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# An Efficient Numerical Method for Solving Systems of Fractional Order Differential Equations

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**Abstract.** In this work, we apply an efficient method based on hybrid functions for solving linear and non-linear systems of fractional order differential equations (SFDEs). Here, we consider the fractional derivatives in the Caputo sense. By using the present method, a system of

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FDEs is reduced to a system of algebraic equations which can be solved by a proper numerical method. In convergence discussion of the method, an upper bound of the error is obtained. To show the efficiency and the accuracy of this method, some examples are simulated and then some comparisons between the outputs with those of several other methods are carried out.

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# 1 Introduction

Integer order differential and integral equations and their systems due to the varied and widespread applications in physics, medicine, engineering, biology, and other fields [32, 17, 10, 13, 15, 44, 3, 49] have always been considered by researchers. But the problem is that the behavior of many dynamical systems and consequently their mathematical models cannot be described based on the differential and integral calculus of integer order. Hence, fractional calculus (FC) as a generalization of classical calculus has been noticed especially extensively by researchers [41, 37, 43, 29, 7]. The significant role of FC in describing phenomena with memory effects and hereditary properties in diverse fields [46, 21, 33, 12, 42, 48, 14] has made it a powerful instrument in the mathematical modeling of these phenomena.

Since that the exact solutions of FDEs and their systems are not available in general, the numerical methods such as the homotopy perturbation method [2, 28], homotopy analysis method [9, 45], the residual power series [5, 18], the Adomian decomposition method [39, 24], the variational iteration method [19, 40], the differential transform method [16, 26], spectral collocation method [27, 8] and some other methods [4, 51, 1, 25] have been introduced to solve the fractional models.

In recent years, by using hybrid functions consisting of the combination of polynomials such as Legendre, Chebyshev, Lagrange, Bernoulli, Bernstein, and Taylor or Fourier series with the Block-pulse functions (BPFs), various numerical methods have been proposed for solving mathematical models of integer and fractional orders [6, 23, 22, 50, 38, 36,

47, 35, 20].

In this work, an attempt is made to solve the system of FDEs by using the hybrid Legendre Block-pulse functions (HLBPFs). To do this, we consider the general form of the system of FDEs as

$$\begin{cases} {}_0^C\mathcal{D}_t^\alpha u_1 = f_1(t, u_1, u_2, ..., u_n), \\ {}_0^C\mathcal{D}_t^\alpha u_2 = f_2(t, u_1, u_2, ..., u_n), \\ \quad \vdots \\ {}_0^C\mathcal{D}_t^\alpha u_n = f_n(t, u_1, u_2, ..., u_n), \end{cases} \tag{1}$$

where $0 < \alpha \leq 1$ and the initial conditions are $u_1(0) = \lambda_1$, $u_2(0) = \lambda_2$, $\ldots$, $u_n(0) = \lambda_n$. In addition to the present section, this paper is divided into four other sections. In Section 2, some FC concepts and some definitions and properties of HLBPFs are reviewed. In Section 3, we apply the hybrid Legendre Block-pulse functions method (HLBPM) to construct approximate solutions for linear and non-linear systems of FDEs. The error and convergence analysis of the method is studied in Section 4. Finally, in Section 5, the applicability and effectiveness of the method are illustrated by means of some examples.

## 2 Preliminaries and Definitions

In this section, some required concepts of FC and some definitions are presented.

### 2.1 Fractional calculus

Although, there is not a uniform definition for the fractional derivative and the fractional integral, and several definitions have been presented by Grünwald-Letnikov, Riemann-Liouville, Hadamard, Erdélyi-Kober, Caputo, and others, two common and accepted definitions which have defined by Riemann-Liouville and Caputo are demonstrated as follows:

**Definition 2.1.** [43] The Riemann-Liouville fractional integral operator

${}_0\mathcal{I}_t^\alpha$ for a function $\omega \in L^1(a,b)$ is defined as

$$({}_0\mathcal{I}_t^\alpha \omega)(t) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t \frac{\omega(\varsigma)}{(t-\varsigma)^{1-\alpha}}\, \mathrm{d}\varsigma, & \alpha > 0, \\ f(t), & \alpha = 0, \end{cases}$$

where $\alpha$ is the order of operator and $\Gamma(.)$ denotes the well-known Gamma function.

**Definition 2.2.** [43] The Caputo fractional derivative operator of order $\alpha > 0$ for $t > 0$ is defined as

$$({}_0^C\mathcal{D}_t^\alpha \omega)(t) = ({}_0\mathcal{I}_t^{n-\alpha}\ {}_0^C\mathcal{D}_t^n \omega)(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\varsigma)^{n-\alpha-1} \omega^{(n)}(\varsigma)\, \mathrm{d}\varsigma,$$

where $n-1 < \alpha \leq n \in \mathbb{N}$. For any constant $C \in \mathbb{R}$, it yields ${}_0^C\mathcal{D}_t^\alpha C = 0$.

**Property 2.3.** *The above-defined fractional operators satisfy the following properties:*

*(i)* $({}_0^C\mathcal{D}_t^{\alpha_1}\ {}_0^C\mathcal{D}_t^{\alpha_2} \omega)(t) = ({}_0^C\mathcal{D}_t^{\alpha_1+\alpha_2} \omega)(t)$,

*(ii)* $({}_0\mathcal{I}_t^{\alpha_1}\ {}_0\mathcal{I}_t^{\alpha_2} \omega)(t) = ({}_0\mathcal{I}_t^{\alpha_2}\ {}_0\mathcal{I}_t^{\alpha_1} \omega)(t) = ({}_0\mathcal{I}_t^{\alpha_1+\alpha_2} \omega)(t)$,

*(iii)* ${}_0^C\mathcal{D}_t^\alpha t^\beta = \begin{cases} 0, & \beta \in \mathbb{Z}^+ \quad \textit{and} \quad \beta < \alpha, \\ \dfrac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}, & \textit{otherwise,} \end{cases}$

*(iv)* $({}_0\mathcal{I}_t^\alpha\ {}_0^C\mathcal{D}_t^\alpha \omega)(t) = \omega(t) - \sum_{k=0}^{\lceil\alpha\rceil-1} \frac{t^k}{k!} \omega^{(k)}(0^+), \quad n-1 < \alpha \leq n \in \mathbb{N}.$

## 2.2 Hybrid functions

**Definition 2.4.** A $P$-set of BPFs $b_p(t)$ on the interval $[0,1)$ is defined as

$$b_p(t) = \begin{cases} 1, & \dfrac{p-1}{P} \leq t < \dfrac{p}{P}, \\ 0, & \text{o.w}, \end{cases}$$

where $p = 1, 2, \ldots, P$, is the order of BPFs. The set $\{b_p(t)\}$ has orthogonality and disjointness properties on $[0,1)$.

