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On Upper and Lower $gp\alpha$-Continuous Multifunctions

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Abstract. In this paper, we introduced and studied the properties of weaker form of multifunctions such as upper and lower $gp\alpha$-continuous multifunctions. We obtain their characterizations and preservation theorems. Further, we have investigated the properties of upper and lower $gp\alpha$-irresolute multifunctions.

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1 Introduction

Many authors have continued their research and looked into different types of continuous functions and multifunctions, both stronger and weaker. It has been observed that continuity and multifunctions are basic concepts in general topology, set valued analysis, and several

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branches of mathematics. Particularly in general topology, continuous functions and continuous multifunctions are two of the most important and closely related topics.

In the literature, the concept of upper and lower continuity for multifunctions was first studied and introduced by Berge[2]. After his work, many authors turned their direction to investigate several work regarding weak and strong forms of continuity. In 1999 , Mahmoad[7] introduced the concept of pre-irresolute multi-valued functions. Neubrunn [10] introduced and studied the notion of upper(lower) α-continuous multifunctions. Some properties of weakly α-continuous multifunctions were studied by Popa[18]. Recently, multifunctions in topological spaces have been used in applications to study graphs, which are used in physics and smart cities.

The purpose of this paper is to give a new weaker form of some types of continuous functions called upper $gp\alpha$-continuous multifunctions and lower $gp\alpha$-continuous multifunctions. Moreover, properties and their preservation theorems of an upper $gp\alpha$-continuous (resp. lower$gp\alpha$-continuous) multifunctions are introduced and studied. Further, the notion of upper $gp\alpha$-irresolute (resp. lower $gp\alpha$-iresolute) multifunctions are studied and their properties are investigated. we further develop the topological aspects of the multifunctions. We provide discussions for several new as well as already existing topologies for continuous multifunctions. We have adopted the theoretic approach to discuss continuous convergence for the topology of multifunctions

Let (M, τ), (N, σ) and (O, η) are nonempty topological spaces on which no separation axioms are assumed unless expressly indicated, and they are written M, N and O respectively, throughout this study. If M is a topological space and A is a subset of M, gpα-closed set [16] is defined as $pcl(A) \subseteq U$ whenever $A \subseteq U$ and U is α-open in M. gpα-closure of A, gpα-interior of A are also denoted by gpα-$cl(A)$, gpα-$int(A)$ and are defined as gpα-$cl(A) = \cap\{G : A \subseteq G$, G is gp$\alpha$-closed in M $\}$, gpα-$int(A) = \cup\{G : G \subseteq A$, G is gp$\alpha$-open in $M\}$ respectively.

A multifunction [2] $P : M \to N$ is a point to set correspondence and always we assume that $P(m) \neq \phi$ for every point $m \in M$, provided for any subset A of M and B be any subset of N, then $P(A) = \cup\{P(m) : m \in A\}$, $P^+(B) = \{m \in M : P(m) \subset B\}$ and $P^-(B) = \{m \in M :$

$P(m) \cap B \neq \phi\}$, where $P^+(B)$, $P^-(B)$ denotes upper and lower inverse of any subset B of N. In particular, $P^-(n) = \{m \in M : n \in P(m)\}$ for each point $n \in N$. Then P is said to be surjection if $P(M) = N$ or equivalently, if for each $n \in N$ there exists $m \in M$ such that $n \in P(m)$.

For multifunction $P : M \to N$, the graph multifunction [19] $G_P : M \to M \times N$ is defined as $G_P(m) = \{m\} \times P(m)$ for each $m \in M$ and the subset $\{\{m\} \times P(m) : m \in M\} \subset M \times N$ is called the multifunction of P and is denoted by $G(P)$.

2 Generalized pre α-Continuous Multifunctions

***Definition* 2.1.** *A multifunction $P : M \to N$ is called a*

(i) upper $gp\alpha$-continuous (briefly u.$gp\alpha$-c) at a point $m \in M$, if for each open subset V in N with $p(m) \subseteq V$, there exists an $gp\alpha$-open set V containing m such that $P(U) \subseteq V$.

(ii) lower $gp\alpha$-continuous (briefly l.$gp\alpha$-c) at a point $m \in M$, if for every open subset V in N with $P(m) \cap V \neq \phi$, there exists an $gp\alpha$-open set V containing m such that $P(z) \cap V \neq \phi$ for each $z \in U$.

(iii) upper(resp. lower) $gp\alpha$-continuous, if it is upper(resp. lower) $gp\alpha$-continuous at every point of M.

Example 2.2. Let $M = \{a_1, a_2, a_3\}$, $N = \{a_1, a_2, a_3\}$, $\tau = \{M, \phi, \{a_1\}\}$ and $\sigma = \{N, \phi, \{a_1, a_2\}\}$. Let $P : M \to N$ be an identity function. Then
$gp\alpha - O(M)$=M, ϕ, $\{a_1\}$, $\{a_1, a_2\}$, $\{a_1, a_3\}$ and
$gp\alpha - O(N)$=N, ϕ, $\{a_1\}$, $\{a_2\}$, $\{a_1, a_2\}$, $\{a_1, a_3\}$, $\{a_2, a_3\}$
Let $a_1 \in M$ and $V = \{a_1, a_2\}$ be an open set of N with $P(\{a_1\}) = \{a_1\} \subseteq V$. Then by definition, there exists $gp\alpha$-open set $U = \{a_1, a_2\}$ in M containing the point a_1 such that $P(U) = P(\{a_1, a_2\}) = \{a_1, a_2\} \subseteq V$. Therefore, $P(U) \subseteq V$. Hence, P is u.$gp\alpha$-c.

Example 2.3. Let $M = N = \{a_1, a_2, a_3\}$, let $\tau = \{M, \phi, \{a_1\}, \{a_1, a_2\}\}$, be a topology on M, Then $gp\alpha - O(M)$ are : $M, \phi, \{a_1\}, \{a_2\}, \{a_1, a_2\}$
$\sigma = \{N, \phi, \{a_1\}\}$ be a topology on N, then

$gp\alpha - O(N)$ are : $N, \phi, \{a_1\}, \{a_1, a_2\}, \{a_1, a_3\}$
Let $P : M \to N$ be an identity function and $M = \{a_2\} \in M$.
Here $gp\alpha$-open sets in N containing a_2 are :$N, \{a_1, a_2\}$
Let us consider $V = N$,
$V = N = \{a_1, a_2, a_3\} \to P(\{a_2\}) \cap V = \{a_2\} \cap \{a_1, a_2, a_3\} = \{a_2\} \neq \phi$
Now consider $V = \{a_1, a_2\}$
$V = \{a_1, a_2\} \to P(\{a_2\}) \cap V = \{a_2\} \cap \{a_1, a_2\} = \{a_2\} \neq \phi$
$gp\alpha$-open set containing $\{a_2\}$ are: $N, \{a_1, a_2\}$, then

(i) $\{a_2\} \cap \{a_1, a_2\} = \{a_2\} \neq \phi$

(ii) $\{a_2\} \cap N = \{a_2\} \cap \{a_1, a_2, a_3\} = \{a_2\} \neq \phi$
Therefore P is a $l.gp\alpha$-c.

