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A Generalization of Order Continuous Operators

M. Bakhshi
University of Mohaghegh Ardabili

K. Haghnejad Azar*
University of Mohaghegh Ardabili

Abstract. Let E be a sublattice of a vector lattice F. A net $\{x_\alpha\}_{\alpha\in\mathcal{A}} \subseteq E$ is said to be F-order convergent to a vector $x \in E$ (in symbols $x_\alpha \xrightarrow{Fo} x$ in E), whenever there exists a net $\{y_\beta\}_{\beta\in\mathcal{B}}$ in F satisfying $y_\beta \downarrow 0$ in F and for each β, there exists α_0 such that $|x_\alpha - x| \leq y_\beta$ whenever $\alpha \geq \alpha_0$. In this manuscript, first we study some properties of F-order convergence nets and we extend some results to the general cases. Let E and G be sublattices of vector lattices F and H, respectively. We introduce FH-order continuous operators, that is, an operator T between two vector lattices E and G is said to be FH-order continuous, if $x_\alpha \xrightarrow{Fo} 0$ in E implies $Tx_\alpha \xrightarrow{Ho} 0$ in G. We will study some properties of this new classification of operators and its relationships with order continuous operators.

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*Corresponding Author

1 Introduction

To state our result, we need to fix some notations and recall some definitions. A vector lattice E is an ordered vector space in which $\sup(x, y)$ exists for every $x, y \in E$. A subspace E of a vector lattice F is said to be a sublattice if for every pair of elements a, b of E the supremum of a and b taken in F belongs to E. A vector lattice is said to be Dedekind complete (resp. σ-complete) if every nonempty subset (resp. countable subset) that is bounded from above has a supremum.

A sublattice E of a vector lattice F is said to be:

1. order dense if for every $0 < x \in F$ there exists $0 < y \in E$ such that $y \leq x$.
2. majorizing if for every $x \in F$ there exists $y \in E$ such that $x \leq y$.
3. regular if for every subset A of E, $\inf A$ is the same in F and in E whenever $\inf A$ exists in E.

A Dedekind complete space F is said to be a Dedekind completion of the vector lattice E whenever E is lattice isomorphic to a majorizing order dense sublattice of F. Recall that a non-zero element $a \in E^+$ is an atom iff the ideal I_a consists only of the scalar multiples of a. Let E be a vector lattice. A net $\{x_\alpha\}_{\alpha \in \mathcal{A}} \subseteq E$ is said to be order convergent (in short o-convergent) to a vector $x \in E$ (in symbols $x_\alpha \xrightarrow{o} x$), whenever there exists a net $\{y_\beta\}_{\beta \in \mathcal{B}}$ in E satisfying $y_\beta \downarrow 0$ and for each β there exists α_0 such that $|x_\alpha - x| \leq y_\beta$ whenever $\alpha \geq \alpha_0$. Let $\{x_n\}$ be a sequence in a vector lattice E. Consider the sequence $\{a_n\}$ of Cesáro means of $\{x_n\}$, defined by $a_n = \frac{1}{n} \sum_{k=1}^{n} x_k$. Let E, G be vector lattices. An operator $T : E \to G$ is said to be order bounded if it maps each order bounded subset of E into order bounded subset of G. The collection of all order bounded operators from a vector lattice E into a vector lattice G will be denoted by $L_b(E, G)$. The vector space $E^\sim$ of all order bounded linear functionals on vector lattice E is called the order dual of E, i.e., $E^\sim = L_b(E, \mathbb{R})$. Let A be a subset of vector lattice E and Q_E be the natural mapping from E into $E^{\sim\sim}$. If $Q_E(A)$ is order bounded in $E^{\sim\sim}$, then A is said to b-order bounded in E. The concept of b-order bounded was first time itroduced by Alpay, Altin and Tonyali, see [5].

It is clear that every order bounded subset of E is b-order bounded. However, the converse is not true in general. For example, the standard basis of c_0, $A = \{e_n \mid n \in \mathbb{N}\}$ is b-order bounded in c_0 but A is not order bounded in c_0. A linear operator between two vector lattices is order continuous (resp. σ-order continuous) if it maps order null nets (resp. sequences) to order null nets (resp. sequences). The collection of all order continuous (resp. σ-order continuous) linear operators from vector lattice E into vector lattice G will be denoted by $L_n(E,G)$ (resp. $L_c(E,G)$). For unexplained terminology and facts on Banach lattices and positive operators, we refer the reader to [2, 3].

