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Interpolation in Spaces of Functions

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Abstract: In this paper we consider the interpolation by certain functions such as trigonometric and rational functions for finite dimensional linear space X. Then we extend this to infinite dimensional linear spaces.

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1. Introduction

Interpolation theory constitutes a fundamental part of numerical analysis and almost all such books treat this topic in full detail (see [1, 4, 7]). Even though the finite dimensional case is more popular the extension to the infinite dimension has gained some interest. Here we first state the finite dimensional results and then deal with its extensions. Let X be a linear space with $dim\ X = n$. Suppose $L_1, L_2, \cdots, L_n$ are n linear functionals on X and $y_1, y_2, \cdots, y_n$ are n scalars. We would like to find

$x \in X$ such that

$$L_i x = y_i \quad ; \quad i = 1, 2, \cdots, n. \tag{1}$$

Problems of this type are called generalized interpolation problems. At this stage we should give a criteria for the linear independence of n linear functionals. This is stated in the next theorem.

Theorem 1.1. *Let X be a linear space with dim $X = n$. Choose a basis $\{x_1, x_2, \cdots, x_n\}$ of X. Then the linear functionals $L_1, L_2, \cdots, L_n$ are linearly independent if and only if the matrix $[L_i, x_j]_{i,j=1}^n$ has nonzero determinant.*

Theorem 1.2. *Let X be a linear space with dim $X = n$. Then the interpolation problem (1) has a unique solution if and only if the L_i are linearly independent in X^*.*

Sometimes we are interested in finding a polynomial whose value and derivatives up to certain order take on prescribed values at distinct points. Problems of this type are called *general Hermite interpolation.* It deals with finding a polynomial $p(x)$ that satisfies

$$p^{(i)}(x_1) = y_1^{(i)}; \qquad i = 0, 1, \cdots, m_1 - 1$$

$$\vdots$$

$$p^{(i)}(x_n) = y_n^{(i)}; \qquad i = 0, 1, \cdots, m_1 - 1.$$

The numbers $y_1^{(i)}, \cdots, y_n^{(i)}$ will already be given and we have $m_j, j = 1, \cdots, n$ conditions on $p(x)$ at the point x_j. If we define $N = m_1 + m_2 + \cdots + m_n$, then there is a polynomial $p(x)$, unique among those of degree $\leqslant N - 1$ which is a solution to the above equations.

Trigonometric interpolation

A function f is said to be periodic with period 2π if

$$f(t + 2\pi) = f(t), \qquad -\infty < t < \infty.$$

It is customary to approximate such functions $f(t)$ using trigonometric polynomials

$$p_n(t) = \sum_{k=-n}^{n} c_k e^{ikt}.$$

We study interpolation problems with $p_n(t)$ as a solution, since $p_n(t)$ contains $2n + 1$ coefficients c_k we must impose $2n + 1$ interpolating conditions. We also assume the existence of the interpolation nodes

$$0 \leqslant t_0 < t_1 < \cdots < t_{2n} < 2\pi$$

and the polynomial p_n should satisfy

$$p_n(t_i) = f(t_i) \;\; i = 0, 1, \cdots, 2n.$$

It is known that this problem has a unique solution.

Interpolation by rational functions

The rational function

$$r(x) = \frac{p(x)}{q(x)} = \frac{a_0 + a_1 x + \cdots + a_m x^m}{b_0 + b_1 x + \cdots + b_n x^n}.$$

is determined by its $m + n + 2$ coefficients

$$a_0, a_1, \cdots, a_m, b_0, b_1, \cdots, b_n.$$

On the other hand $r(x)$ determines these coefficients only up to a common factor $\lambda \neq 0$. Therefore $r(x)$ is fully determined by $m + n + 1$ conditions

$$r(x_i) = r_i, \qquad i = 0, 1, \cdots, m + n.$$

It is therefore necessary that the coefficients a_j, b_j of $r(x)$ solve the homogeneous system of linear equations

$$p(x) - r_i q(x_i) = 0, \qquad i = 0, 1, \cdots, m + n \tag{2}$$

which can also be written as

$$a_0 + a_1 x_1 + \cdots + a_m x_i^m - r_i(b_0 + b_1 x_i + \cdots + b_n x_i^n) = 0.$$