Theorem 2.4. *If M and N be any two topological spaces. Then for a multifunction $P : M \to N$ the following properties are equivalent:*

(i) P is $u.gp\alpha$-c

(ii) for each $m \in M$, and $V \in O(N)$ such that $m \in P^+(V)$ there exists $U \in gp\alpha - O(M, m)$ such that $U \subseteq P^+(V)$

(iii) for each $m \in M$ and $K \in C(N)$ such that $m \in P^+(N - K)$, there exists $gp\alpha$-closed set H in M such that $m \in M - H$ and $P^-(K) \subseteq H$

(iv) for each $V \in O(N)$, $P^+(V)$ is $gp\alpha$-open

(v) for each $K \in C(N)$, $P^-(V)$ is $gp\alpha$-closed

(vi) for each $m \in M$ and for each neighbourhood V of $P(M)$, $P^+(V)$ is $gp\alpha$-neighbourhood of m

(vii) for each $m \in M$ and for each neighbourhood V of $P(M)$, there exists $gp\alpha$-neighbourhood U of m such that $P(U) \subseteq V$

Proof. $(i) \to (ii)$: Follows from definition.
$(ii) \to (iii)$: Let $m \in M$ and K be a closed set in N with $m \in P^+(N - K)$, from (ii) there exists $gp\alpha$-open set U containing m such that $U \subseteq P^+(N - K)$. Take $H = N - K$, then H is $gp\alpha$-closed with $m = M - H$.

Also, $U \subseteq P^+(N-K) = M - P^-(K)$, that is $P^-(K) \subseteq M - V = H$.
$(iii) \to (ii)$: Let $m \in M$ and $V \in O(N)$ with $m \in P^+(V)$, put $K = N - V$ where $K \in C(N)$ with $m \in P^+(N-K)$. From (iii), there exists $gp\alpha$-closed set H such that $m \in M - H$ and $P^-(K) \subseteq H$. Let $U = M - H$, then $U \in gp\alpha - O(M,m)$ and also $P^-(K) \subseteq H$. Thus, $M - P^-(N-K) \subseteq H$ and so $M - H \subseteq P^-(N-K)$. Hence $U \subseteq P^+(V)$.
$(ii) \to (iv)$: Let $V \in O(N)$ and $m \in P^-(V)$. From (ii), there exists $U \in gp\alpha - O(M,m)$ with $U \subseteq P^+(V)$. So $P^+(V) = \bigcup_{m \in P^+(V)} U$. We know that, arbitrary union of $gp\alpha$-open sets is $gp\alpha$-open [15] and so $P^+(V) \in gp\alpha - O(M)$.
$(iv) \to (ii)$: Let $m \in M$ and $V \in O(N)$ with $m \in P^+(V)$. From (iv), $P^+(V) \in gp\alpha - O(M)$. Let $U = P^+(V)$ then $U \in gp\alpha - O(M,m)$. Hence, $U \in P^+(V)$.
$(iv) \to (v)$: Let $U \in C(N)$ and so $N - U = O(N)$. From (iv), $P^+(N - U) \in gp\alpha - O(M)$, as $P^+(N-U) = M - P^-(U)$ and so $M - P^-(U) \in gp\alpha - O(M)$. Thus $P^-(U) \in gp\alpha - O(M)$.
$(v) \to (vi)$:Let $V \in O(N)$ and so $N - V \in C(N)$. But from (vi), $P^-(N-V) \in gp\alpha - C(M)$, since $P^-(N-V) = M - P^+(V)$ and so $M - P^+(U) \in gp\alpha - C(M)$. Thus, $P^+(V) \in gp\alpha - O(M)$
$(iv) \to (vi)$: Let $m \in M$ and V be a neighbourhood of $P(m)$. Then, there exists open set U in N such that $P(m) \subseteq U \subseteq V$, that is $m \in P^+(U) \in P^+(V)$. But from (iv), $P^+(U) \in gp\alpha - O(M)$ and so $P^+(V)$ is $gp\alpha$-neighbourhood of m.
$(vi) \to (vii)$: Let $m \in M$ and V be a neighbourhood of $P(m)$, from (vi), $P^+(V)$ is a $gp\alpha$-neighbourhood of the point $m \in M$. Put $U = P^+(V)$, then U is a $gp\alpha$-neighbourhood of m with $P(U) \subseteq V$.
$(vii) \to (i)$: Let $m \in M$ and $V \in O(N)$ with $P(m) \subseteq V$, then V is a neighbourhood of $P(m)$. By (vii), there exists $gp\alpha$-neighbourhood U of m such that $P(U) \subseteq V$. Thus there exists $gp\alpha$-open set G with $m \in G \subseteq U$ and so $P(m) \in P(G) \subseteq P(U) \subseteq V$. Hence, P is u.$gp\alpha$-c for each point $m \in M$. $\square$

***Theorem* 2.5.** *The following properties are equivalent for a multifunction $P : M \to N$:*

(i) P is l.$gp\alpha$-c.

(ii) *for each* $m \in M$,*and* $V \in O(N)$ *with* $m \in P^{-}(V)$ *there exists* $U \in gp\alpha - O(M)$ *containing* m *such that* $U \subseteq P^{-}(V)$.

(iii) *for each* $m \in M$ *and* $K \in C(N)$ *with* $m \in P^{-}(N-K)$ *there exists* $H \in gp\alpha - C(M)$ *such that* $m \in M - H$ *and* $P^{+}(K) \subseteq H$.

(iv) *for each* $V \in O(N)$, $P^{-}(V) \in gp\alpha - O(M)$.

(v) *for each* $K \in C(N)$, $P^{+}(K) \in gp\alpha - C(M)$.