2 F-order Convergent and Its Properties

In this section, E is a sublattice of a vector lattice F. A net $\{x_\alpha\}_{\alpha\in\mathcal{A}} \subseteq E$ is said to be F-order convergent (in short Fo-convergent) to a vector $x \in E$ (in symbols $x_\alpha \xrightarrow{Fo} x$), whenever there exists a net $\{y_\beta\}_{\beta\in\mathcal{B}}$ in F satisfying $y_\beta \downarrow 0$ and for each β there exists α_0 such that $|x_\alpha - x| \leq y_\beta$ whenever $\alpha \geq \alpha_0$. If $A \subseteq E$ is order bounded in F, we say that A is F-order bounded, in case $F = E^{\sim\sim}$, we say that A is b-order bounded. It is clear that if E is regular in F, then every order convergence net (or order bounded set) in vector lattice E is F-order convergent (or F-order bounded), but as following example the converse in general not holds. On the other hand, there is a sequence in E that is order convergent in E and F, but is not F-order convergent in E.

Example 2.1.

1. Suppose that $E = c_0$ and $F = \ell^\infty$. The standard basis of c_0, $\{e_n\}_{n=1}^\infty$ is not order convergence to zero, but $\{e_n\}_{n=1}^\infty$ is ℓ^∞-order convergent to zero. On the other hand $\{e_n\}_{n=1}^\infty$ is not order bounded in c_0, but is ℓ^∞-order bounded in c_0.

2. Assume that F is a set of real valued functions on $[0,1]$ of form $f = g + h$ where g is continuous and h vanishes except at finitely many point. Let $E = C([0,1])$ and $f_n(t) = t^n$ where $t \in [0,1]$. It is obvious that $f_n \downarrow 0$ in E and $f_n \downarrow \chi_{\{1\}}$ in F, but $\{f_n\}$ is not F-order convergent.

It can easily be seen that a net in vector lattice E have at most one

F-order limit. The basic properties of Fo-convergent are summarized in the next theorem.

Theorem 2.2. *Assume that the nets $\{x_\alpha\}$ and $\{z_\gamma\}$ of a vector lattice E satisfy $x_\alpha \xrightarrow{Fo} x$ and $z_\gamma \xrightarrow{Fo} z$. Then we have*

1. $|x_\alpha| \xrightarrow{Fo} |x|$; $x_\alpha^+ \xrightarrow{Fo} x^+$ *and* $x_\alpha^- \xrightarrow{Fo} x^-$.
2. $\lambda x_\alpha + \mu z_\gamma \xrightarrow{Fo} \lambda x + \mu z$ *for all* $\lambda, \mu \in \mathbb{R}$.
3. $x_\alpha \vee z_\gamma \xrightarrow{Fo} x \vee z$ *and* $x_\alpha \wedge z_\gamma \xrightarrow{Fo} x \wedge z$.
4. *For each* $y \in F$, *if* $x_\alpha \leq y$ *for all* $\alpha \geq \alpha_0$, *then* $x \leq y$.
5. *If* $0 \leq x_\alpha \leq z_\alpha$ *for all* α, *then* $0 \leq x \leq z$.
6. *If* P *is order projection, then* $Px_\alpha \xrightarrow{Fo} Px$.

Proof. These follow immediately by definition. □

Theorem 2.3. *Let G be a sublattice of E and E be an ideal of F. Then the following statements hold:*

1. *If* $\{x_\alpha\}_{\alpha\in\mathcal{A}} \subset G$ *and* $x_\alpha \xrightarrow{Eo} 0$ *in* G, *then* $x_\alpha \xrightarrow{Fo} 0$ *in* G.
2. *If* $\{x_\alpha\}_{\alpha\in\mathcal{A}} \subset G$ *is order bounded net in* E *and* $x_\alpha \xrightarrow{Fo} 0$ *in* G, *then* $x_\alpha \xrightarrow{Eo} 0$ *in* G.
3. *If* $\{x_\alpha\}_{\alpha\in\mathcal{A}} \subset G$, $x_\alpha \xrightarrow{o} 0$ *in* G *and* $x_\alpha \xrightarrow{Fo} 0$ *in* G, *then* $x_\alpha \xrightarrow{Eo} 0$ *in* G.