**Definition 2.5.** [34] A $PQ$-set of HLBPFs $\varphi_{pq}(t)$ on the interval $[0,1)$ is defined as

$$\varphi_{pq}(t) = \begin{cases} L_q(2Pt - 2p + 1), & \frac{p-1}{P} \leq t < \frac{p}{P}, \\ 0, & \text{o.w}, \end{cases}$$

where $L_q$ is the well-known Legendre polynomial of order $q = 0, 1, 2, \ldots, Q-1$, which is obtained with the following formulas:

$$L_0(t) = 1, \quad L_1(t) = t,$$
$$(q+1)L_{q+1}(t) = (2q+1)tL_q(t) - qL_{q-1}(t), \quad t \in [-1, 1].$$

Note that since the BPFs and Legendre polynomials are both orthogonal and complete, then the set $\{\varphi_{pq}(t)\}$ forms an orthogonal complete system in $L^2[0,1)$. We can expand a function $\omega \in L^2[0,1)$ using the basis functions $\{\varphi_{pq}(t)\}$ as [34]

$$\omega(t) = \sum_{p=1}^{\infty} \sum_{q=0}^{\infty} c_{pq}\, \varphi_{pq}(t).$$

**Theorem 2.6.** [31] *Let* $\mathrm{Y}$ *be a finite dimensional subspace of a strictly convex normed space* $\Theta$*. Then, for each* $\omega \in \Theta$ *there exists a unique best approximation* $\bar{\omega} \in \mathrm{Y}$*.*

We let $\Theta = L^2[0,1)$ and consider the normed space $\Theta$ with the following norm:

$$||\omega||_2 = \langle \omega, \omega \rangle^{\frac{1}{2}} = \left( \int_0^1 |\omega(t)|^2 \, \mathrm{d}t \right)^{\frac{1}{2}},$$

where $\langle ., . \rangle$ denotes the inner product.

**Remark 2.7.** Since $\Theta$ is a Hilbert space, it is also strictly convex.

Let

$$\mathrm{Y} = \mathrm{span}\{\varphi_{10}(t), \ldots, \varphi_{1(Q-1)}(t), \varphi_{20}(t), \ldots, \varphi_{2(Q-1)}(t), \ldots, \varphi_{P0}(t), \ldots, \varphi_{P(Q-1)}(t)\}.$$

Since Y is a finite dimensional subspace of $\Theta$, using Theorem 2.6 we have

$$\omega(t) \simeq \bar{\omega}(t) = \omega_{PQ}(t) = \sum_{p=1}^{P} \sum_{q=0}^{Q-1} c_{pq}\, \varphi_{pq}(t) = \mathbf{C}^T \mathbf{\Phi}(t) = \mathbf{\Phi}^T(t)\mathbf{C}, \quad (2)$$

where

$$\mathbf{\Phi}(t) = [\varphi_{10}(t), \ldots, \varphi_{1(Q-1)}(t), \varphi_{20}(t), \ldots, \varphi_{2(Q-1)}(t), \ldots, \varphi_{P0}(t), \ldots, \varphi_{P(Q-1)}(t)]^T,$$

and

$$\mathbf{C} = [c_{10}, c_{11}, \ldots, c_{1(Q-1)}, c_{20}, c_{21}, \ldots, c_{2(Q-1)}, \ldots, c_{P0}, c_{P1}, \ldots, c_{P(Q-1)}]^T.$$

The hybrid coefficients $c_{pq}$ are obtained by

$$c_{pq} = \frac{\langle \omega(t), \varphi_{pq}(t) \rangle}{\langle \varphi_{pq}(t), \varphi_{pq}(t) \rangle}, \qquad p = 1, 2, \ldots, P, \quad q = 0, 1, \ldots, Q-1.$$

### 2.3 Operational matrices

The integration of the vector $\mathbf{\Phi}(t)$ is approximated as [34]

$$\int_0^t \mathbf{\Phi}(\varsigma)\, \mathrm{d}\varsigma \simeq \Upsilon\, \mathbf{\Phi}(t), \quad (3)$$

where $\Upsilon$, i.e. the operational matrix of integration is defined as

$$\Upsilon = \begin{bmatrix} \mathrm{M} & \mathrm{S} & \mathrm{S} & \ldots & \mathrm{S} \\ \mathbf{0} & \mathrm{M} & \mathrm{S} & \ldots & \mathrm{S} \\ \mathbf{0} & \mathbf{0} & \mathrm{M} & \ldots & \mathrm{S} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \ldots & \mathrm{M} \end{bmatrix}_{\theta \times \theta},$$

where $\theta = PQ$, $\mathbf{0}$ is the $Q \times Q$ zero matrix, and S and M have the following forms:

$$\mathrm{S} = \frac{1}{P} \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}_{Q \times Q},$$

$$\mathrm{M} = \frac{1}{2P}\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{5} & 0 & \frac{1}{5} & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{7} & 0 & \frac{1}{7} & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{9} & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & \frac{1}{2Q-9} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & \frac{-1}{2Q-7} & 0 & \frac{1}{2Q-7} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & \frac{-1}{2Q-5} & 0 & \frac{1}{2Q-5} & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \frac{-1}{2Q-3} & 0 & \frac{1}{2Q-3} \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \frac{-1}{2Q-1} & 0 \end{bmatrix}_{Q\times Q}.$$

We note that $\mathbf{\Phi}(t)$ can be approximated by using the BPFs as

$$\mathbf{\Phi}(t) \simeq \Psi_{\theta\times\theta}\, B(t), \tag{4}$$

where

- $\Psi_{\theta\times\theta} = [\mathbf{\Phi}(\tau_1) \quad \mathbf{\Phi}(\tau_2) \quad \dots \quad \mathbf{\Phi}(\tau_\theta)]$, where $\tau_s = \frac{2s-1}{2\theta}$ for $s = 1, 2, \dots, \theta$ are the collocation points.
- $B(t) = [b_1(t),\, b_2(t),\, \dots,\, b_\theta(t)]^T$, where $b_p(t)$ is the $p$th BPF.