It is well known that the homogeneous linear system of equations (2) always has nontrivial solutions. For each such solution

$$r(x) = p(x)/q(x) \quad , \quad q(x) \not\equiv 0.$$

2. Extensions to Infinite Dimension

We have already considered finite dimensional linear spaces. It is now time to extend these ideas to infinte dimensional linear spaces. To avoid pathological situations we let X be a normed linear space such that X is complete in this norm, that is X is a Banach space. Examples of Banach spaces are $C[0,1]$ the space of continuous functions on [0,1] with the supermum norm, the $L^p[0,1]$ spaces and the space ℓ^1 of summable sequences. If $c = \{c_i\} \in \ell^1$ then $||c||_1 = \sum_{i=1}^{n} |c_i| < \infty$. Now let X be a Banach space. A sequence $\{f_n\}$ in X is called the unit vector basis of ℓ^1 (or an ℓ^1-basis) if there are constants $a, b > 0$ so that

$$a\sum_{i=1}^{n}|c_i| \leqslant ||\sum_{i=1}^{n} c_i f_i|| \leqslant b\sum_{i=1}^{n}|c_i| \tag{3}$$

for any scalars $c_1, \cdots, c_n$ and any n. If $\{f_n\}$ in X is the unit vector basis of ℓ^1 then one has an isomorphic copy of ℓ^1 inside of X.

Proposition 2.1. *If $\{f_n\}$ is an ℓ^1-basis of the Banach space X then the closed linear span on $\{f_n : n \in \mathbf{N}\}$ is isomorphic (linearly homeomorphic) to ℓ^1.*

Proof. Let L be the closed linear span of the sequence f_n. We define

$T : \ell^1 \to L$ by

$$T(\{c_i\}) = \sum_{i=1}^{\infty} c_i f_i.$$

Then T is continuous since $||T(\{c_i\})|| = ||\sum_{i=1}^{\infty} c_i f_i|| \leqslant b \sum_{i=1}^{\infty} |c_i| < \infty$. We now show that T is one to one. If $T(\{c_i\}) = 0$ then $\sum_{i=1}^{\infty} c_i f_i = 0$. Since $\{c_i\}$ is in ℓ^1, we have

$$a \sum_{i=1}^{\infty} |c_i| \leqslant ||\sum_{i=1}^{\infty} c_i f_i|| = 0.$$

Hence $\sum_{i=1}^{\infty} |c_i| = 0$ and $c_i = 0$ therefore T is one to one.

Now let $N = \{\sum_{i=1}^{\infty} c_i f_i : \sum_{i=1}^{\infty} |c_i| < \infty\}$. To show that N is closed we use the fact that T is bounded below, *i.e.*, $||Tc|| \geqslant \; ||c||_1$ for all $c = \{c_i\}$ in ℓ^1. From this we have $||Tc - Td|| \geqslant a||c - d||$. Now let $g_k = Tc^{(k)} = \sum_{i=1}^{\infty} c_i^{(k)} f_i$ be in N and $g_k \to g$, as $k \to \infty$ we must show that g is in N. Then $||g_k - g_l|| = ||Tc^{(k)} - Tc^{(l)}|| \geqslant a||c^{(k)} - c^{(l)}||$. Since $\{g_k\}$ is a Cauchy sequence we conclude that $\{c^{(k)}\}$ is Cauchy in ℓ^1. Hence $\{c^{(k)}\} \to c$ in ℓ^1. Therefore $g = \sum_{i=1}^{\infty} c_i f_i$.

Also $T^{-1} : L \to \ell^1$ given by $\sum_{i=1}^{\infty} c_i f_i \longmapsto \{c_i\}_{i=1}^{\infty}$ is continuous since

$$||T^{-1}(\sum_{i=1}^{\infty} c_i f_i)|| = ||\{c_i\}|| = \sum_{i=1}^{\infty} |c_i| \leqslant a^{-1}||\sum_{i=1}^{\infty} c_i \, f_i||.$$

This completes the proof. $\square$

Definition 2.2. *Let X be a Banach space. A sequence $\{f_n\}$ in X is said to be a weak-Cauchy sequence if $\lim_{n\to\infty} x^*(f_n)$ exists for all $x^* \in X^*$, the dual of X.*

The extension to infinite dimension can be accomplished via the important result of Rosenthal. This theorem was proved by Rosenthal [6] when X is a real Banach space and for a complex Banach space the theorem was proved by Dor [3].

Theorem 2.3. **(Rosenthal-Dor)** *In order that each bounded sequence in the Banach space X have weakly Cauchy subsequence, it is both necessary and sufficient that X contains no isomorphic copy of ℓ^1.*

The Rosenthal-Dor Theorem can also be stated as follows.

Theorem 2.4. *Suppose X is a Banach space and $\{f_n\}$ is a bounded sequence in X. Then there exists a subsequence $\{f_{n_k}\}$ such that either*

i) the map $\{c_k\} \to \sum_{k=1}^{\infty} c_k f_{n_k}$ is an isomorphism of ℓ^1 into X.

ii) $\{f_{n_k}\}$ is a weak-Cauchy sequence.