Proof. (1) Suppose $\{x_\alpha\}_{\alpha\in\mathcal{A}} \subset G$ and $x_\alpha \xrightarrow{Eo} 0$ in G, there exists $\{y_\beta\}_{\beta\in\mathcal{B}} \subset E$ with $y_\beta \downarrow 0$ in E such that

$$\forall\beta, \exists\alpha_0 \quad s.t. \quad \forall\alpha \geq \alpha_0 : |x_\alpha| \leq y_\beta.$$

We show that $y_\beta \downarrow 0$ in F. Let $u \in F$ and $0 \leqslant u \leqslant y_\beta$ for all β. Since $\{y_\beta\} \subset E$ and E is an ideal in F, it follows that $u \in E$ and hence $u = 0$. Thus $y_\beta \downarrow 0$ in F. This means that $x_\alpha \xrightarrow{Fo} 0$ in G.
(2) By the assumption, there exists $\{y_\beta\} \subset F$ satisfying, $y_\beta \downarrow 0$ and for

each β there exists α_0 such that $|x_\alpha| \leq y_\beta$ whenever $\alpha \geq \alpha_0$. Let $u \in E^+$ such that $|x_\alpha| \leq u$. Then $\{u \wedge y_\beta\} \subset E$ and $u \wedge y_\beta \leq y_\beta$. Thus for each β there exists α_0 that $|x_\alpha| \leq u \wedge y_\beta$ whenever $\alpha \geq \alpha_0$. It follows that $x_\alpha \xrightarrow{Eo} 0$ in G.

(3) Suppose $\{x_\alpha\}_{\alpha\in\mathcal{A}} \subset G$ and $x_\alpha \xrightarrow{o} 0$ in G, there exists $\{y_\beta\}_{\beta\in\mathcal{B}} \subset G$ with $y_\beta \downarrow 0$ in G such that

$$\forall\beta, \exists\alpha_0 \quad s.t \quad \forall\alpha \geq \alpha_0 : |x_\alpha| \leq y_\beta.$$

By assumption, since $x_\alpha \xrightarrow{Fo} 0$ in G, there exists $\{z_\gamma\}_{\gamma\in\mathcal{C}} \subset F$ with $z_\gamma \downarrow 0$ in F such that

$$\forall\gamma, \exists\alpha_0' \quad s.t \quad \forall\alpha \geq \alpha_0' : |x_\alpha| \leq z_\gamma.$$

For fixed $\beta_0 \in \mathcal{B}$, there exists $\alpha_0 \in \mathcal{A}$ such that $|x_\alpha| \leq y_{\beta_0}$ whenever $\alpha \geq \alpha_0$. Let $w_\gamma = z_\gamma \wedge y_{\beta_0}$ for every $\gamma \in \mathcal{C}$. since E is an ideal of F, $\{w_\gamma\} \subset E$ and $w_\gamma \downarrow 0$ in E. On the other hand, for every $\gamma \in \mathcal{C}$, there exists $\alpha_0' \in \mathcal{A}$ such that $|x_\alpha| \leq z_\gamma$ whenever $\alpha \geq \alpha_0'$. For $\alpha_0, \alpha_0' \in \mathcal{A}$ there exists $\alpha_0'' \in \mathcal{A}$ such that $|x_\alpha| \leq z_\gamma \wedge y_{\beta_0}$ for all $\alpha \geq \alpha_0''$. It follows that $x_\alpha \xrightarrow{Eo} 0$ in G. $\square$

Corollary 2.4. *Suppose that E is an ideal of vector lattice F. If $\{x_\alpha\}_{\alpha\in\mathcal{A}}$ is order bounded in E, then*

$$x_\alpha \xrightarrow{Fo} x \quad in \quad E \quad iff \quad x_\alpha \xrightarrow{o} x \quad in \quad E \quad .$$

As Example 2.1, the condition of boundedness for nets in above corollary is necessary. Now the following example and part (2) of Example 2.1 show that the ideal condition is also necessary.