To obtain the operational matrix of the fractional integration $\Upsilon^{\alpha}_{\theta\times\theta}$ for HLBPFs, we put

$$(\mathcal{I}^{\alpha}\mathbf{\Phi})(t) \simeq \Upsilon^{\alpha}_{\theta\times\theta}\, \mathbf{\Phi}(t). \tag{5}$$

Using Eq. (4), we can write

$$(\mathcal{I}^{\alpha}\mathbf{\Phi})(t) \simeq (\mathcal{I}^{\alpha}\Psi_{\theta\times\theta}\, B)(t) = \Psi_{\theta\times\theta}\, (\mathcal{I}^{\alpha}\, B)(t). \tag{6}$$

From Eqs. (5) and (6), we get

$$\Upsilon^{\alpha}_{\theta\times\theta}\, \mathbf{\Phi}(t) \simeq \Psi_{\theta\times\theta}\, (\mathcal{I}^{\alpha} B)(t). \tag{7}$$

Furthermore,

$$(\mathcal{I}^{\alpha} B)(t) \simeq \mathcal{F}^{\alpha} B(t), \tag{8}$$

where $\mathcal{F}^\alpha$ has the following shape [30]:

$$\mathcal{F}^\alpha = \frac{1}{\theta^\alpha\,\Gamma(\alpha+2)} \begin{bmatrix} 1 & \mu_1 & \mu_2 & \dots & \mu_{\theta-1} \\ 0 & 1 & \mu_1 & \dots & \mu_{\theta-2} \\ 0 & 0 & 1 & \dots & \mu_{\theta-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}_{\theta\times\theta}.$$

where $\mu_s = (s+1)^{\alpha+1} - 2s^{\alpha+1} + (s-1)^{\alpha+1}$; $s = 1, 2, \dots, \theta-1$. By substituting Eq. (8) in Eq. (7), and considering Eq. (4), the following relation is derived:

$$\Upsilon^\alpha_{\theta\times\theta}\Psi_{\theta\times\theta}\,B(t) \simeq \Psi_{\theta\times\theta}\,\mathcal{F}^\alpha B(t).$$

And finally, $\Upsilon^\alpha_{\theta\times\theta}$ is obtained as

$$\Upsilon^\alpha_{\theta\times\theta} \simeq \Psi_{\theta\times\theta}\,\mathcal{F}^\alpha\,\Psi^{-1}_{\theta\times\theta}.$$

## 3 Method Implementation

In this section, we consider system (1) in which the fractional derivative operator ${}^C_0\mathcal{D}^\alpha_t$ is in the Caputo sense. For solving this system by the HLBPM, it is necessary that ${}^C_0\mathcal{D}^\alpha_t u_k(t)$ be approximated by the HLBPFs. To do this, we put

$${}^C_0\mathcal{D}^\alpha_t u_k(t) \simeq \mathrm{U}_k^T\mathbf{\Phi}(t), \qquad k = 1, 2, \dots, n, \tag{9}$$

where $\mathrm{U}_k = [\mathrm{u}_{k,1}, \mathrm{u}_{k,2}, \dots, \mathrm{u}_{k,\theta}]^T$ is the unknown vector.
If we apply the Riemann-Liouville fractional integral operator ${}_0\mathcal{I}^\alpha_t$ defined in Definition 2.1 to both sides of Eq. (9), then we get

$$({}_0\mathcal{I}^\alpha_t\ {}^C_0\mathcal{D}^\alpha_t u_k)(t) \simeq \mathrm{U}_k^T({}_0\mathcal{I}^\alpha_t\mathbf{\Phi})(t).$$

By using item (iv) in property (2.3) and Eq. (5), we rewrite the last equation as

$$u_k(t) \simeq \sum_{j=0}^{\lceil\alpha\rceil-1} \frac{t^j}{j!}u_k^{(j)}(0^+) + \mathrm{U}_k^T\,\Upsilon^\alpha_{\theta\times\theta}\,\mathbf{\Phi}(t).$$

Particularly for $0 < \alpha \leq 1$, we have

$$u_k(t) \simeq u_k(0^+) + \mathrm{U}_k^T \, \Upsilon_{\theta\times\theta}^{\alpha} \, \mathbf{\Phi}(t).$$

If we expand the constant $u_k(0^+)$ in terms of the HLBPFs as

$$u_k(0^+) \simeq \mathrm{U}_{k(0)}^T \, \mathbf{\Phi}(t)$$

where $\mathrm{U}_{k(0)}$ is defined as [34]

$$\mathrm{U}_{k(0)} = \Big[\underbrace{u_k(0^+) \overbrace{0 \;\; 0 \;\; \dots \;\; 0}^{Q-1} \; u_k(0^+) \overbrace{0 \;\; 0 \;\; \dots \;\; 0}^{Q-1} \; \dots \; u_k(0^+) \overbrace{0 \;\; 0 \;\; \dots \;\; 0}^{Q-1}}_{\theta}\Big]^T,$$

then, we will have

$$u_k(t) \simeq (\mathrm{U}_{k(0)}^T + \mathrm{U}_k^T \, \Upsilon_{\theta\times\theta}^{\alpha})\mathbf{\Phi}(t) = \mathrm{U}_{k,\alpha}^T \mathbf{\Phi}(t), \qquad k = 1, 2, \dots, n. \quad (10)$$

Now, for implementation of the method, the system (1) is considered in two forms. At first, we rewrite the system in linear form as follows:

$$\begin{cases} {}_0^C\mathcal{D}_t^{\alpha} u_1(t) = x_1(t) + \sum_{r=1}^{n} y_{1,r}(t)\, u_r(t), \\ {}_0^C\mathcal{D}_t^{\alpha} u_2(t) = x_2(t) + \sum_{r=1}^{n} y_{2,r}(t)\, u_r(t), \\ \quad \vdots \\ {}_0^C\mathcal{D}_t^{\alpha} u_n(t) = x_n(t) + \sum_{r=1}^{n} y_{n,r}(t)\, u_r(t), \end{cases} \quad (11)$$

where the known functions $x_k$ and $y_{k,r}$ for $k, r = 1, 2, \dots, n$, can be approximated by the HLBPFs as

$$x_k(t) \simeq \mathrm{X}_k^T \mathbf{\Phi}(t), \quad (12)$$

$$y_{k,r}(t) \simeq \mathrm{Y}_{k,r}^T \mathbf{\Phi}(t). \quad (13)$$

For approximating the term $y_{k,r}(t)\, u_k(t)$, we use Eqs. (10) and (13), and get

$$y_{k,r}(t)\, u_k(t) \simeq (\mathrm{Y}_{k,r}^T \mathbf{\Phi}(t))(\mathrm{U}_{k,\alpha}^T \mathbf{\Phi}(t)) = \mathrm{Y}_{k,r}^T \mathbf{\Phi}(t)\mathbf{\Phi}^T(t)\mathrm{U}_{k,\alpha} = \mathrm{Y}_{k,r}^T \tilde{\mathrm{U}}_{k,\alpha} \mathbf{\Phi}(t), \quad (14)$$

where the evaluation procedure of $\mathbf{\Phi}(t)\mathbf{\Phi}^T(t)$ and matrix $\tilde{\mathrm{U}}_{k,\alpha}$ are given in [34].
If we subtitute Eqs. (9), (12) and (14) into Eq. (11), then by replacing $\simeq$ with $=$, we will have the following linear system of algebraic equations:

$$\left(\mathrm{U}_k^T - \mathrm{X}_k^T - \sum_{r=1}^{n} \mathrm{Y}_{k,r}^T \tilde{\mathrm{U}}_{k,\alpha}\right) \mathbf{\Phi}(t) = 0, \qquad k = 1, 2, \ldots, n. \tag{15}$$