***Theorem* 2.6.** *If a multifunction* $P : M \to N$ *is upper pre-continuous and* N *is submaximal then* P *is u.*$gp\alpha$*-c.*

Proof. Let $A \in p - O(N)$. As N is submaximal, then $A \in O(N)$ [3]. Since, P is upper pre-continuous, $P^{+}(A)$ is pre-open in M and hence $P^{+}(A)$ is $gp\alpha$-open in M. Thus, P is u.$gp\alpha$-c. □

***Theorem* 2.7.** *A multifunction* $P : (M, \tau) \to (N, \sigma)$ *is u.*$gp\alpha$*-c if and only if for each open set* B *in* N, $P^{-}(B)$ *is open in* M.

Proof. Let $B \in O(N)$ and $m \in P^{+}(B)$. Then by u.$gp\alpha$-c, there exists $V \in gp\alpha - O(M)$ with $P(V) \subseteq B$, where $P^{+}(B) \in O(M)$. Let $P^{+}(B)$ is open and $m \in P^{-}(B)$, then $P^{+}(B)=\{m \in B : P(M) \subseteq B\}$. Thus, P is u.$gp\alpha$-c. □

***Theorem* 2.8.** *A multifunction* $P : M \to N$ *is l.*$gp\alpha$*-c if and only if for each open set* B *in* N, $P^{-}(B)$ *is open in* M.

Proof. Let $B \in O(N)$ and $m \in P^{+}(B)$ Then by $l.gp\alpha$-c, there exists $V \in gp\alpha - O(M)$ with $P(V) \cap B \neq \phi$. As $v \in V$ and so $P^{-}(B) \in O(M)$. Suppose, $P^{-}(B)$ is open and $m \in P^{-}(B)$. Then $P^{-}(B)=\{m \in M : P(M) \cap B \neq \phi\}$. Hence, P is $l.gp\alpha$-c. □

***Theorem* 2.9.** *The following statements are equivalent for a multifunction* $P : (M, \tau) \to (N, \sigma)$

(i) P *is u.*$gp\alpha$*-c.*

(ii) $P(gp\alpha - cl(B)) \subseteq cl(P(B))$ *for each* $B \subset M$.

(iii) $gp\alpha - cl(P^{+}(A)) \subseteq P^{+}cl(A))$ *for each* $A \subset N$.

(iv) $P^-(int(A)) \subseteq gp\alpha - int(P^-(A))$ *for each* $A \subset N$.

(v) $int(P(B)) \subseteq P(gp\alpha - int(B))$ *for each* $B \subset M$.

Proof. $(i) \to (ii)$: Let $B \subseteq M$, we have $P(B) \subset cl(P(B))$, where $cl(P(B))$ is closed in N. As, P is u.$gp\alpha$-c, $B \subset P^+cl(P(B))$. From Theorem 2.4, $P^+cl(P(B))$ is $gp\alpha$-closed in M. Thus $gp\alpha - cl(B) \subseteq P^+cl(P(B)))$ and so $P(gp\alpha - cl(B)) \subseteq cl(P(B))$.
$(ii) \to (iii)$: Let $A \subset N$, so $P^+(A) \subset M$. From (ii), $P(gp\alpha - cl(P^+(A))) \subseteq cl(P(P^+(A))) = cl(A)$, thus $gp\alpha - cl(P^+(A)) \subseteq P^+cl(A)$.
$(iii) \to (iv)$: Let $A \subset N$, Apply (iii) to $N \setminus A$ then $gp\alpha - cl(P^+(N \setminus A)) \subseteq P^+cl(N \setminus A)) \Leftrightarrow gp\alpha - cl(M \setminus P^-(A)) \subseteq P^+(N \setminus int(A)) \Leftrightarrow M \setminus gp\alpha - int(P^-(A)) \subseteq M \setminus P^-(int(A)) \Leftrightarrow P^-(int(A)) \subseteq gp\alpha - int(P^-(A))$.
$(iv) \to (v)$: Let $B \subset M$, so $P(B) \subset N$. From (iv), $P^-(int(P(A)) \subseteq gp\alpha - int(P^-(P(A))) = gp\alpha - int(A)$. Thus, $int(P(A)) \subseteq P(gp\alpha - int(A))$.
$(v) \to (i)$: Let $m \in M$ and $A \in O(N, P(m))$. So, $m \in P^+(A)$ where $P^+(A) \subset M$. From (v), we have $int(P(P^+(A))) \subseteq P(gp\alpha - int(P^+(A)))$. Then, $int(A) \subseteq P(gp\alpha - int(P^+(A)))$. As A is open, $A \subseteq P(gp\alpha - int(P^+(A)))$, that is $P^+(A) \subseteq gp\alpha - int(P^+(A))$. Thus $P^+(A) \in gp\alpha - O(M, m)$ and $P(P^+(A)) \subseteq A$. Hence P is u.$gp\alpha$-c. $\square$

***Theorem* 2.10.** *Let* $P : M \to N$ *be u.*$gp\alpha$*-c. If* N *is closed subset of* Q*, then* $P : M \to Q$ *is u.*$gp\alpha$*-c.*

Proof. Let $A \subset C(Q)$ and so $A \cap Q$ is closed in Q. Then $A \cap Q$ is $gp\alpha$-closed. By u.$gp\alpha$-c $P^+(A \cap Q) \in gp\alpha - C(M)$ with $P(m) \in N$ for all $m \in M$. Thus, $P^+(A) = P^+(A \cap Q)$ is $gp\alpha$-closed in M. From Theorem 2.4, $P : M \to Q$ is u.$gp\alpha$-c. $\square$

***Theorem* 2.11.** *Let* $P : M \to N$ *be u.*$gp\alpha$*-c and* $A \in gp\alpha - C(M)$*. Then* $P \setminus A : A \to N$ *is u.*$gp\alpha$*-c.*

Proof. Let $B \in C(N)$. As P is u.$gp\alpha$-c then $P^+(B) \in gp\alpha - C(M)$. As intersection of two $gp\alpha$-closed sets is closed, then $P^+(B) \cap A = A_1$ where $A_1 \in gp\alpha - C(M)$, and so $(P \setminus A)^+(B) = A_1$ is $gp\alpha$-closed in M. Thus $P \setminus A$ is u.$gp\alpha$-c. $\square$

***Theorem* 2.12.** *If* $P_1 : M \to N$*,* $P_2 : N \to Q$ *be two multifunctions. Then* $P_1 \circ P_2 : M \to Q$ *is u.*$gp\alpha$*-c if* P_2 *is pre-irresolute and* P_1 *is u.*$gp\alpha$*-c.*