Example 2.5. Assume that E is a set of real valued continuous functions on $[0, 1]$ except at finitely many point and F is Lebesgue integrable real valued functions on $[0, 1]$. Obviously, E is a sublattice in F, but is not ideal in F. Let $I_1 = (\frac{1}{3}, \frac{2}{3})$, $I_2 = (\frac{1}{9}, \frac{2}{9}) \cup (\frac{3}{9}, \frac{6}{9}) \cup (\frac{7}{9}, \frac{8}{9})$, ..., the segments that we remove them for constructing of the Contor set P. It is obvious that $\chi_{I_n} \in E$ and $\chi_{I_n} \uparrow \chi_{P^c}$ in F, but $\{\chi_{I_n}\}$ is not F-order convergent in E.

Definition 2.6. 1. E is said to be F-Dedekind complete (or F-order complete), if every nonempty $A \subseteq E$ that is bounded from above in F has supermum in E. In case $F = E^{\sim\sim}$, we say that E is b-Dedekind complete.

2. If each F-order bounded subset of E is order bounded in E, then E is said to have the property (F). In case $F = E^{\sim\sim}$, we say that E has property (b).

Obviosly, if E is F-Dedekind complete, then E has property (F).

Remark 2.7. Let F be a Dedekind complete AM-space with order unit e. If E is a Dedekind complete closed in F contain e, then E has property (F), see [[11], p.110]. We obvious that every majorizing sublattice E of F has the property (F). Since $E^{\sim}$ has property (b), $E^{\sim}$ is b-Dedekind complete. If E is F-Dedekind complete, then E is Dedekind complete. The converse of last assertion in general not holds, of course c_0 is Dedekind complete, but is not ℓ^{∞}-Dedekind complete. It is easy to show that a vector lattice E has property (F) if and only if for each net $\{x_\alpha\}$ in E with $x_\alpha \uparrow y$ for some $y \in F$, follows that $\{x_\alpha\}$ is bounded above in E.

Theorem 2.8. *Let E and F both be Banach lattices. For each sequence $\{x_n\}_{n\in\mathbb{N}}$ in E the following statements hold:*

1. *If F has an order continuous norm and $x_n \xrightarrow{Fo} 0$ in E, then there exists a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \xrightarrow{o} 0$ holds in E.*
2. *If F is a Banach lattice and $\{x_n\}$ is norm convergent to $x \in E$, then there exists a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \xrightarrow{Fo} x$ holds in E.*
3. *If E has property (F) and $x_n \xrightarrow{Fo} 0$ in E, then $\{x_n\}_{n\in\mathbb{N}}$ is order bounded in E.*
4. *If E is an ideal of F and $x_n \xrightarrow{Fo} 0$ in E, then $\{x_n\}_{n\in\mathbb{N}}$ is order bounded in E.*

Proof.

1. There exists $\{y_m\}_{m\in\mathbb{N}}$ in F satisfying, $y_m \downarrow 0$ and for every m there exists n_0 such that $|x_n| \leq y_m$ whenever $n \geq n_0$. By the assumption $||y_m|| \longrightarrow 0$, it follows that $||x_n|| \longrightarrow 0$. Pick subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $||x_{n_k}|| < \frac{1}{2^k}$ for all k. Set $z_k = \sum_{i=k}^{\infty} |x_{n_i}|$. Since E is a Banach lattice, $z_k \in E$. For some k_0, we have $|x_{n_k}| \leq z_k \downarrow 0$ whenever $n_k \geq k_0$. This implies that $x_{n_k} \xrightarrow{o} 0$ holds in E.
2. By our hypothesis, there exists a subsequence $\{x_{n_k}\}$ such that $||x_{n_k} - x|| \leq \frac{1}{k2^k}$ for all k. Since $\sum_{k=1}^{\infty} k|x_{n_k} - x|$ is norm convergence to some $u \in F$, then $k|x_{n_k} - x| \leq u$ for all k. Clearly, $\{\frac{1}{k}u\}$ is a sequence in F such that $\frac{1}{k}u \downarrow 0$ and $|x_{n_k} - x| \leq \frac{1}{k}u$ and the proof is complete.
3. There exists a sequence $\{y_m\}_{m\in\mathbb{N}}$ in F satisfying, $y_m \downarrow 0$ and for every m there exists n_0 such that $|x_n| \leq y_m$ whenever $n \geq n_0$. Fix $m \in \mathbb{N}$ such that $|x_n| \leq y_m$ for all $n \geq n_0$. Put $z = \sup\{|x_1|, |x_2|, \dots, |x_{n_0-1}|, y_m\}$. Thus $|x_n| \leq z$ for all $n \in \mathbb{N}$, and so z is an upper bound of $\{x_n\}$ in F. Since E has property (F), it follows that $\{x_n\}$ is bounded in E.
4. Obviously.