Now, before further details about this system, we also obtain a system of algebraic equations for the non-linear form of system (1). Here, in addition to the previously obtained approximations, we approximate non-linear terms by the HLBPFs as

$$\begin{aligned} u_k(t)u_r(t) \simeq (\mathrm{U}_{k,\alpha}^T \mathbf{\Phi}(t))(\mathrm{U}_{r,\alpha}^T \mathbf{\Phi}(t)) &= \mathrm{U}_{k,\alpha}^T \mathbf{\Phi}(t)\mathbf{\Phi}^T(t)\mathrm{U}_{r,\alpha} \\ &= \mathrm{U}_{k,\alpha}^T \tilde{\mathrm{U}}_{r,\alpha} \mathbf{\Phi}(t), \quad r, k = 1, 2, \ldots, n, \end{aligned}$$

and

$$\begin{aligned} u_k^3(t) = u_k^2(t).u_k(t) \simeq (\mathrm{U}_{k,\alpha}^T \tilde{\mathrm{U}}_{k,\alpha} \mathbf{\Phi}(t))(\mathrm{U}_{k,\alpha}^T \mathbf{\Phi}(t)) &= \mathrm{U}_{k,\alpha}^T \tilde{\mathrm{U}}_{k,\alpha} \mathbf{\Phi}(t)\mathbf{\Phi}^T(t)\mathrm{U}_{k,\alpha} \\ &= \mathrm{U}_{k,\alpha}^T \tilde{\mathrm{U}}_{k,\alpha}^2 \mathbf{\Phi}(t). \end{aligned}$$

Likewise, we have

$$u_k^m(t) \simeq \mathrm{U}_{k,\alpha}^T \tilde{\mathrm{U}}_{k,\alpha}^{m-1} \mathbf{\Phi}(t),$$

where $m$ is a positive integer number. By substituting the obtained approximations in terms of the vector $\mathbf{\Phi}(t)$ into function $f_k$, we can write

$$f_k(t, u_1, u_2, \ldots, u_n) \simeq \bar{\mathbf{F}}_{k,\alpha}^T \mathbf{\Phi}(t), \qquad k = 1, 2, \ldots, n.$$

Hence, the non-linear system of algebraic equations associated with the non-linear form of system (1) can be written as

$$\left(\mathrm{U}_k^T - \bar{\mathbf{F}}_{k,\alpha}^T\right) \mathbf{\Phi}(t) = 0, \qquad k = 1, 2, \ldots, n. \tag{16}$$

Now, we collocate systems of algebraic equations (15) and (16) at the points $t = \tau_s = \frac{2s-1}{2\theta}$; $s = 1, 2, \ldots, \theta$, and solve them by a proper numerical method such as Newton's method. After obtaining the vectors $\mathrm{U}_k$ for $k = 1, 2, \ldots, n$, the approximate solution of system (1) are determined by using Eq. (2).

## 4 Error Analysis

In current section, by using a Hilbert space called the Sobolev space as well as the relevant norm, an upper bound of the error for the present method is obtained.

**Definition 4.1.** [11] Let $\gamma \geq 0$ be an integer and $(a, b)$ be a bounded real interval. The vector space of the functions $v \in L^2(a, b)$ with the following definition, is called the Sobolev space.

$$H^{\gamma}(a, b) = \left\{ v |\, v^{(m)} \in L^2(a, b), \text{ for } 0 \leq m \leq \gamma \right\}.$$

For the Sobolev space, the relevant norm is defined as

$$\|v\|_{H^{\gamma}(a,b)} = \left( \sum_{m=0}^{\gamma} \|v^{(m)}\|^2_{L^2(a,b)} \right)^{\frac{1}{2}}.$$

**Remark 4.2.** Some properties of the Sobolev spaces are considered as follows:

- $H^0(a, b) \equiv L^2(a, b)$,
- $\ldots H^{\gamma+1}(a, b) \subset H^{\gamma}(a, b) \subset \ldots \subset H^1(a, b) \subset L^2(a, b)$.

**Lemma 4.3.** [11] *Let $\{L_q\}_{q=0}$ be the sequence of Legendre polynomials. Also, assume that $\omega_J(t) = \sum_{j=0}^{J} d_j L_j(t)$ be the best polynomial approximation of degree $J$ for $\omega \in L^2(-1, 1)$. Then, for $\gamma \geq 1$, there exists a positive constant $\delta_0 > 0$ such that*

$$\|\omega - \Omega_J\|_{L^{\infty}(-1,1)} \leq \delta_0 J^{\frac{3}{4}-\gamma} \|\omega\|_{H^{\gamma}(-1,1)},$$

*for all functions $\omega$ in $H^{\gamma}(-1, 1)$.*

**Lemma 4.4.** *Let $\omega \in H^{\gamma}[0, 1)$ and $\omega_{PQ}$ be the approximation of $\omega$ defined in Eq. (2). Then,*

$$\|\omega - \omega_{PQ}\|_{L^{\infty}[0,1)} \leq \delta_0 (PQ)^{\frac{3}{4}-\gamma} \max_{1 \leq p \leq P} \|\omega\|_{H^{\gamma}(I_p)},$$

*where $I_p = [\frac{p-1}{P}, \frac{p}{P})$.*

**Proof.** It can be obviously concluded by using Lemma 4.3. □

Before we express the following theorem, we apply operator ${}_0\mathcal{I}_t^\alpha$ to both sides of equations in system (1) and obtain

$$\begin{cases} {}_0\mathcal{I}_t^\alpha \ {}_0^C\mathcal{D}_t^\alpha u_1 = {}_0\mathcal{I}_t^\alpha f_1(t, u_1, u_2, \dots, u_n), \\ {}_0\mathcal{I}_t^\alpha \ {}_0^C\mathcal{D}_t^\alpha u_2 = {}_0\mathcal{I}_t^\alpha f_2(t, u_1, u_2, \dots, u_n), \\ \vdots \\ {}_0\mathcal{I}_t^\alpha \ {}_0^C\mathcal{D}_t^\alpha u_n = {}_0\mathcal{I}_t^\alpha f_n(t, u_1, u_2, \dots, u_n). \end{cases} \tag{17}$$

