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A method of weighted residuals for solving fractional boundary value problems

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Abstract. In this paper, we proposed an approximation scheme for solving boundary value problems of fractional order with a finite element called the method of Weighted residuals. The fractional differential operators are taken in the Riemann-Liouvill and Coputo sense. Numerical examples are provided to show that the numerical method is easy to apply and computationally efcient.

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1 Introduction

In the last few decades, fractional-order models are found to be more adequate than integer-order models for some real world problems. Fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. This is the main advantage of fractional differential equations in comparison with classical integer-order models[1]. Fractional differential equations arise in many engineering and scientific disciplines as the mathematical

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modelling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, and so forth, involves derivatives of fractional order. In consequence, the subject of fractional differential equations is gaining much importance and attention. For examples and details, see([1]-[7], [11]-[14]) and the references therein.

In this paper we use Weighted residuals method with simple base, to solve linear and nonlinear boundary value problems of fractional order. Different kind of Examples of linear and nonlinear boundary value problems of fractional order are given to demonstrate the ability of the proposed method.

This paper has been organized as follows: section 2 gives notations and basic definitions. Section 3 consists of main results of this paper, in which Weighted residuals method has been applied on the boundary value problems of fractional order. Finally two illustrative examples are given in section 4.

2 Preliminaries and notations

In order to proceed, we need the following definitions of fractional derivatives and integrals. We first introduce the Riemann-Liouville definition of fractional derivative operator J_a^α.

Definition 2.1. *Let $\alpha \in R^+$. The operator J_a^α, defined on the usual Lebesgue space $L_1[a,b]$ by*

$$J_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds, \tag{1}$$

$$J_a^0 f(t) = f(t),$$

for $a \leq t \leq b$, is called the Riemann-Liouville fractional integral operator of order α.

Properties of the operator J_a^α can be found in [11]. For $f \in L_1[a,b], \alpha, \beta \geq 0$ and $\gamma > -1$, we mention only the following:

$J_a^\alpha f(t)$ *exists for almost every* $t \in [a,b]$,

$$J_a^\alpha J_a^\beta f(t) = J_a^{\alpha+\beta} f(t),$$

$$J_a^\alpha J_a^\beta f(t) = J_a^\beta J_a^\alpha f(t),$$

$$J_a^\alpha (t-a)^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}(t-a)^{\alpha+\gamma}.$$

Definition 2.2. *The fractional derivative of $f(t)$ in the Riemann-Liouville sense is defined as*

$$D_a^\alpha f(t) = D^m J_a^{m-\alpha} f(t) = \frac{d^m}{dt^m}\frac{1}{\Gamma(m-\alpha)}\int_a^t (t-s)^{m-\alpha-1} f(s)ds, \quad (2)$$

where $m \in N$ and satisfies the relation $m-1 < \alpha \leq m$, and $f \in L_1[a,b]$. Properties of the operator D_a^α can be found in [15, 16]. For $m-1 < \alpha \leq m$, $t > a$ and $\gamma > -1$ we mention only the following:

$$D_a^\alpha k = \frac{k(t-a)^{-\alpha}}{\Gamma(1-\alpha)},$$

$$D_a^\alpha (t-a)^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)}(t-a)^{\gamma-\alpha},$$

$$D_a^\alpha J_a^\alpha f(t) = f(t).$$

In passing, we remark that the definition of Riemann-Liouville fractional derivative, which dose certainly play an important role in the development of theory of fractional derivatives and integrals, could hardly produce the physical interpretation of the initial conditions required for the initial value problems involving fractional differential equations. The same applies to the boundary value problems of fractional differential equations. It was Caputo definition of fractional derivative $D_*^\alpha f(t)$ which solved this problem. In fact, the Caputo derivative becomes the conventional nth derivative of the function $f(t)$ as $\alpha \to n$ and the initial conditions for fractional differential equations retain the same form as that of ordinary differential equations with integer derivatives. Another difference is that the Caputo derivative for a constant is zero while the Riemann-Liouville fractional derivative of a constant is nonzero. For more details, see [6, 11, 12].

Definition 2.3. *The fractional derivative of $f(t)$ in the Caputo sense is defined as*

$$D_*^\alpha f(t) = J^{m-\alpha} D^m f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-s)^{m-\alpha-1} f^{(m)}(s) ds, \quad (3)$$

$$D_*^\alpha J^\alpha f(t) = f(t),$$

for $m-1<\alpha\leq m, m\in N, t>0$. Also, if $m-1<\alpha\leq m$, $t>a$ then

$$D_*^\alpha k = 0,$$

$$D_*^\alpha (J_a^\alpha f(t)) = f(t),$$

$$J_a^\alpha (D_*^\alpha f(t)) = f(t) - \sum_{k=0}^{m-1} f^{(k)}(a) \frac{(t-a)^k}{k!}.$$