Proof. Let $A \in pcl(Q)$ and so $P_2^+(A) \in pcl(N)$ as P_2 is pre-irresolute. Hence $P_2(A) \in gp\alpha - C(N)$, since P_1 is u.$gp\alpha$-c. Then $P_1^+(P_2^+(A)) \in gp\alpha - C(M)$ and so $P_2 \circ P_1$ is u.$gp\alpha$-c. $\square$

***Theorem* 2.13.** *If $P : M \to N$ is u.$gp\alpha$-c injective with N is T_1-space then M is $gp\alpha - T_1$.*

Proof. Let N be T_1 space. Then for each $m_1, m_2 \in M$ such that $m_1 \neq m_2$, there exist $A, B \in O(N)$ such that $P_1(m_1) \in A$, $P_1(m_2) \notin A$ and $P_1(m_1) \notin B$, $P_1(m_2) \in B$. Since, P is u.$gp\alpha$-c there exist $U, V \in gp\alpha - O(M)$ with $m_1 \in U$, $m_1 \notin V$ and $m_2 \notin U$, $m_2 \in V$, that is $m_1 \in U$, $m_2 \in V$ and $P_1(U) \subseteq A$, $P_1(V) \subseteq B$. Thus M is $gp\alpha - T_1$. $\square$

***Theorem* 2.14.** *Let $P : M \to N$ is u.$gp\alpha$-c injective and N is T_2. Then M is $gp\alpha - T_2$.*

Proof. Let $m_1, m_2 \in M$ such that $m_1 \neq m_2$. Then, there exist $G, H \in O(N)$ such that $P(m_1) \in G$, $P(m_2) \in H$. As P is u.$gp\alpha$-c there exist $A, B \in gp\alpha - O(M)$ such that $P(G) \subseteq A$, $P(H) \subseteq B$ with $A \cap B = \phi$ and so $G \cap H = \phi$. Hence M is $gp\alpha - T_2$. $\square$

***Theorem* 2.15.** *For a multifunction $P : M \to N$ is u.$gp\alpha$-c image of $gp\alpha$-connected space is $gp\alpha$-connected.*

Proof. Let $P : M \to N$ is u.$gp\alpha$-c. Suppose N is not connected and $N = A \cup B$ with a partition of N, where $A \in O(N)$ and $B \in C(N)$. Since P is u.$gp\alpha$-c, $P^+(A), P^+(B) \in gp\alpha - O(M)$ where $P^+(A) \cap P^+(B) = \phi$ and $M = P^+(A) \cup P^+(B)$ is a partition of M, which contradicts that, M is $gp\alpha$-connected. Hence N must be connected. $\square$

***Definition* 2.16.** *For a multifunction $P : M \to N$, the graph multifunction $G_p : M \to M \times N$ is defined as $G_p(m) = \{m\} \times P(m)$ holds for every $m \in M$.*

***Lemma* 2.17.** *[12] For a multifunction $P : M \to N$,*

(i) $G_p^+(A \times B) = A \cap P^+(B)$

(ii) $G_p^-(A \times B) = A \cap P^-(B)$ *for any subsets $A \subseteq M$ and $B \subseteq N$.*

***Theorem* 2.18.** *Let $P : M \to N$ be a multifunction. If the graph multifunction G_p is u.$gp\alpha$-c, then P is u.$gp\alpha$-c, where $G_p : M \to M \times N$ is defined as $G_p(m) = \{m\} \times P(m)$.*

Proof. Let $m \in M$ and $V \in O(N, P(m))$, then $M \times V \in O(M \times N)$ and $G_p(m) \subseteq M \times V$ As G_p is u.$gp\alpha$-c there exists $U \in gp\alpha - O(M, m)$ such that $G_p(U) \subseteq M \times V$. Thus $U \subseteq G_p^+(M \times V)$, and by Lemma 2.17, $G_p^+(M \times V) = M \cap P^+(V) = P^+(V)$ and so $U \subseteq P^+(V)$. Thus P is u.$gp\alpha$-c. □

***Theorem* 2.19.** *Let $P : M \to N$ be a multifunction. If the graph multifunction G_p is $l.gp\alpha$-c then P is $l.gp\alpha$-c.*

Proof. Let $m \in M$ and $V \in O(N)$ with $m \in P^-(V)$. Then $M \times V$ is open in $M \times N$, and also, we have $G_p(m) \cap (M \times V) = (\{m\}) \times P(m)) \cap (M \times V) = \{m\}) \times P(m) \cap V \neq \phi$. As G_p is $l.gp\alpha$-c, there exists $U \in gp\alpha - O(M, m)$ such that $U \in G_p^-(M \times V)$. From Lemma 2.17 $G_p^-(M \times V) = M \cap P^-(V) = P^-(V)$ and so $U \subseteq P^-(V)$. Thus P is $l.gp\alpha$-c. □

***Theorem* 2.20.** *Let (M, τ), (N, σ), (Q, η) be topological spaces and $P_1 : M \to N$ and $P_2 : N \to Q$ be multifunctions. Let $P_1 \times P_2 : M \to N \times Q$ be a multifunction defined by $(P_1 \times P_2)(m) = P_1(m) \times P_2(m)$ holds for each $m \in M$ and $(P_1 \times P_2)$ is u.$gp\alpha$-c, then P_1 and P_2 are u.$gp\alpha$-c.*

Proof. Let $m \in M$ and $V \subseteq N$ and $W \subseteq Q$ be open sets with $m \in P_1^+(V)$ and $m \in P_2^+(W)$ and so $P_1(m) \subseteq V$ and $P_2(m) \subseteq W$. Thus $(P_1 \times P_2)(m) = P_1(m) \times P_2(m) \subseteq V \times W$ and so $m \in (P_1 \times P_2)^+(V \times W)$. As $P_1 \times P_2$ is u.$gp\alpha$-c, there exists $gp\alpha$-open set U containing m such that $U \subseteq (P_1 \times P_2)^+(V \times W)$, that is $U \subseteq P_1^+(V)$ and $U \subseteq P_2^+(W)$. Thus P_1 and P_2 are u.$gp\alpha$-c. □

***Theorem* 2.21.** *Let $P : M \to N$ be compact multifunction. Then G_p is u.$gp\alpha$-c if and only if P is u.$gp\alpha$-c.*