Now, by considering Definitions 2.1, 2.2 and Property 2.3, system (17) can be written as

$$\begin{cases} u_1(t) - u_1(0^+) = \displaystyle\int_0^t \frac{(t-\varsigma)^{\alpha-1}}{\Gamma(\alpha)} f_1(\varsigma, u_1(\varsigma), u_2(\varsigma), \dots, u_n(\varsigma))\, \mathrm{d}\varsigma, \\ u_2(t) - u_2(0^+) = \displaystyle\int_0^t \frac{(t-\varsigma)^{\alpha-1}}{\Gamma(\alpha)} f_2(\varsigma, u_1(\varsigma), u_2(\varsigma), \dots, u_n(\varsigma))\, \mathrm{d}\varsigma, \\ \vdots \\ u_n(t) - u_n(0^+) = \displaystyle\int_0^t \frac{(t-\varsigma)^{\alpha-1}}{\Gamma(\alpha)} f_n(\varsigma, u_1(\varsigma), u_2(\varsigma), \dots, u_n(\varsigma))\, \mathrm{d}\varsigma. \end{cases} \tag{18}$$

We represent the system (18) in matrix form as

$$U(t) = U(0^+) + \int_0^t \frac{(t-\varsigma)^{\alpha-1}}{\Gamma(\alpha)} F(\varsigma, U(\varsigma))\, \mathrm{d}\varsigma, \tag{19}$$

where

$$U(t) = [u_1(t), u_2(t), \dots, u_n(t)]^T,$$

and

$$F(\varsigma, U(\varsigma)) = \begin{bmatrix} f_1(\varsigma, U(\varsigma)) \\ f_2(\varsigma, U(\varsigma)) \\ \vdots \\ f_n(\varsigma, U(\varsigma)) \end{bmatrix}.$$

**Theorem 4.5.** *Let $U \in H^\gamma[0,1)$ be the exact solution of Eq. (19), $\hat{U}(t) = U_{PQ}(t)$ be the approximate solution obtained by the HLBPM and $e(t) = U(t) - \hat{U}(t)$ be the error term. Moreover, assume that*

*(a)* $f_k(\varsigma, U(\varsigma))$, $k = 1, 2, \ldots, n$ *be a continuous function for* $0 \le \varsigma \le t < 1$ *and satisfies the Lipschitz condition*

$$\left| f_k(\varsigma, U(\varsigma)) - f_k(\varsigma, W(\varsigma)) \right| \le L_k \left\| U - W \right\|_\infty,$$

*where* $L_k > 0$, $k = 1, 2, \ldots, n$, *is Lipschitz constant.*

*(b)* $\mathcal{K}_\alpha = \frac{1}{\Gamma(\alpha)} \sup\limits_{0 \le t < 1} \int_0^t (t - \varsigma)^{\alpha - 1} \, \mathrm{d}\varsigma$.

*Then, there exists a positive constant* $\delta$ *such that*

$$\|e\|_\infty = \|U - \hat{U}\|_\infty \le \mathcal{K}_\alpha \delta \max_{1 \le p \le P} \|u_r\|_{H^\gamma(I_p)}.$$

**Proof.** Let $\tilde{u}_k(t) = u_{k(PQ)}(t)$, $k = 1, 2, \ldots, n$, be the approximate solution of the system defined in Eq. (19), and $e_k(t) = u_k(t) - \tilde{u}_k(t)$ be the error term. Then,

$$e_k(t) = u_k(t) - \tilde{u}_k(t) = \int_0^t \frac{(t - \varsigma)^{\alpha - 1}}{\Gamma(\alpha)} \left( f_k(\varsigma, U(\varsigma)) - f_k(\varsigma, \tilde{U}(\varsigma)) \right) \mathrm{d}\varsigma.$$

By considering assumptions a and b and $0 \le t < 1$, we obtain

$$|e_k(t)| \le \mathcal{K}_\alpha L_k \left\| U - \tilde{U} \right\|_\infty = \mathcal{K}_\alpha L_k \max_{1 \le k \le n} \|u_k - \tilde{u}_k\|_\infty.$$

Let $L = \max\limits_{1 \le k \le n} L_k$, then

$$\|e_k\|_\infty \le \mathcal{K}_\alpha L \left\| u_r - \tilde{u}_r \right\|_\infty, \qquad r \in \{1, 2, \ldots, n\}. \tag{20}$$

Now, by using Lemma 4.4 and Eq. (20), we have

$$\|e_k\|_\infty \le \mathcal{K}_\alpha L \left\| u_r - \tilde{u}_r \right\|_\infty = \mathcal{K}_\alpha L \left\| u_r - \tilde{u}_r \right\|_{L^\infty[0,1)} \le \mathcal{K}_\alpha \delta \max_{1 \le p \le P} \|u_r\|_{H^\gamma(I_p)}, \tag{21}$$

where $\delta = L \delta_0 (PQ)^{\frac{3}{4} - \gamma}$.
If $e(t) = U(t) - \tilde{U}(t)$, then using (21), we get

$$\|e\|_\infty = \|U - \tilde{U}\|_\infty \le \mathcal{K}_\alpha \delta \max_{1 \le p \le P} \|u_r\|_{H^\gamma(I_p)}.$$

□

## 5 Numerical Examples

In this section, the effectiveness of the present method is studied by applying the HLBPM on several examples of linear and non-linear systems of FDEs. For all computations, a Matlab 2017a software package on a laptop with core i5-3210M CPU Intel processor and 4GB RAM is used.

**Example 5.1.** As the first example, a linear system of FDEs is considered as [51, 1]

$$\begin{cases} {}_0^C\mathcal{D}_t^\alpha u_1(t) = u_1(t) + u_2(t), \\ {}_0^C\mathcal{D}_t^\alpha u_2(t) = -u_1(t) + u_2(t), \end{cases} \tag{22}$$

subject to the initial conditions $u_1(0) = 0$, $u_2(0) = 1$. For $\alpha = 1$, the exact solution of the system (22) is $u_1(t) = e^t \sin(t)$, $u_2(t) = e^t \cos(t)$. By taking $P = 2, Q = 10$, we apply the HLBPM for solving this system when $\alpha = 1$. Figure 1 shows the obtained absolute errors of $u_1$ and $u_2$. Also, in Table 1 a comparison between the absolute errors of $u_1$ and $u_2$ obtained by the present method and the method in [51] is shown. It is noted that the maximum absolute errors of $u_1$ and $u_2$ on $[0, 1]$ by Haar wavelet collocation method (HWCM) [1] are $4.03865E-07$ and $1.14139E-07$, respectively. It can be obtained from Figure 1 and the tabulated data in Table 1 that the obtained numerical solutions by the HLBPM are in high agreement with the exact solutions.

