_\alpha \mathbf{Z} \simeq C^*(M_z, \lambda_1).$$

Remark 2.2. *Set $U = M_z$ and $V = \lambda_1$. Then U and V are unitaries satisfying*

$$UV = e^{2\pi i\theta} VU. \tag{$*$}$$

Lemma 2.3. *Assume that U and V are two unitaries in $B(L^2(T))$, satisfying the relation $(*)$. Let π be the representation of $C(T)$ on $B(L^2(T))$ taking ι to U, where $\iota(z) = z$ for all z in T. Also let Λ be the representation of $\mathbf{Z}$ on $B(L^2(T))$ taking 1 to V. Then (π, Λ) is a covariant representation of $(C(T), \mathbf{Z}, \alpha)$.*

Proof. It is clear that π is a non-degenerate representation and Λ is a unitary representation. It suffices to show that

$$\Lambda_n \pi(g) \Lambda_n^* = \pi(\alpha_n(g))$$

for all $g \in C(T)$ and $n \in \mathbb{Z}$. Since U is unitary, Proposition 4.1.1 (iii) of [5] implies that $sp(U) \subset T$. On the other hand, $sp(U)$ is invariant under the rotation R_θ through the irrational angle $2\pi\theta$, because

$$e^{2\pi i\theta} sp(U) = sp(e^{2\pi i\theta}U) = sp(V^*UV) = sp(UV^*V) = sp(U).$$

Thus considering the fact that θ is irrational, we conclude that $sp(U) = T$. So we can use the Functional Calculus. For any polynomial $p(z) = \sum_{k=-N}^{N} a_k z^k$, one has

$$Vp(U)V^* = \sum_{k=-N}^{N} a_k (VUV^*)^k = \sum_{k=-N}^{N} e^{-2\pi ik\theta} a_k U^k = \alpha_1(p)U.$$

Similarly, we have $V^*p(V)V = \alpha_{-1}(p)U$. It is easily verified by induction that

$$V^n p(U) V^{n^*} = \alpha_n(p)U \quad \text{for all } n \in \mathbb{Z}.$$

So we have

$$\Lambda_n \pi(p) \Lambda_n^* = V^n p(U) V^{n^*} = \alpha_n(p)U = p(e^{-2\pi in\theta}U) = \pi(\alpha_n(p)).$$

Since $C(T)$ is the closure of such these polynomials, the result follows.□

In [1] K. R. Davidson has defined the irrational rotation algebra $\mathcal{A}_\theta$ as the following:

Definition 2.4. *The universal C^*-algebra $\mathcal{A}_\theta$ satisfying $(*)$ is called the irrational rotation algebra.*

Recall that $\mathcal{A}_\theta$ is universal for the relation $(*)$ provided that it is generated by two unitaries $\tilde{U}$ and $\tilde{V}$ satisfying $(*)$, and whenever $\mathcal{A} = C^*(U, V)$ is another C^*-algebra satisfying $(*)$, there is a homomorphism of $\mathcal{A}_\theta$ onto $\mathcal{A}$ which carries $\tilde{U}$ to U and $\tilde{V}$ to V.

Remark 2.5. *Let $(\mathcal{A}, G, \alpha)$ be a C^*-dynamical system. Then the crossed product $\mathcal{A} \times_\alpha G$ has the universal property [1]. That is, if (π, Λ) is any covariant representation of $(\mathcal{A}, G, \alpha)$, then there is a representation of $\mathcal{A} \times_\alpha G$ into $C^*(\pi(\mathcal{A}), \Lambda(G))$ obtained by setting*

$$\sigma(f) = \sum_{t \in G} \pi(A_t)\Lambda_t \quad \textit{for} \quad f = \sum_{t \in G} A_t \delta_t \in \mathcal{A}G$$

and then extending by continuity. In the unital case, this map is surjective.

Theorem 2.6. *The crossed product $C(T) \times_\alpha \mathbf{Z}$ can be identified with* *the irrational rotation algebra* $\mathcal{A}_\theta$*.*

Proof. By Remark 2.2, M_z and λ_1 are unitaries satisfying $(*)$. Now, since $\mathcal{A}_\theta$ is simple, by Theorem VI.1.4 of [1], $C^*(M_z, \lambda_1)$ is isomorphic to $\mathcal{A}_\theta$. Thus Lemma 2.1 implies that $C(T) \times_\alpha \mathbf{Z} \simeq \mathcal{A}_\theta$. $\square$

Remark 2.7. *There is another proof of the theorem due to Davidson [1] which we bring here.*

Suppose that $\mathcal{A}_\theta = C^*(\tilde{U}, \tilde{V})$ such that $\tilde{U}$ and $\tilde{V}$ are unitaries satisfying $(*)$. Then by Lemma 2.3, (π, Λ) is a covariant representation of $(C(T), \mathbf{Z}, \alpha)$, where $\pi : \iota \mapsto \tilde{U}$ and $\Lambda : 1 \mapsto \tilde{V}$. By Remark 2.5, there is a homomorphism of $C(T) \times_\alpha \mathbf{Z}$ onto $C^*(\pi(C(T)), \Lambda(\mathbf{Z})) = C^*(\pi(\iota), \Lambda(1)) = C^*(\tilde{U}, \tilde{V}) = \mathcal{A}_\theta$. Conversely, by Remark 2.2, M_z and λ_1 are unitaries satisfying $(*)$. Therefore, the universal property of $\mathcal{A}_\theta$ implies that there is a homomorphism of $\mathcal{A}_\theta$ onto $C^*(M_z, \lambda_1)$, and so by Lemma 2.1, there is a homomorphism of $\mathcal{A}_\theta$ onto $C(T) \times_\alpha \mathbf{Z}$. Clearly these homomorphims are inverses.

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