Proof. Suppose $G_p : M \to N$ be u.$gp\alpha$-c. Let $m \in M$ and $V \in O(N, P(m))$. Since, $M \times V$ is open in $M \times N$ and $G_p(m) \subseteq M \times N$, so there exists $U \in gp\alpha - O(M, m)$ such that $G_p(U) \subseteq M \times N$. From Lemma 2.17 $U \subseteq G_p^+(M \times V) = P^+(V)$ and $P(U) \subseteq V$. Hence P is

u.$gp\alpha$-c.
Conversely, let P be u.$gp\alpha$-c. Let $m \in M$ and W be any open set in $M \times N$ containing $G_p(m)$. Then for each $n \in P(m)$, there exist $U(n) \subseteq M$ and $V(n) \subseteq N$ such that $(m,n) \in U(n) \times V(n) \subseteq W$. Then the family $\{V(n); n \in P(m)\}$ is an open cover of $P(m)$. As $P(m)$ is compact there exists finite number of points say $n_1, n_2, ...n_k$ in $P(m)$ such that $P(m) \subseteq \{V(n_i); i = 1, 2, ...k\}$. Let us put $U = \cap\{U(n_i); i = 1, 2....k\}$ and $V = \cup\{V(n_i); i = 1, 2, ...k\}$, then U and V are open sets in M and N respectively and $\{m\} \times P(m) \subseteq U \times V \subseteq W$. As P is u.$gp\alpha$-c there exists $U \in gp\alpha - O(M,m)$ such that $P(U_1) \subseteq V$. From Lemma 2.17 $U \cap U_1 \subseteq U \cap P^+(V) = G_p^+(U \times V) \subseteq G_p^+(W)$. Thus $U \cap U_1 \in gp\alpha - O(M,m)$ and $G_p(U \cap V) \subseteq W$. Thus G_p is u.$gp\alpha$-c. □

3 Applications of Generalized pre α-Continuous Multifunctions

***Definition* 3.1.** [15] *Let (M,τ) be a topological space. Then (M,τ) is said to be $gp\alpha$-T_2 space if for each $m,n \in M$ with $m \neq n$, there exist disjoint $gp\alpha$-open sets U and V such that $m \in U$ and $n \in V$.*

***Theorem* 3.2.** *Let P_1 and P_2 be u.$gp\alpha$-c and closed multifunctions from a space M to a normal space N. Then the set $A = \{m : P_1(m) \cap P_2(m) \neq \phi\}$ is closed in M.*

Proof. Let $m \in M - A$. Then $P_1(m) \cap P_2(m) = \phi$. As P_1 and P_2 are closed multifunctions and N is normal, there exist $U, V \in O(N)$ with $U \cap V = \phi$ containing $P_1(m)$ and $P_2(m)$ respectively. As P_1 and P_2 are u.$gp\alpha$-c, the sets $P_1^+(U)$ and $P_2^+(V) \in O(m)$. Let $W = P_1^+(U) \cap P_2^+(V)$. Then $W \in gp\alpha - O(M,m)$ with $W \cap A = \phi$. Thus A is closed. □

***Theorem* 3.3.** *Let $P : M \to N$ be u.$gp\alpha$-c and closed multifunction from a space M to a normal space N with $P(m) \cap P(n) = \phi$, where $m,n \in M$ with $m \cap n = \phi$. Then M is $gp\alpha$-T_2.*

Proof. Let $m,n \in M$ with $m \neq n$ and so $P(m) \cap P(n) = \phi$. As N is normal, there exist $U \in O(N, P(m))$, $V \in O(N, P(n))$ with $U \cap V = \phi$.

So, $P^+(U) \in gp\alpha - O(N,m)$ and $P^+(V) \in gp\alpha - O(N,n)$. Thus, M is $gp\alpha$-T_2. $\square$

***Definition* 3.4.** *The graph $G(P)$ of the multifunction $P : M \to N$ is $gp\alpha$-closed with respect to M, if for each $(m,n) \notin G(P)$ there exist $U \in gp\alpha - O(M,m)$ and $V \in gp\alpha - O(M,n)$ wirh $(U \times V) \cap G(P) = \phi$.*

***Definition* 3.5.** *A subset A of a space M is called α-paracompact [21] if every open cover of A in M has a locally finite open refinement in M which covers A.*

***Theorem* 3.6.** *Let $P : M \to N$ be u.$gp\alpha$-c and α-paracompact multifunction into a T_2- space N. Then $G(P)$ is $gp\alpha$-closed with respect to M.*

Proof. Let $m_0, n_0 \notin G(P)$. Then $n_0 \notin P(m_0)$, holds for every $n \in P(m_0)$. Then there exist $V_n \in O(N,n)$ and $W_n \in O(N,n_0)$ with $V_n \cap W_n = \phi$. Then $\{V_n : n \in P(m_0)\}$ is an open cover of $P(m_0)$. Thus there is a locally finite open cover $\xi = \{U_\beta : \beta \in \Lambda\}$ of $P(m_0)$ which refines $\{V_n : n \in P(m_0)\}$. Thus there exists an open neighbourhood W_0 of n_0 such that W_0 intersect only finitely many members $U_{\beta_1}, U_{\beta_2}, U_{\beta_3}, ...U_{\beta_i}$ of ξ, and there exist finitely many points $n_1, n_2, n_3, ...n_i$ of $P(m_0)$ such that $U_{\beta_k} \subset V_{(n_k)}$ where $1 \leq k \leq i$. Hence the set $W = W_0 \cap [\bigcap_{k=1}^{i} W_{(n_k)}]$ becomes an open neighbourhood of n_0 such that $W \cap (\cup\xi) = \phi$. As P is $u.gp\alpha - c$, there exists $U \in gp\alpha - O(M, m_0)$ such that $P(U) \subset \cup\xi$. Thus $(U \times W) \cap G(P) = \phi$ and so $G(P)$ is $gp\alpha$-closed with respect to M. $\square$

***Definition* 3.7.** *A topological space M is $gp\alpha$-compact if every $gp\alpha$-open cover of M has a finite subcover.*

***Theorem* 3.8.** *Let $P : M \to N$ be compact surjection, u.$gp\alpha$-c. If M is $gp\alpha$-compact Then N is compact.*