□

Example 2.1 shows that F-Dedekind condintion for E in part (iii) of Theorem 2.8 is necessary. Also ℓ^∞ (with sup norm) does not have order continuous norm and $\{e_n\}$ is ℓ^∞-order convegent to zero but there is no subsequence of $\{e_n\}$ that is convegent to zero, therefore, having order continuous norms in part (1) is necessary.

Remark 2.9. It is easy to see that for an order bounded net $\{x_\alpha\}$ in a Dedekind complete vector lattice E,

$$x_\alpha \xrightarrow{o} x \quad \text{in E iff} \quad x = \inf_\alpha \sup_{\beta\geq\alpha} x_\beta = \sup_\alpha \inf_{\beta\geq\alpha} x_\beta \quad \text{in E}$$

$$\text{iff} \quad 0 = \inf_\alpha \sup_{\beta\geq\alpha} |x_\beta - x| \text{ in E.}$$

The following fact is straightforward.

Lemma 2.10. *Let E be a sublattice of a Dedekind complete vector lattice F. Then*

$$x_\alpha \xrightarrow{Fo} x \quad in \quad E \quad iff \quad x = \inf_\alpha \sup_{\beta \geq \alpha} x_\beta = \sup_\alpha \inf_{\beta \geq \alpha} x_\beta \ in\ F,$$

for every order bounded net $\{x_\alpha\}$ in E.

A net $\{x_\alpha\}$ in E is a F-order Cauchy, if the double net $\{(x_\alpha - x_\beta)\}_{(\alpha,\beta)}$ is a F-order convergent to zero in E. The following proposition follows from the double equality of Lemma 2.10 and the proof is straightforward.

Proposition 2.11. *Every F-order Cauchy net in an Dedekind complete vector sublattice E is order convergent.*

For a vector lattice E, we write E^δ for its order (or Dedekind) completion. Recall from Theorem 1.41 of [2] that E^δ is the unique (up to a lattice isomorphism) order complete vector lattice that contains E as a majorizing and order dense sublattice. In particular, E is regular sublattice of E^δ.

Theorem 2.12. *[10] Let E be a regular sublattice of a vector lattice F. Then*

1. *E^δ is a regular sublattice of F^δ.*
2. *$x_\alpha \xrightarrow{o} 0$ in E iff $x_\alpha \xrightarrow{o} 0$ in F for every order bounded net $\{x_\alpha\}$ in E.*

Corollary 2.13. *Let E be a regular sublattice of a vector lattice F. Then $x_\alpha \xrightarrow{Fo} 0$ in E for every order bounded net $\{x_\alpha\}$ in E, when one of the following conditions hold:*

1. *$x_\alpha \xrightarrow{o} 0$ in E.*
2. *$x_\alpha \xrightarrow{o} 0$ in F.*

The proof of the following Theorem is similar to Theorem 2.8 of [10].

Theorem 2.14. *Suppose that E is an order dense and majorizing sublattice of F. Then the order convergence and F-order convergence are equivalent.*

Corollary 2.15. *For every net $\{x_\alpha\}$ in E, $x_\alpha \xrightarrow{o} 0$ in E iff $x_\alpha \xrightarrow{E^\delta o} 0$ in E.*

Corollary 2.16. *If E is regular sublattice of F, then $x_\alpha \xrightarrow{E^\delta o} 0$ in E iff $x_\alpha \xrightarrow{F^\delta o} 0$ in E for every order bounded net $\{x_\alpha\}$ in E.*

Proof. From Corollary 2.15 and Theorem 2.12, it should be obvious. □

Theorem 2.17. *Suppose that E is a regular sublattice of a vector lattice F. Then $x_\alpha \xrightarrow{o} 0$ iff $x_\alpha \xrightarrow{Fo} 0$ for every order bounded net $\{x_\alpha\}$ in E.*