| t | HLBPM ($P = 2$, $Q = 4$) | | HLBPM ($P = 2$, $Q = 6$) | | Method of [51] ($M = 3$, $k = 4$) | |
|---|---|---|---|---|---|---|
| | $e_{u_1}$ | $e_{u_2}$ | $e_{u_1}$ | $e_{u_2}$ | $e_{u_1}$ | $e_{u_2}$ |
| 0.1 | 2.22e-05 | 8.00e-05 | 3.86e-08 | 7.59e-09 | 5.09E-04 | 6.79E-05 |
| 0.2 | 6.95e-06 | 4.02e-05 | 1.38e-08 | 4.59e-10 | 2.02E-04 | 7.59E-05 |
| 0.3 | 1.62e-05 | 5.00e-05 | 2.80e-08 | 1.03e-08 | 2.50E-04 | 2.85E-05 |
| 0.4 | 1.83e-05 | 7.51e-05 | 5.32e-08 | 1.62e-08 | 2.16E-04 | 4.26E-05 |
| 0.5 | 4.07e-05 | 1.55e-04 | 1.54e-07 | 3.96e-08 | 8.34E-04 | 6.53E-04 |
| 0.6 | 9.78e-05 | 1.01e-04 | 3.40e-08 | 2.64e-08 | 5.48E-03 | 1.39E-04 |
| 0.7 | 3.53e-05 | 4.59e-05 | 6.96e-09 | 4.86e-10 | 1.39E-03 | 3.69E-04 |
| 0.8 | 6.86e-05 | 6.54e-05 | 4.40e-08 | 4.81e-08 | 2.72E-03 | 7.74E-04 |
| 0.9 | 8.13e-05 | 8.96e-05 | 7.62e-08 | 7.68e-08 | 1.71E-03 | 7.66E-03 |
| 1.0 | 1.42e-04 | 1.52e-04 | 9.84e-08 | 9.04e-08 | 2.81E-03 | 1.43E-03 |

**Table 1:** Comparison between the absolute errors of $u_1$ and $u_2$ in Example 5.1 for $\alpha = 1$.

Also, for fractional derivatives, we take $P = 2, Q = 10$ and use the HLBPM for $\alpha_j = 0.7 + 0.1j; j = 0, 1, 2, 3$ on $[0, 1]$. The obtained results

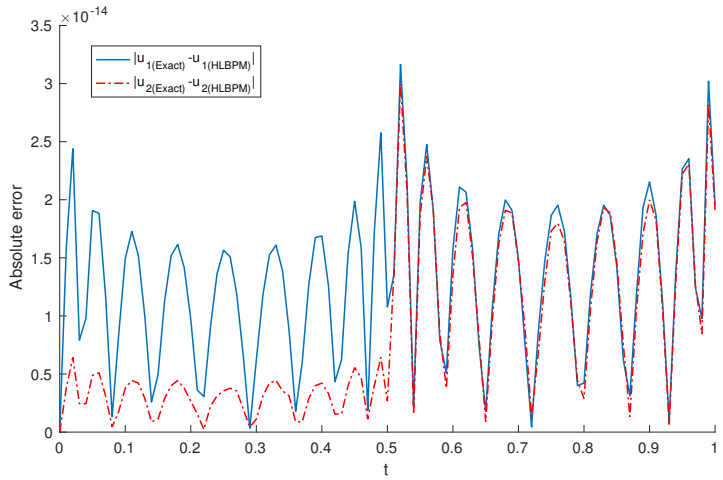
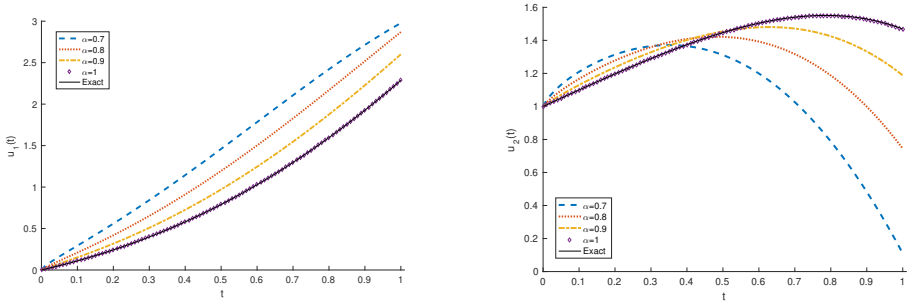



**Figure 1:** The absolute errors of $u_1$ and $u_2$ for $\alpha = 1$ in Example 5.1.




**(a)** **(b)**

**Figure 2:** The solution behavior of Example 5.1 for $\alpha_j = 0.7+0.1j; j = 0, 1, 2, 3$.

are shown in Figures 2a and 2b. In Figure 2, we see that when $\alpha \to 1$, the numerical solutions converge to the solutions of system (22) for $\alpha = 1$, which indicate that the presented method is practicable.

**Example 5.2.** Stiff systems are observed in the study of electrical circuit theory, ballistics, aerodynamics, chemical kinetics and other areas

of applications. Here, we consider a non-linear stiff system of FDEs as [18, 1]

$$\begin{cases} {}_0^C\mathcal{D}_t^\alpha u_1(t) = -1002u_1(t) + 1000u_2^2(t), \\ {}_0^C\mathcal{D}_t^\alpha u_2(t) = u_1(t) - u_2(t) - u_2^2(t), \end{cases} \tag{23}$$

where the initial conditions are $u_1(0) = 1$, $u_2(0) = 1$. When $\alpha = 1$, the exact solution of the system (23) is $u_1(t) = e^{-2t}$, $u_2(t) = e^{-t}$. The absolute errors of $u_1$ and $u_2$ obtained by the HLBPM (for $P = 4$, $Q = 12$) and the fractional residual power series method (FRPS) [18] are shown in Table 2. In Table 3, we give a comparison between the maximum absolute errors by HLBPM, FRPS [18] and HWCM [1] on $[0, 2]$.

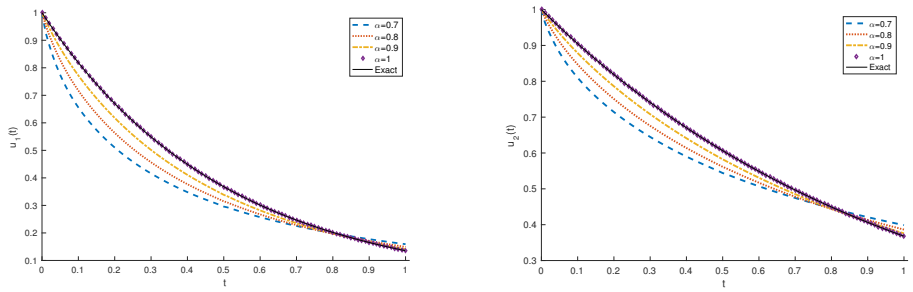
| t | | Absolute error of $u_1$ | | | Absolute error of $u_2$ | |
|---|---|---|---|---|---|---|
| | Exact $u_1(t)$ | HLBPM | FRPS [18] | Exact $u_2(t)$ | HLBPM | FRPS [18] |
| 0.2 | 0.670320046035 | 2.22e-16 | 0.00e+00 | 0.818730753077 | 0.00e+00 | 0.00e+00 |
| 0.4 | 0.449328964117 | 1.11e-16 | 5.55e-17 | 0.670320046035 | 0.00e+00 | 0.00e+00 |
| 0.6 | 0.301194211912 | 2.22e-16 | 0.00e+00 | 0.548811636094 | 0.00e+00 | 0.00e+00 |
| 0.8 | 0.201896517994 | 1.94e-16 | 5.55e-16 | 0.449328964117 | 0.00e+00 | 5.55e-17 |
| 1.0 | 0.135335283236 | 1.67e-16 | 3.76e-14 | 0.367879441171 | 5.55e-17 | 5.55e-17 |
| 1.2 | 0.090717953289 | 2.50e-16 | 1.70e-12 | 0.301194211912 | 1.11e-16 | 0.00e+00 |
| 1.4 | 0.060810062625 | 2.08e-16 | 4.26e-11 | 0.246596963941 | 1.39e-16 | 1.39e-16 |
| 1.6 | 0.040762203978 | 1.73e-16 | 6.93e-10 | 0.201896517994 | 8.33e-17 | 5.55e-16 |
| 1.8 | 0.027323722447 | 1.70e-16 | 8.09e-09 | 0.165298888221 | 8.33e-17 | 4.22e-15 |
| 2.0 | 0.018315638888 | 1.77e-16 | 7.28e-08 | 0.135335283236 | 5.55e-17 | 3.76e-14 |