Proof. Let ξ be an open cover of N. If $m \in M$, then $P(m) \in \cup\xi$ and so ξ is an open cover of $P(m)$. As $P(m)$ is compact, there exists a finite subfamily $\xi_{n(m)}$ of ξ such that $P(m) \subseteq \cup\xi_{n(m)} = V_m$ and so V_m is open set in N. Since P is u.$gp\alpha$-c. $P^+(V_m) \in gp\alpha - O(M)$

and so $\Lambda = \{P^+(V_m) : m \in M\}$ is a $gp\alpha$-open cover of M. By $gp\alpha$-compactness of M, there exist points $m_1, m_2, m_3, ...m_n \in M$ such that $M \subset \cup\{P^+(V_{m_i}) : m_i \in M, i = 1, 2, 3, ...n\}$ and so $N = P(M) = P(\cup\{P^+(V_{m_i}) : i = 1, 2, 3, ...n\}) \subset \cup\{V_{m_i} : i = 1, 2, 3, ...n\} \subset \{\xi_{n(m_i)} : i = 1, 2, 3, ...n\}$. Hence N is compact. $\square$

4 Upper (Lower) Generalized pre α-Irresolute Multifunctions

***Definition* 4.1.** *A multifunction $P : M \to N$ is called a*

(i) upper $gp\alpha$-irresolute (briefly u.$gp\alpha$-I) if for each $m \in M$ and each $V \in gp\alpha - O(N, P(m))$, there exists $U \in gp\alpha - O(M, m)$ such that $P(U) \subseteq V$.

(ii) lower $gp\alpha$-irresolute (briefly l.$gp\alpha$-I) if for each $m \in M$ and each $gp\alpha$-open set V with $P(m) \cap V \neq \phi$, there exists $U \in gp\alpha - O(M, m)$ such that $U \subseteq P^-(V)$.

***Theorem* 4.2.** *For a multifunction $P : M \to N$ the following statements are equivalent:*

(i) P is u.$gp\alpha$-I.

(ii) for each $m \in M$, for each $gp\alpha$-neighbourhood V of $P(m)$, $P^+(V)$ is $gp\alpha$-neighbourhood of m.

(iii) for each $m \in M$, for each $gp\alpha$-neighbourhood V of $P(m)$,there exists $gp\alpha$-neighbourhood U of m with $P(U) \subseteq V$.

(iv) $P^+(V) \in gp\alpha - O(M)$ for each $V \in gp\alpha - O(N)$.

(v) $P^-(V) \in gp\alpha - C(M)$ for each $V \in gp\alpha - C(N)$.

(vi) $gp\alpha - cl(P^-(B)) \subset P^-(gp\alpha - cl(B))$ for each $B \subset N$.

Proof. $(i) \to (ii)$: Let $m \in M$ and W be a $gp\alpha$-neighbourhood of $P(m)$. Then there exists $V \in gp\alpha - O(N)$ with $P(m) \subset V \subset W$. As P is u.$gp\alpha$-I, there exists $U \in gp\alpha - O(M, m)$ such that $P(U) \subseteq V$. Thus,

$m \in U \subset P^+(V) \subset P^+(W)$ and so $P^+(W)$ is a $gp\alpha$-neighbourhood of m.
$(ii) \to (iii)$: Let $m \in M$ and V be a $gp\alpha$-neighbourhood of $P(m)$. Let $U = P^+(V)$. Then by (ii), U is $gp\alpha$-neighbourhood of m with $P(U) \subseteq V$.
$(iii) \to (iv)$: Let $V \in gp\alpha - O(N)$ and $m \in P^+(V)$. Then there exists $gp\alpha$-neighbourhood G of m with $P(G) \subseteq V$. Thus for some $U \in gp\alpha - O(M, m)$ with $U \subseteq G$ and $P(U) \subseteq V$. So, $m \in U \subset P^+(V)$ and hence $P^+(V) \in gp\alpha - O(N)$
$(iv) \to (v)$: Let $A \in gp\alpha - C(N)$, then $M \setminus P^-(A) = P^+(N \setminus K) \in gp\alpha - O(M)$. Thus $P^-(A) \in gp\alpha - C(N)$.
$(v) \to (vi)$: Let $B \subset N$. Since $gp\alpha - cl(B)$ is $gp\alpha$-closed in N. So $P^-(gp\alpha - cl(B))$ is $gp\alpha$-closed in M with $P^-(B)) \subset P^-(gp\alpha - cl(B)$. Thus $gp\alpha - cl(P^-(B)) \subset P^-(gp\alpha - cl(B))$.
$(vi) \to (i)$: Let $m \in M$ and $V \in gp\alpha - O(N)$ with $P(m) \subseteq V$. So $P(m) \cap (N \setminus V) = \phi$. Hence $m \notin P^-(N \setminus V)$. From (vi) $m \in gp\alpha - cl(P^-(N \setminus V))$ and so there exists $U \in gp\alpha - O(M, m)$ such that $U \cap P^-(N \setminus V) = \phi$. Thus $P(U) \subseteq V$ and so P is u.$gp\alpha$-I. $\square$

***Theorem* 4.3.** *The following statements are equivalent for a multifunction $P : M \to N$:*

(i) *P is l.$gp\alpha$-I.*

(ii) *for each $V \in gp\alpha - O(N)$ and each $m \in P^-(V)$, there exists $U \in gp\alpha - O(M, m)$ such that $U \subset P^-(V)$.*

(iii) *$P^-(V) \in gp\alpha - O(M)$ for each $V \in gp\alpha - O(N)$.*

(iv) *$P^+(K) \in gp\alpha - C(M)$ for each $K \in gp\alpha - C(N)$.*

(v) *for each $A \subset M$, $P(gp\alpha - cl(A)) \subset gp\alpha - cl(P(A))$.*

(vi) *$gp\alpha - cl(P^+(B)) \subset P^+(gp\alpha - cl(B))$, for each $B \subset N$.*

Proof. $(i) \to (ii)$: Follows from definition 4.1.
$(ii) \to (iii)$: Let $V \in gp\alpha - O(N)$ with $m \in P^-(V)$. From (ii) there exists $U \in gp\alpha - O(M, m)$ such that $U \subset P^-(V)$. Thus, $m \in U \subset cl(int(U)) \cup int(cl(U)) \subset cl(int(P^-(U))) \cup int(cl(P^-(U)))$. So $P^-(V) \in$

