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Notes on the Hypercyclic Operator

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Abstract: In this paper by using a nice criterion, we show that the perturbation of identity operators by some multiples of the standard backward shift is hypercyclic. This gives a new proof for Salas Theorem in ([10], Theorem 3.3).

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1. Introduction

Suppose that X is a separable topological vector space and T is a continuous linear mapping on X. If $x \in X$, then the orbit of x under T is denoted by $Orb(T,x)$ and is defined by $Orb(T,x) = \{x, Tx, T^2x, \cdots\}$. An operator T is called a hypercyclic if there is a vector x such that $Orb(T,x)$ is dense in X and in this case x is called a hypercyclic vector for T.

It is interesting that many continuous linear mapping can actually be hypercyclic. The first example was constructed by Rolewicz in 1969

[9]. He showed that if B is the backward shift on $\ell^2(\mathbb{N})$, then λB is hypercyclic if and only if $|\lambda| > 1$.

A nice criterion namely the Hypercyclicity Criterion, was developed independently by Kitai [8] and, Gethner and Shapiro [6]. This criterion has been used to show that hypercyclic operators arise within the classes of composition operators [4], weighted shifts [10], adjoints of multiplication operators [5], and adjoints of subnormal and hyponormal operators [3].

The Hypercyclicity Criterion. *Suppose X is a separable Banach space and T is a continuous linear mapping on X. If there exists two dense subsets Y and Z in X and a sequence $\{n_k\}$ such that:*

1. $T^{n_k} y \to 0$ for every $y \in Y$, and

2. There exists functions $S_{n_k} : Z \to X$ such that for every $z \in Z, S_{n_k} z \to 0$, and $T^{n_k} S_{n_k} z \to z$, then T is hypercyclic.

The above formulation of the Hypercyclicity Criterion was given by J.Bes in the Ph.D thesis [1] (see also [2]).

In the present paper we give a nice criterion that reduce the question of hypercyclicitiy to a study of the eigenvectors of the operators. This is a specially interesting when the eigenvectors are easy compute.

2. Main Results

We will denote by H an infinite-dimensional separable complex Hilbert space. For any $T \in B(H)$, let $N_+(T)$ be the vector space spanned by the kernels $ker(T - \lambda I)$ with $|\lambda| > 1$, and $N_-(T)$ be the space spanned by the kernels $ker(T - \lambda I)$ with $|\lambda| < 1$.

The following result is used by Godetroy and Shapiro in their paper [7], but is not stated explicitly there. For this reason, we sketch a proof of it.

Theorem 2.1. *For any bounded operator T on H, if $N_+(T)$ and $N_-(T)$ are dense subspace of H, then T is hypercyclic. Morever T satisfies the Hypercyclicity* Criterion.

Proof. Put $X = N_-(T)$ and $Y = N_+(T)$. Let $(x_i)_{i \in I}$ be an algebric basis of X such that for every $i \in I$, there exist an eigenvalue λ_i with $|\lambda_i| < 1$ such that $Tx_i = \lambda_i x_i$. Every vector x in X can be written as a finite sum $x = \sum a_i x_i$ and for every $k \geqslant 0, T^k x = \sum a_i \lambda_i^k x_i$, which obviously converge to zero as $k \to +\infty$. Also we can choose the algebric basis $(y_j)_{j \in J}$ of Y such that foe every $j \in J$, there exists an eigenvalue β_j with $|\beta_j| > 1$ such that $Ty_j = \beta_j y_j$. Define the sequence of mapping

$\{S_k\}$ on Y by

$$S_k(y_j) = \frac{1}{(b_j)^k} y_j$$

for every $j \in J$. Like as above, $S_k y \to 0$ for every $y \in Y$, and if $y = \sum b_j y_j$ is any vector of Y, then $T^k S_k y = T(\sum \frac{b_j}{\beta_j} y_j) = y$. $\square$

H. Salas in ([10], Theorem 3.3) shows that, the perturbation of identity operators by weighted backward shift is hypercyclic. In the following Theorem, we give a simple proof for a special case.

Theorem 2.2. *Let $(e_i)_{i\geqslant 1}$ be an orthonormal basis of H, and B be a the backward shift defined by $Be_1 = 0$ for $i \geqslant 2, Be_i = e_{i-1}$, then the operator $I + wB$ is hypercyclic when $|w| > 1$.*

Proof. Let $x = \sum_{i\geqslant 1} x_i e_i$ be a vector of H. If λ is any complex number, then $x \in ker(I + wB - \lambda I)$ if and only if for every $i \geqslant 0$, $(1-\lambda)x_i + wx_{i+1} = \theta$, which means that $x_{i+1} = \frac{\lambda-1}{w} x_i$. If $\frac{|\lambda|+1}{|w|} < 1$, then λ is an eigenvector of $I + wB$ and the eigenspace $ker(I + wB - \lambda I)$ is spanned by the vector

$$x_\lambda = e_1 + \sum_{n\geqslant 2} (\frac{\lambda - 1}{w})^{n-1} e_n.$$

The disc $B(0, R)$, where $R = |w| - 1$, entirely consists of the eigenvalues of $I + wB$. If $y = \sum_{i\geqslant 1} y_i e_i$ belongs to the orthogonal complement of

$N_+(I+wB)$, then the function

$$\varphi(\lambda) = \overline{y_1} + \sum_{n=2}^{+\infty} \overline{y_n}(\frac{\lambda - 1}{w})^{n-1}$$

vanishes on the annulus $1 < |\lambda| < R$. Because for every λ with $1 < |\lambda| < R$, $x_\lambda \in N_+(I+wB)$ and

$$\varphi(\lambda) =< x_\lambda, y >= 0.$$

Since the sequence $(\overline{y_i})_{i \geqslant 1}$ is bounded, this function is analytic in the open disc $B(0,R)$, and this implies that φ is identically zero on the whole disc $B(0,R)$. Note that $R > 1$, because $R = |w| - 1$ and $|w| > 2$. So $\varphi(d_1) = 0$ and $\overline{y_1} = 0$. This implies that φ can be written as

$$\varphi(\lambda) = (\lambda - d_1)(\overline{y_2} + \sum_{n=2}^{+\infty} \overline{y}_n(\frac{\lambda - 1}{w})^{n-1}) = (\lambda - d_1)\varphi_1(\lambda),$$

where φ_1 is analytic on the disck of radius R. Now we repeat the above process for φ_1 instend of φ, so the result is $\varphi_1(d_2) = \bar{y}_2 = 0$. Continuing this way, we see that all coordinates $\bar{y}_n = 0$ for every positive integer n. This implies that $y = 0$ and $N_-t(I+wB)$ is dense in H and by a similar way, $N_+t(I+wB)$ is also dense in H. Thus indeed $I + wB$ is hypercyclic. $\Box$

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