3 Weighted residuals method

Suppose we have the boundary value problem of fractional order

$$D^\alpha [y(t)] + L[y(t)] + N[y(t)] = f(t), \qquad m-1<\alpha\leq m, \quad (4)$$

$$y(a)=\alpha, \qquad y(b)=\beta, \qquad a\leq t\leq b,$$

where α and β are constants. The term $D^\alpha[y(t)]$ denotes a linear fractional differential operator, $L[y(t)]$ is a linear differential operator, $N[y(t)]$ is a nonlinear operator and $f(t)$ is a given function.
We will approximate the solution $y(t)$ as

$$\widehat{y}(t) = \sum_{i=0}^{n} c_i \phi_i(t), \quad (5)$$

where n is the number of unknown parameters, and each ϕ_i is an independent basis function. Hence, we denote $\widehat{y}(t)$ as the trial functions. The goal in method of Weighted residuals is the determination of the $(n+1)$ scalars $\{c_i\}_{i=0}^n$.
Hence an error or residual will exist

$$E(t) = R(t) = D^\alpha[\widehat{y}(t)] + L[\widehat{y}(t)] + N[\widehat{y}(t)] - f(t) \neq 0. \quad (6)$$

The notion in the method of Weighted residuals is to force the residual to zero in some average sense over the domain $T = [a, b]$. That is

$$\int_T R(t)W_i(t)dt = 0, \qquad i = 0, 1, \cdots, n, \tag{7}$$

where $\{W_i\}_{i=0}^n$ are the test functions or weights. A good choice of basis functions for boundary value problems of fractional order are the fractional power polynomials

$$\varphi_i(t) = \left\{t^{\frac{c}{d}+\frac{e}{f}i}\right\}_{i=0}^n$$

where c, d, e and f are constants. The result is a set of $(n+1)$ algebraic equations for the unknown constants c_i. There are (at least) three method of Weighted residuals sub-methods, according to the choices for the W_i's. These three methods are:

1. *Least Squares method*
2. $Sub - domain\ method$,
3. *Galerkin method.*

Each of these will be explained below [8, 10].

3.1 Least Squares method

If the continuous summation of all the squared residuals is minimized, the rationale behind the name can be seen. In other words, a minimum of

$$S = \int_T R(t)R(t)dt = \int_T R^2(t)dt. \tag{8}$$

In order to achieve a minimum of this scalar function, the derivatives of S with respect to all the unknown parameters must be zero. That is,

$$\begin{aligned} 0 &= \frac{\partial S}{\partial c_i} \\ &= 2\int_T R(t)\frac{\partial R}{\partial c_i}dt. \end{aligned} \tag{9}$$

Comparing with 3.1, the weight functions are seen to be

$$W_i = 2\frac{\partial R}{\partial c_i},$$

however, the 2 can be dropped, since it cancels out in the equation. Therefore the weight functions for the Least Squares method are just the derivatives of the residual with respect to the unknown constants:

$$W_i = \frac{\partial R}{\partial c_i}.$$

3.2 Sub-domain method

This method doesn't use weighting factors explicity, so it is not, strictly speaking, a member of the Weighted residuals family. However, it can be considered a modification of the collocation method. The idea is to force the weighted residual to zero not just at fixed points in the domain, but over various subsections of the domain. To accomplish this, the weight functions are set to unity, and the integral over the entire domain is broken into a number of subdomains sufficient to evaluate all unknown parameters.
That is,

$$\int_T R(t)W_i(t)dt = \sum_i \left(\int_{T_i} R(t)dt \right) = 0, \quad i = 0, 1, \cdots, n. \tag{10}$$

3.3 Galerkin method

In this method, the weight functions are chosen from the fractional power polynomials. That is,

$$W_i = t^{\frac{c}{d}+\frac{e}{f}i}, \quad i = 0, 1, \cdots, n.$$

In the event that the basis functions for the approximation (the $\varphi_i' s$) were chosen as fractional power polynomials.

4 Numerical examples

In this section, two examples are presented in order to show the ability and efficiency of the proposed method. The algorithms are performed by Maple 12 with 10 digits precision. For two examples, we take $t_0 = 0, t_i = t_0 + 0.05\ i$ for $i = 1, \cdots, 20$ and results for $n = 1, 2$ and 3 are reported.