$gp\alpha - O(M)$
$(iii) \to (iv)$: Let $K \in gp\alpha - C(N)$, then $M \setminus P^+(K) = P^-(N \setminus K) \in gp\alpha - O(M)$ and so $P^+(K) \in gp\alpha - C(M)$.
$(iv) \to (v)$: Follows from the definition.
$(v) \to (vi)$: Follows from the definition.
$(vi) \to (i)$:Let $m \in M$ and $V \in gp\alpha - O(N)$ with $P(m) \cap V \neq \phi$. So $P(m) \cap (N \setminus V) = \phi$. Then $P(m) \nsubseteq N \setminus V$ and $m \notin P^+(N \setminus V)$. Since $N \setminus V \in gp\alpha - C(N)$, and from (vi), $m \notin gp\alpha - cl(P^+(N \setminus V))$. So there exists $U \in gp\alpha - O(M, m)$ with $U \cap P^-(N \setminus V) = U \cap (M \setminus P^-(V)) = \phi$. Thus $U \subset M \setminus (M \setminus P^-(V)) = P^-(V)$, that is $U \subset P^-(V)$. Hence P is $l.gp\alpha$-I. $\square$

***Lemma* 4.4.** *Let $P : M \to N$ be a multifunction. Then $(gp\alpha - cl(P))^-(V) = P^-(V)$, for each $V \in gp\alpha - O(N)$.*

Proof. Let $V \in gp\alpha - O(N)$ with $m \in (gp\alpha - cl(P))^-C(V)$. So $V \cap (gp\alpha - cl(P))(m) \neq \phi$, as $V \in gp\alpha - O(N)$ and so $V \cap P(m) \neq \phi$. Thus $m \in P^-(V)$
Conversely, let $m \in P^-(V)$. Then $V \cap P(m) \subseteq (gp\alpha - cl(P))(m) \cap V \neq \phi$, and so $m \in (gp\alpha - cl(P))^-(V)$. Thus $(gp\alpha - cl(P))^-(V) = P^-(V)$. $\square$

***Lemma* 4.5.** *Let $A, B \subseteq M$. Then*

(i) If $A \in gp\alpha - O(M)$ and $B \in M$ then $A \cap B \in gp\alpha - O(B)$

(ii) If $A \in gp\alpha - O(B)$ and $B \in gp\alpha - O(M)$ then $A \in gp\alpha - O(M)$.

***Theorem* 4.6.** *Let $P : M \to N$ be a multifunction, and $U \in O(M)$. If P is u.gpα-I(resp $l.gp\alpha - I$) then $P_{l_U} : U \to N$ is an $u.gp\alpha - I(respl.gp\alpha - I)$.*

Proof. Let $V \in gp\alpha - O(N)$, let $m \in U$ and $m \in P^-_{l_U}(V)$. As P is $l.gp\alpha$-I. there exists $G \in gp\alpha - O(M, m)$ with $G \subseteq P^-(V)$ and so $m \in G \cap U \in gp\alpha - O(U)$ and $G \cap U \subseteq P_{l_U}(V)$. Thus P_{l_U} is $l.gp\alpha$-I. $\square$

Similarlly, we can prove for u.$gp\alpha$-I.

***Theorem* 4.7.** *Let $\{U_i : i \in \Lambda\}$ be an open cover of M. A multifunction $P : M \to N$ is u.gpα-I if and only if the restriction $P_{l_{U_i}} : U_i \to N$ is u.gpα-I for each $i \in \Lambda$.*

Proof. Suppose P is u.$gp\alpha$-I. Let $i \in \Lambda$, $m \in U_i$ and $V \in gp\alpha - O(N, P_{l_{U_i}}(m))$. As P is u.$gp\alpha$-I and $P(m) = P_{l_{U_i}}(m)$ there exists $G \in gp\alpha - O(M, m)$ with $P(G) \subseteq V$, put $V = G \cap U_i$ then $m \in U \in gp\alpha - O(U_i, m)$ and $P_{l_{U_i}}(U) = P(U) \subseteq V$ thus $P_{l_{U_i}}$ is $ugp\alpha - I$. On the other hand let $m \in M$ and $V \in gp\alpha - O(N)$ containing $P(m)$ thus there exists $i \in \Lambda$ with $m \in U_i$, since $P_{l_{U_i}}$ is u.$gp\alpha$-I, and $P(m) = P_{l_{U_i}}(m)$ there exists $U \in gp\alpha - O(U_i, m)$ with $P_{l_{U_i}}(U) \subseteq V$ and so $U \in gp\alpha - O(M, m)$ with $P(U) \subseteq V$. Hence P is u.$gp\alpha$-I. □
Similarlly, we can prove for $l.gp\alpha$-I.

***Definition* 4.8.** *A subset B of a space M is said to be a*

(i) $gp\alpha$-compact relative to M ($gp\alpha$-Lindelof relative to M) if every cover of B by $gp\alpha$-open sets of M has a finite (countable) subcover.

(ii) $gp\alpha$-compact ($gp\alpha$-Lindelof). If M is $gp\alpha$-compact(resp $gp\alpha$-Lindelof) relative to M.

***Theorem* 4.9.** *If $P : M \to N$ be u.$gp\alpha$-I multifunction and $P(m)$ is $gp\alpha$-compact relative to N for each $m \in M$. If B is $gp\alpha$-compact relative to M, then $P(B)$ is $gp\alpha$-compact relative to N.*

Proof. Let $\{V_i : i \in \Lambda\}$ be a cover of $P(B)$ by $gp\alpha$-open sets in N. Then for each $m \in B$, there exists a finite subset $\Delta(m) \in \Lambda$ with $P(m) \subset \cup\{V_i : i \in \Lambda(m)\}$ and so $P(m) \subset V(m) \in gp\alpha - O(N)$, thus there exist finite number of points of B, $m_1, m_2, m_3, ...m_k$ with $B \subset \cup\{V(m_i) : i = 1, 2, 3, ...k\}$. Thus $P(B) \subset P(\bigcup_{i=1}^{k}\{V_i(m_i)\}) \subset \bigcup_{i=1}^{k} P(V_i(P_i)) \subset \bigcup_{i=1}^{k} V(P_i) \subset \bigcup_{i=1}^{k} \bigcup_{i \in \Lambda(m_i)} V_i$. Thus $P(B)$ is $gp\alpha$-compact relative to N. □

***Corollary* 4.10.** *Let $P : M \to N$ be a multifunction and u.$gp\alpha$-I surjective multifunction and $P(B)$ is $gp\alpha$-compact relative to N for each $m \in M$. If M is $gp\alpha$-compact, then N is $gp\alpha$-compact.*

***Theorem* 4.11.** *If $P : M \to N$ be u.$gp\alpha$-I and $P(m)$ is $gp\alpha$-Lindelof relative to N for each $m \in M$ if B is $gp\alpha$-Lindelof relative to M, then $P(B)$ is $gp\alpha$-Lindelof relative to N.*