Proof. Since, for a bounded net, F-order convergence is equivalent to order convergence in F, thus by Theorem 2.12 the result holds. □

Theorem 2.18. *Let F be Dedekind σ-complete. If $\{x_n\}_{n\in\mathbb{N}}$ is a disjoint sequence in E, then $x_n \xrightarrow{Fo} 0$.*

Proof. Suppose $\{x_n\}$ is a disjoint sequence in E. We claim that $y_n = \sup_{k\geq n} |x_k| \downarrow 0$ in F. Indeed, assume that $y \in F$ and $y_n \geq y \geq 0$ for all $n \geq 1$. Then

$$0 \leq y \wedge |x_n| \leq (|x_n| \wedge \sup_{k\geq n+1} |x_k|) = \sup_{k\geq n+1} (|x_k| \wedge |x_n|) = 0,$$

holds in F. Thus $y \wedge |x_n| = 0$ holds in F for all $n \geq 1$. It follows that

$$y = y \wedge (\sup_{n\geq 1} |x_n|) = \sup_{n\geq 1}(y \wedge |x_n|) = 0 \quad \text{for all} \quad n \geq 1.$$

It follows that $|x_n| \leqslant y_n$ and $y_n \downarrow 0$ holds in F. □

Theorem 2.19. *Suppose that E is a sublattice of a vector lattice F. Assume also F is atomic and has order continuous norm, and $\{x_n\}$ is an order bounded sequence in F. If $x_n \xrightarrow{o} 0$ then $x_n \xrightarrow{Fo} 0$.*

Proof. It can be proven in the same manner of Lemma 5.1 of [9]. □

3 FH-order Continuous Operators

In this section, the basic properties of FH-order continuous operators will be studied. In the following, E and G are vector sublattices of vector lattices F and H, respectively. Let $\{x_n\} \subseteq E$ and $x \in E$. The notation $x_\alpha \downarrow_F x$ means that $x_\alpha \downarrow$ and $\inf\{x_\alpha\} = x$ holds in F. Assume that F is a set of real valued functions on $[0, 1]$ of form $f = g + h$ where g is continuous and h vanishes except at finitely many points. Let $E = C([0, 1])$, $\{f_n\}$ be a decreasing sequence in E^+ such that $f_n(\frac{1}{2}) = 1$ for all $n \in \mathbb{N}$ and $f_n(t) \to 0$ for every $t \neq \frac{1}{2}$. It is clear that $f_n \downarrow 0$ in E, but $f_n \downarrow_F 0$ not holds.

Definition 3.1. An operator $T : E \longrightarrow G$ between two vector lattices is said to be:
(a) FH-order continuous, if $x_\alpha \xrightarrow{Fo} 0$ in E implies $Tx_\alpha \xrightarrow{Ho} 0$ in G.
(b) FH-σ-order continuous, if $x_n \xrightarrow{Fo} 0$ in E implies $Tx_n \xrightarrow{Ho} 0$ in G.

The collection of all FH-order continuous operators will be denoted by $L_{FHn}(E, G)$, that is,

$$L_{FHn}(E, G) = \{T \in L(E, G) : T \text{ is } FH\text{-order continuous}\}.$$

Similarly, the collection of all FH-σ-order continuous operators from E to G will denoted by $L_{FHc}(E, G)$, that is,

$$L_{FHc}(E, G) = \{T \in L(E, G) : T \text{ is } FH\text{-}\sigma\text{-order continuous}\}.$$

Lemma 3.2. *Let E and G are F-Dedekind complete and H-Dedekind complete, respectively. Then we have the following assertions.*

1. *$0 \leq T \in L_{FHn}(E, G)$ if and only if for each net $\{x_\alpha\}$ in E, $x_\alpha \downarrow_F 0$ implies $Tx_\alpha \downarrow_H 0$.*

2. *If E and G are ideals in F and H, respectively, then $L_{FHc}(E, G) = L_c(E, G)$. Moreover, the FH-order continuous operator $0 \leq T$ is an order bounded.*

Proof.