Now if $\{x_n\}$ is a bounded sequence in a Banach space X and $S =$ *ball* $X^* = \{x^* \in X^* : ||x^*|| \leqslant 1\}$, the unit ball of X^*, define $f_n(x^*) =$

$x^*(x_n)\forall x^* \in S, i.e., f_n = \hat{x}_n$ then we can reformulate the Rosenthal-Dor Theorem as follows.

Theorem 2.5. *Let $S = ball X^*$ and $\{f_n\} = \{\hat{x}_n\}$ be a uniformly bounded sequence of real-valued functions defined on S. Then $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ satisfying one of the following alternatives.*

i) $\{f_{n_k}\}$ in equivalent in the supremum norm to the usual ℓ^1-basis.

ii) $\{f_{n_k}\}$ converges pointwise on S.

Let X be a separable Banach space and $Y = X^*$ be a Banach space of functions defined on a bounded domain G. Before continuing we just make a few comments on the topology of these spaces. The topology of the Banach space X is given by its norm and hence it is a metric space. The topology of the dual $Y = X^*$ is however the weak star topology. In fact it is the topology of pointwise convergence. Hence if $\{x^*_\alpha\}$ is a net in X^* then $\{x^*_\alpha\} \to x^*$ weak star in X^* provided $x^*_\alpha(x) \to x^*(x)$ for every $x \in X$. Furthermore assume that for every $\lambda \in G$ the linear functional $e_\lambda : Y \to C$ given by $e_\lambda(\varphi) = \varphi(\lambda)$ is weak star continuous.

Definition 2.6. *We say that a sequence $\{\lambda_n\}$ in a plane domain G is an interpolating sequence for the Banach space $Y = X^*$ if for each* *bounded sequence* $\{a_n\} \subset \mathbf{C}$ *there exists* $\varphi \in X$ *such that* $\varphi(\lambda_n) = a_n$.

Because e_{λ_n} is a weak star continuous linear functional on X there exists $x_n \in X$ with $||x_n|| = 1$ such that $e_\lambda(\varphi) = \varphi(\lambda_n) = \varphi(x_n)$ for all $\varphi \in Y$.

We will prove the following theorem.

Theorem 2.7. *Suppose* $\{\lambda_n\}$ *is a sequence in* G *such that*

$$e_{\lambda_n}(\varphi) = \varphi(\lambda_n) = \varphi(x_n),$$

for all $\varphi \in Y$. *Assume that case (1) of the Rosenthal-Dor Theorem holds, i.e., there is a subsequence of* $\{x_n\}$ *that is an* ℓ^1 *basis for* X. *Then there is a subsequence of* $\{\lambda_n\}$ *that is interpolating for* Y.

Proof. Since $(X^*, wk^*) = X$, and each weak star continuous linear functional comes from some element of the predual we have $\varphi(\lambda_n) = \varphi(x_n)$ for all $\varphi \in X^*$.

Let $\{x_{n_k}\}$ be the subsequence of $\{x_n\}$ given by the Rosenthal-Dor Theorem and suppose that case (1) holds. i.e., the map $\{a_k\} \to \sum_{k=1}^{\infty} a_k x_{n_k}$ is an isomorphism of ℓ^1 into X. We will show that $\{\lambda_{n_k}\}$ is interpolating for Y. For this let $a = \{a_k\}$ be an element of ℓ^∞. By case (1) of Rosenthal-Dor theorem there exists an isomorphism $T : \ell^1 \to X$. Since

T is one to one and has closed range we can define $T^* : X^* \to \ell^\infty$. Since ran T^* is closed and ran $T^* = (kerT)^\perp$ we conclude that T^* is onto. Therefore there exists $\varphi \in X^* = Y$ such that $T^*\varphi = a,\ a \in \ell^\infty$. We know that $T^*\varphi = \varphi \circ T$, therefore $\varphi \circ T = a$. Now apply both sides of this to the vector $e_k = \{0, 0, \cdots, 0, 1, 0, 0, \cdots\}$ where 1 is in the k-th coordinate. Since $T(\{a_k\}) = \sum_{k=1}^{\infty} a_k x_{n_k}$ we have $T_{e_k} = x_{n_k}$. Therefore $\varphi \circ T_{e_k} = \varphi(x_{n_k}) = a_k$. Thus $\varphi(\lambda_{n_k}) = \varphi(x_{n_k}) = a_k$ for every k. Hence $\{\lambda_{n_k}\}$ is interpolating for Y. This completes the proof. $\square$

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