***Definition* 4.12.** [15] *A space M is said to be gpα-normal (briefly gpα-N) if for any pair of distinct gpα-closed sets A and B in M there exists disjoint open sets U and V in M such that $A \subseteq U$, $B \subseteq V$.*

***Theorem* 4.13.** *The set of a point m of M at which a multifunction $P : M \to N$ is not u.gpα-I(resp $l.gp\alpha - I$) is identical with the union of the gpα-frontiers of the upper(lower) inverse image of gpα-open sets containing (respectively meeting) $P(m)$.*

Proof. Let m be a point of M at which P is not u.$gp\alpha - I$ then there exists $V \in gp\alpha - O(N)$ containing $P(m)$ with $U \cap (M \setminus P^+(V)) \neq \phi$ for each $U \in gp\alpha - O(M, m)$ then $m \in gp\alpha - cl(M \setminus P^+(V))$ as $m \in P^+(V)$ then we have $m \in gp\alpha - cl(P^+(N))$ and so $m \in gp\alpha - F_r(P^+(B))$. On the other hand $V \in gp\alpha - O(N)$ containing $P(m)$ and $m \in gp\alpha - F_r(P^+(B))$ let P is u.$gp\alpha$-I then there exists $U \in gp\alpha - O(M, m)$ with $P(U) \subseteq V$, thus $m \in U \subseteq gp\alpha - int(P^+(V))$ which contradicts to the fact that $m \in gp\alpha - F_r(P^+(V))$. Hence P is not u.$gp\alpha$-I. □
Similarlly we can prove the theorem related to $l.gp\alpha$-I.

***Theorem* 4.14.** *Let $P : M \to N$ be an u.$gp\alpha - I$ injective multifunction and point closed from a topological space M to gpα-normal space N, then M is $gp\alpha - T_2$-space.*

Proof. Let $m, n \in M$ with $m \neq n$ then $P(m) \cap P(n) = \phi$ as P is injective. By $gp\alpha$-normality of N, there exist disjoint open sets U and V containing $P(m)$ and $P(n)$ respectively. Thus there exist disjoint $gp\alpha$-open sets $P^+(U)$ and $P^+(V)$ containing m and n respectively such that $G \subseteq P^+(U)$ and $H \subseteq P^+(V)$ and so $G \cap H = \phi$. So M is $gp\alpha - T_2$-space. □

***Definition* 4.15.** *A multifunction $P : M \to N$ is said to have gpα-closed graph if for each $(m, n) \notin G(P)$ there exist $U \in gp\alpha - O(M, m)$ and $V \in gp\alpha - O(N, n)$ with $(U \times V) \cap G(P) = \phi$.*

***Theorem* 4.16.** *Let $P : M \to N$ be a multifunction from a space M into gpα-compact space N if $G(P)$ is gpα-closed. Then P is u.gpα-c.*

Proof. Suppose P is not u.$gp\alpha$-c then there exists non empty closed subset B in N with $P^-(B)$ is not $gp\alpha$-closed in M. Let us assume that

$P^{-}(B) \neq \phi$ then there exists point $m_0 \in gp\alpha - cl(P^{-}(B) \setminus P^{-}(B))$, thus for each point $n \in B$ we have $(m_0, n_0) \notin G(P)$, as P is $gp\alpha$-closed graph there exists $gp\alpha$-open sets $U(n)$ and $V(n)$ containing m_0 and n respectively with $(U(n) \times V(n)) \cap G(P) = \phi$. Then $\{N \setminus B\} \cup \{V(n) : n \in B\}$ is a $gp\alpha$-open cover of N and so it has a subcover $\{N \setminus B\} \cup \{V(n_i) : n_i \in B, 1 \leq i \leq k\}$ put $U = \bigcap\limits_{i=1}^{k} U(n_i)$ and $V = \bigcup\limits_{i=1}^{k} V(n_i)$ then we can verify that $B \subseteq V$ and $(U \times V) \cap G(P) = \phi$ as U is $gp\alpha$-neighbourhood of m_0 $U \cap P^{-}(B) = \phi$ and so $\phi \neq (U \times B) \cap G(P) \subseteq (U \times V) \cap G(P)$ which is contradiction. Thus P is u.$gp\alpha$-c. $\square$

***Definition* 4.17.** *Let $A \subseteq M$ then $P : M \to A$ is called retracting multifunction if $m \in P(m)$ for each $m \in A$.*

***Theorem* 4.18.** *Let $P : M \to M$ be a u.$gp\alpha - I$ multifunction of a T_2-space M into itself if $P(m)$ is α-paracompact for each $m \in M$ then $A = \{m : m \in P(m)\}$ is $gp\alpha$-closed.*

Proof. Let $m_0 \in gp\alpha - cl(A) \setminus A$ then $m_0 \notin P(m_0)$ as M is T_2 for each $m \in P(m_0)$ there exist disjoint open sets $U(m)$ and $V(m)$ containing m_0 and m respectively. Then $\{V(m) : m \in P(m_0)\}$ is an open cover of $P(m_0)$ by α-paracompactness of $P(m_0)$. $\{V(m) : m \in P(m_0)\}$ has a locally finite open refinement say $W = \{W_\beta : \beta \in I\}$ covers $P(m_0)$. Thus there exist open neighbourhood U_0 and m_0 such that U_0 interests only finitely many members $W_{\beta_1}, W_{\beta_2}......W_{\beta_k}$ of W_k. Consider $m_1, m_2....m_k$ of $P(m_0)$ such that $W_{\beta_i} \subset V(m_i)_k$ where $1 \leq i \leq k$ and $U = U_0 \cap (\bigcap\limits_{i=1}^{k} U(m_i))$ and so U is an open neighbourhood of m_0 with $U \cap (W_\beta) = \phi$ as P is u.$gp\alpha$-I there is a $G \in gp\alpha - O(M, m_0)$ with $G \subseteq P^{+}(\bigcup\limits_{\beta \in I} W_\beta)$ then $G \cap U$ is a $gp\alpha$-nbd of m_0 and $(G \cap U) \cap A = \phi$, which contradicts that $m_0 \in gp\alpha - cl(A)$. $\square$

5 Conclusion

In this present work, we have analyzed new weaker forms of new types of continuous functions called upper $gp\alpha$-continuous multifunctions and lower $gp\alpha$-continuous multifunctions. Established the properties and preservation theorems of a upper $gp\alpha$-continuous (resp. lower$gp\alpha$-continuous) multifunctions. Further, upper(lower) $gp\alpha$-irresolute multifunctions are studied here. There is a scope to study and extend these newly defined concepts in topological spaces.

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