1. Assume that $0 \leq T$ and $\{x_\alpha\}$ is a net in E such that $x_\alpha \xrightarrow{Fo} 0$. It follows that there exists a net $\{y_\beta\}_{\beta \in \mathcal{B}}$ in F satisfying, $y_\beta \downarrow 0$ and for each β there exists α_0 such that $|x_\alpha| \leq y_\beta$ whenever $\alpha \geq \alpha_0$. Set $z_\alpha = \bigvee_{\lambda \geq \alpha} |x_\lambda|$. Then we have $|x_\alpha| \leq z_\alpha$ for all α and $z_\alpha \downarrow_F 0$, and so by our assumption we have $|Tx_\alpha| \leq T|x_\alpha| \leq Tz_\alpha \downarrow$ whenever $\alpha \geq \alpha_0$. Since $Tz_\alpha \downarrow_H 0$, it follows that $Tx_\alpha \xrightarrow{Ho} 0$ and therefore $T \in L_{FHn}(E, G)$.
   Conversely, suppose that $0 \leq T \in L_{FHn}(E, G)$ and $x_\alpha \downarrow_F 0$. It follows that $x_\alpha \xrightarrow{Fo} 0$ and so $Tx_\alpha \xrightarrow{Ho} 0$. Then there exists a net $\{z_\beta\}$ in H satisfying, $z_\beta \downarrow 0$ and for each β there exists α_0 such that $|Tx_\alpha| \leq z_\beta$ whenever $\alpha \geq \alpha_0$, which shows that $Tx_\alpha \downarrow_H 0$ and the proof is follows.
2. Suppose that $T \in L_{FHc}(E, G)$ and $x_n \xrightarrow{o} 0$. By using Corollary 2.4, we have $x_n \xrightarrow{Fo} 0$ and by our assumption, we have $Tx_n \xrightarrow{Ho} 0$. Using Corollary 2.4 again, we have $Tx_n \xrightarrow{o} 0$ and so $T \in L_c(E, G)$. The converse is proved in the same manner. For the last part, let $0 \leq T \in L_{FHn}(E, G)$ and $x_0 \in E^+$. If we consider the order interval $[0, x_0]$ as a net $\{x_\alpha\}$ where $x_\alpha = \alpha$ for each $\alpha \in [0, x_0]$, then $x_\alpha \uparrow x_0$ holds in F. It follows that $x_\alpha \xrightarrow{Fo} x_0$ and therefore $Tx_\alpha \uparrow Tx_0$ holds in H. Thus T is order bounded.

□

Theorem 3.3. *Suppose that E and G are sublattices of F and H, respectively. Assume also H is Dedekind complete and $T \in L(E, G)$. If $T \in L_n(F, H)$, then $T \in L_{FHn}(E, G)$.*

Proof. Since T is order bounded, follows that $T = T^+ - T^-$, thus without loss of generality, we assume that T is a positive operator. Suppose that $\{x_\alpha\}_{\alpha \in \mathcal{A}}$ is a net in E which is F-order convergent to zero, then there exists a net $\{y_\beta\}_{\beta \in \mathcal{B}}$ in F satisfying, $y_\beta \downarrow 0$ and for each β there exists α_0 such that $|x_\alpha| \leq y_\beta$ holds whenever $\alpha \geq \alpha_0$. Since T is a positive operator, we have $|Tx_\alpha| \leq T|x_\alpha| \leq T(y_\beta) \downarrow 0$ whenever $\alpha \geq \alpha_0$. Now by using Lemma 3.2, the proof follows. □

An other application of the preceding lemma yields the following theorem, in which the techniques of this theorem are similar argument like as Theorem 1.56 [3], so we omit its proof.

Theorem 3.4. *Let E and G are F-Dedekind complete and H-Dedekind complete, respectively. Then the following assertions are equivalent.*

1. $T \in L_{FHn}(E, G)$.
2. $x_\alpha \downarrow_F 0$ implies $Tx_\alpha \downarrow_H 0$.
3. $x_\alpha \downarrow_F 0$ implies $\inf_H |Tx_\alpha| = 0$.
4. T^+, T^- *and* $|T|$ *belong to* $L_{FHn}(E, G)$.

The next result presents a useful sufficient condition for a set to be order bounded in two vector lattices.