**Table 2:** The numerical results of Example 5.2 for $\alpha = 1$.

| | HLBPM | FRPS [18] | (HWCM) [1] |
|---|---|---|---|
| $u_1$ | 2.50e-16 | 7.28e-08 | 4.48e-06 |
| $u_2$ | 1.39e-16 | 3.76e-14 | 1.37e-06 |

**Table 3:** The maximum absolute errors of $u_1$ and $u_2$ for Example 5.2 for $\alpha = 1$.

For fractional derivatives, the HLBPM (for $P = 2$, $Q = 8$) is used for $\alpha_j = 0.7 + 0.1j; j = 0, 1, 2, 3$ on $[0, 1]$. The obtained results are shown in Figures 3a and 3b. As the Example 5.1, we see that when $\alpha \to 1$, the numerical solutions converge to the solutions of system (23) for $\alpha = 1$.




**(a)** **(b)**

**Figure 3:** The solution behavior of Example 5.2 for $\alpha_j = 0.7+0.1j$; $j = 0, 1, 2, 3$.

**Example 5.3.** In this example, an other non-linear system of FDEs is considered as [51, 1]

$$\begin{cases} {}_0^C\mathcal{D}_t^{\alpha} u_1(t) = \dfrac{3}{4}u_1^2(t), \\ {}_0^C\mathcal{D}_t^{\alpha} u_2(t) = u_1(t)u_2(t) - \dfrac{1}{8}u_2^4(t) + 2, \end{cases} \tag{24}$$

with the initial conditions $u_1(0) = 0$, $u_2(0) = 0$. When $\alpha = 1$, the exact solution of this system is $u_1(t) = t^3$, $u_2(t) = 2t$. Here, we take $P = 2$, $Q = 4$ and apply the present method for solving system (24). Some numerical results including a comparison between the absolute errors of $u_1$ and $u_2$ obtained by the HLBPM and the method in [51] are shown in Table 4. Also, it is noted that the maximum absolute errors of $u_1$ and $u_2$ on $[0, 1]$ by HWCM [1] are $7.10139E{-}06$ and $4.90903E{-}06$, respectively. It can be found from Figure 4 and Table 4 that the approximate solution by HLBPM and the exact solution are in excellent agreement. Also, Figures 5a and 5b show the solution behavior obtained by the present method at the fractional orders $\alpha_j = 0.5 + 0.1j; j = 0, 1, \ldots, 5$ on $[0, 1]$ which indicate that when $\alpha \to 1$, the numerical solutions converge to the solutions of system (24) for $\alpha = 1$.

**Example 5.4.** As the last example, we consider a variable coefficients

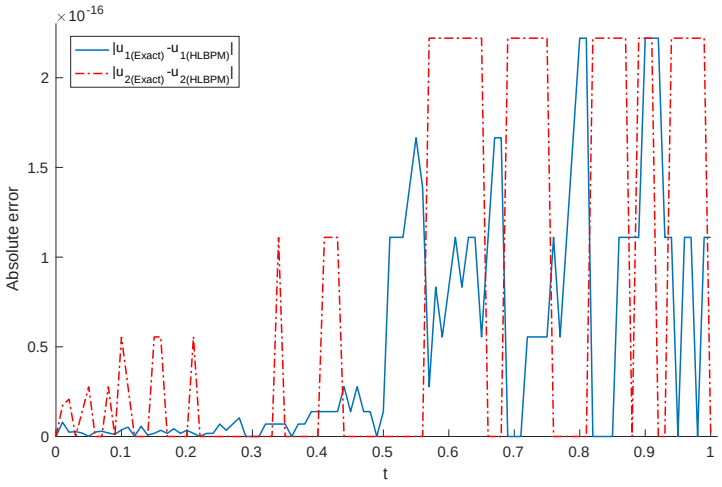
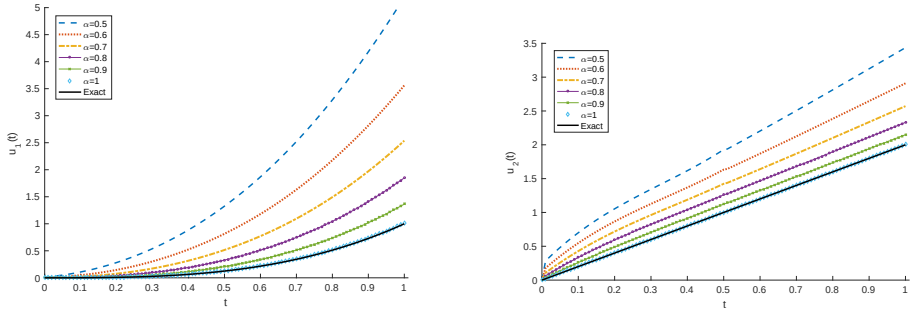



**Figure 4:** The absolute errors of $u_1$ and $u_2$ for $\alpha = 1$ in Example 5.3.




**(a)** **(b)**

**Figure 5:** The solution behavior of Example 5.3 for $\alpha_j = 0.5+0.1j$; $j = 0, 1, 2, 3$.

| t | Exact | | HLBPM ($P=2$, $Q=4$) | | Method of [51] ($M=3$, $k=4$) | |
|---|---|---|---|---|---|---|
| | $u_1$ | $u_2$ | $e_{u_1}$ | $e_{u_2}$ | $e_{u_1}$ | $e_{u_2}$ |
| 0.2 | 0.008 | 0.4 | 1.56E-17 | 5.55E-17 | 4.19E-05 | 1.16E-06 |
| 0.4 | 0.064 | 0.8 | 4.16E-17 | 2.22E-16 | 8.52E-05 | 9.21E-06 |
| 0.6 | 0.216 | 1.2 | 1.39E-16 | 4.44E-16 | 1.39E-04 | 3.03E-05 |
| 0.8 | 0.512 | 1.6 | 1.11E-16 | 2.22E-16 | 2.02E-04 | 6.69E-05 |
| 1.0 | 1.000 | 2.0 | 0.00E+00 | 2.22E-16 | 2.76E-04 | 1.14E-04 |