RMS errors

A reasonable scalar index for the closeness of two functions is the L_2 norm, or Euclidian norm. This measure is often called the root-mean squared (RMS) error in engineering. The RMS error can be defined as

$$E_{RMS} = \frac{\sqrt{\int (y(t) - \widehat{y}(t))^2 dt}}{\int dt},$$

which in discrete terms can be evaluated as

$$E_{RMS} = \sqrt{\frac{\sum_{i=1}^{n} (y_i - \widehat{y}_i)^2}{n}}.$$

Example 4.1. Consider the linear boundary value problem of Riemann-Liouvill fractional order

$$y''(t) + \sin t D^{0.5} y(t) + t y(t) = f(t), \quad 0 < t < 1, \tag{11}$$

with the boundary condition:

$$y(0) = y(1) = 0,$$

where

$$f(t) = t^9 - t^8 + 56t^6 - 42t^5 + \sin t(\frac{32768}{6435\sqrt{\pi}} t^{7.5} - \frac{2048}{429\sqrt{\pi}} t^{6.5}), \tag{12}$$

and the exact solution is $y(t) = t^8 - t^7$[5].
Let's solve the above example by the method of Weighted residuals using

Table 1: Numerical results for Example 1 using $\varphi_i(t) = \{t^{6+\frac{95}{100}i}\}_{i=0}^{n}$

n	$E_{RMS} - Galerkin$	$E_{RMS} - LeastSquares$	$E_{RMS} - Subdomain$
1	0.06337614509	0.06201871147	0.05913693352
2	0.01942054388	0.02013470606	0.01609329029
3	0.02147222606	0.02395010463	0.01667516339

Table 2: Numerical results for Example 1 using $\varphi_i(t) = \{t^{8-\frac{95}{100}i}\}_{i=0}^{n}$

n	$E_{RMS} - Galerkin$	$E_{RMS} - LeastSquares$	$E_{RMS} - Subdomain$
1	0.02403628953	0.02342287470	0.01948231563
2	0.01895034797	0.01994715241	0.01654689139
3	0.02192845645	0.02441210961	0.01653205299

a fractional power polynomial functions as a basis. That is, let the approximating function $\widehat{y}(t)$ be

$$\widehat{y}(t) = \sum_{i=0}^{n} c_i t^{\frac{c}{d}+\frac{e}{f}i}.$$

By applying the boundary condition and calculating the second derivative and 0.5 Riemann-Liouvill derivative of $\widehat{y}(t)$ the residual $R(t)$ could be found:

$$R(t) = \widehat{y}''(t) + \sin t D^{0.5}\, \widehat{y}(t) + t\widehat{y}(t) - f(t), \tag{13}$$

The computational results are summarized in Tables 1 and 2.

Example 4.2. Consider the nonlinear boundary value problem of Coputo fractional order

$$D_*^{0.25} y(t) + t y^2(t) = f(t), \quad 0 < t < 1, \tag{14}$$

with the boundary condition

$$y(0) = 0, \quad y(1) = 1,$$

where

$$f(t) = \frac{32}{21\Gamma(0.75)} t^{1.75} + t^5,$$

Table 3: Numerical results for Example 2 using $\varphi_i(t) = \{t^{\frac{175}{100}+\frac{25}{100}i}\}_{i=0}^{n}$

n	$E_{RMS} - Galerkin$	$E_{RMS} - LeastSquares$	$E_{RMS} - Subdomain$
1	0.0	0.0	0.000003244042158
2	0.0	0.0	0.000006365831408
3	0.0	0.0	0.000007910210400

and the exact solution is $y(t) = t^2$[9].
We solve the mentioned example by the method of Weighted residuals using a fractional power polynomial function as a basis. That is, let the approximating function $\widehat{y}(t)$ be

$$\widehat{y}(t) = \sum_{i=0}^{n} c_i t^{\frac{c}{d}+\frac{e}{f}i}.$$

By applying the boundary condition and calculating 0.25 Coputo derivative of $\widehat{y}(t)$ the residual $R(t)$ could be found:

$$R(t) = D_*^{0.25}\widehat{y}(t) + t\widehat{y}^2(t) - \frac{32}{21\Gamma(0.75)}t^{1.75} - t^5. \tag{15}$$

The computational results are summarized in Tables 3 and 4.

In the Weighted residuals solutions, the basis of $\{t^{2-\frac{95}{100}i}\}_{i=0}^{n}$ form for $n = 3, \cdots$ are not used because the Coputo derivative of order 0.25 of the approximated function $\widehat{y}(t)$

$$\widehat{y}(t) = \sum_{i=0}^{n} c_i t^{2-\frac{95}{100}i}, \qquad n = 3, 4, \cdots$$

does not exist.

5 Conclusion

In this paper, the method of Weighted residuals for approximate solution of linear and nonlinear boundary value problems of fractional order

Table 4: Numerical results for Example 2 using $\varphi_i(t) = \{t^{2-\frac{95}{100}i}\}_{i=0}^{n}$

n	$E_{RMS} - Galerkin$	$E_{RMS} - LeastSquares$	$E_{RMS} - Subdomain$
1	0.0	0.00001625833120	0.000001214985793
2	0.00001292837411	0.00001901002242	0.000005589105048
3	-	-	-

is introduced and proposed. Moreover, a comparison between the exact solution and three sub-method of Weighted residuals method using the fractional power polynomials basis, shows that the error of the approximation is small and usually only a few iterations leading to very accurate solutions. Two examples of boundary value problems of fractional order were solved by Weighted residuals to illustrate the efficiency and accuracy of the method. By this method in Example 2, we found the exact solution.

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