Theorem 3.5. *Let I be a sublattice of E and E be F-Dedekind complete. Then subset A of I is E-order bounded if and only if it is F-order bounded.*

An operator $T : E \to G$ is said to be FH-order bounded if it maps each F-order bounded subset of E into H-order bounded a subset of G. The collection of all FH-order bounded operators from a vector lattice E into a vector lattice G will be denoted by $L_{FHb}(E, G)$.

The following example shows that, there are operators $T : E \to G$ between Riesz spaces that are FH-order bounded, but are not an order bounded operators.

Example 3.6. Let $T : L^1[0, 1] \longrightarrow c_0$ be defined by

$$T(f) = (\int_0^1 f(x)\sin(x)dx, \int_0^1 f(x)\sin(2x)dx, \ldots).$$

Then T is a $L^\infty[0, 1]\ell^\infty$-order bounded but is not an order bounded operator.

Theorem 3.7. *For two vector lattices E and F, we have the following:*

1. $L_{FHc}(E, G) \subseteq L_{FHb}(E, G)$.
2. *If E has property (F), then* $L_b(E, G) \subseteq L_{FHb}(E, G)$.
3. *If G has property (H), then* $L_{FHb}(E, G) \subseteq L_b(E, G)$.

4. *If E and G have property (F) and (H), respectively, then*

$$L_b(E,G) = L_{FHb}(E,G).$$

Proof.

1. The proof is clear.
2. Suppose that $T \in L_b(E,G)$ and $A \subset E$ is an F-order bounded subset of E. From our hypothesis, A is an order bounded subset of E and $T(A)$ is an order bounded subset of G. Therefore $T(A)$ is an H-order bounded subset of G and hence $T \in L_{FHb}(E,G)$.
3. Assume that $T \in L_{FHb}(E,G)$ and $A \subset E$ is an order bounded subset of E. Then A is an F-order bounded subset of E and from our hypothesis, $T(A)$ is an H-order bounded subset of G. Since G has property (H), $T(A)$ is an order bounded subset of G and therefore $T \in L_b(E,G)$.
4. It is obvious by (1) and (2).

□

Corollary 3.8. *Let E and G be F-Dedekind complete and H-Dedekind complete, respectively, then $L_b(E,G) = L_{FHb}(E,G)$.*

Corollary 3.9. *Let E be an F-Dedekind complete ideal of F. Assume also G is an H-Dedekind complete ideal of H. Then $L_{FHn}(E,G)$ and $L_{FHc}(E,G)$ are both bands of $L_{FHb}(E,G)$.*

Proof. Corollary 3.8 and part 2 of Lemma 3.2, show that $L_{FHn}(E,G)$ and $L_{FHc}(E,G)$ are both subspaces of $L_{FHb}(E,G)$ and the rest of the proof has a similar argument like as Theorem 1.57 [3]. □

Proposition 3.10. *Let T and S be FF-order bounded operators on Riesz space E. Then ToS is also FF-order bounded.*

Proposition 3.11. *Let E be a Riesz space with property (F) and T is an operator on E that it has order bounded left inverse. Then T^n for each $2 \leq n$, be a FF-order bounded operator if and only if T be FF-order bounded.*

Proof. Suppose that T is a FF-order bounded operator on E, clearly T^n, for each $n \in \mathbb{N}$, is FF-order bounded.
For the converse, suppose that A is a F-order bounded subset of E. By hypothesis $T^n(A)$, for each $n \in \mathbb{N}$, is order bounded subset of E and so, there exists $x \in E^+$ such that $T^n(A) \subset [-x, x]$. Since the left inverse of T is an order bounded, there exists $a, b \in E$ such that $T^{n-1}(A) = T^{-1}oT^n(A) \subset T^{-1}[-x, x] \subset [a, b]$. By continuing this process, it is easy to see that $T(A)$ is order bounded set in E. Therefore, T is FF-order bounded operator. $\square$

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Mehrdad Bakhshi
Department of Mathematics
Ph.D. Candidate
University of Mohaghegh Ardabili
Ardabil, Iran
E-mail: mehrdad.bakhshi@uma.ac.ir

Kazem Haghnejad Azar
Department of Mathematics
Professor of Mathematics
University of Mohaghegh Ardabili
Ardabil, Iran
E-mail:haghnejad@uma.ac.ir