**Table 4:** Numerical results for Example 5.3 for $\alpha = 1$.

system of FDEs as

$$\begin{cases} {}_0^C\mathcal{D}_t^\alpha u_1(t) = t\,u_1(t), \\ {}_0^C\mathcal{D}_t^\alpha u_2(t) = 2t\,u_1^2(t), \\ {}_0^C\mathcal{D}_t^\alpha u_3(t) = 3t\,u_1(t)u_2(t), \end{cases} \tag{25}$$

subject to the initial conditions $u_1(0) = 1$, $u_2(0) = 1$, $u_3(0) = 1$. The exact solutions of this system for $\alpha = 1$ are given by $u_1(t) = e^{\frac{1}{2}t^2}$, $u_2(t) = e^{t^2}$, $u_3(t) = e^{\frac{3}{2}t^2}$.

We apply the HLBPM (for $P = 2$, $Q = 8$) for solving this system and report the obtained absolute errors for $\alpha = 1$ in Table 5. The tabulated results show that the approximate solutions are in a desired agreement with the exact solutions. Also, to show the solution behavior for some fractional orders, the reader is referred to Figures 6a, 6b and 6c.

| t | Exact | | | HLBPM | | | Absolute error | | |
|---|---|---|---|---|---|---|---|---|---|
| | $u_1$ | $u_2$ | $u_3$ | $u_1$ | $u_2$ | $u_3$ | $e_{u_1}$ | $e_{u_2}$ | $e_{u_3}$ |
| 0.1 | 1.005013 | 1.010050 | 1.015113 | 1.005013 | 1.010050 | 1.015113 | 2.63e-10 | 5.56e-09 | 3.61e-08 |
| 0.2 | 1.020201 | 1.040811 | 1.061837 | 1.020201 | 1.040811 | 1.061837 | 8.33e-11 | 1.98e-09 | 1.40e-08 |
| 0.3 | 1.046028 | 1.094174 | 1.144537 | 1.046028 | 1.094174 | 1.144537 | 5.75e-12 | 4.12e-10 | 4.17e-09 |
| 0.4 | 1.083287 | 1.173511 | 1.271249 | 1.083287 | 1.173511 | 1.271249 | 1.80e-10 | 3.44e-09 | 2.05e-08 |
| 0.5 | 1.133148 | 1.284025 | 1.454991 | 1.133148 | 1.284025 | 1.454991 | 4.47e-10 | 9.04e-09 | 5.66e-08 |
| 0.6 | 1.197217 | 1.433329 | 1.716007 | 1.197217 | 1.433329 | 1.716006 | 1.29e-09 | 6.20e-08 | 7.71e-07 |
| 0.7 | 1.277621 | 1.632316 | 2.085482 | 1.277621 | 1.632316 | 2.085482 | 5.97e-10 | 3.07e-08 | 4.01e-07 |
| 0.8 | 1.377128 | 1.896481 | 2.611696 | 1.377128 | 1.896481 | 2.611696 | 2.90e-10 | 1.67e-08 | 2.35e-07 |
| 0.9 | 1.499303 | 2.247908 | 3.370294 | 1.499302 | 2.247908 | 3.370294 | 4.28e-10 | 1.73e-08 | 1.82e-07 |
| 1.0 | 1.648721 | 2.718282 | 4.481689 | 1.648721 | 2.718282 | 4.481689 | 6.25e-11 | 4.91e-09 | 8.02e-08 |

**Table 5:** Numerical results of Example 5.4 for $\alpha = 1$.

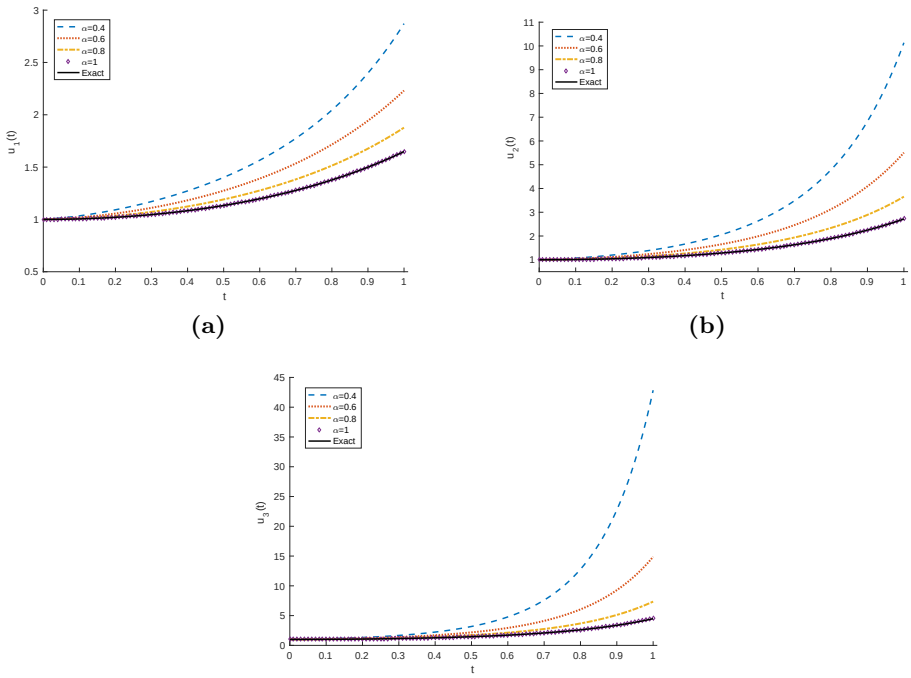



**Figure 6:** The solution behavior of Example 5.4 for $\alpha_j = 0.4+0.2j$; $j = 0, 1, 2, 3$.

## Conclusion

In the present work, the hybrid Legendre Block-pulse method (HLBPM) has been successfully used for solving both linear and non-linear systems of fractional ordinary differential equations. By using this method, systems of FDEs were reduced to linear or non-linear systems of algebraic equations which can be solved by a proper method such as Newton's method. Also, an upper bound of the error was obtained for the proposed method. Finally, some numerical examples were simulated to show the effectiveness of the method. It can be found from these examples that the obtained numerical solutions by the HLBPM are in high agreement with the exact solutions. Also, for the fractional orders $\alpha$ when $\alpha \to 1$, the obtained numerical solutions converge to the solution of systems for $\alpha = 1$, which indicate that the presented method is practicable.

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