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Limit Points of Trigonometric Sequences

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Abstract: In this article, we find the set of all limit points of sequences of polynomials with real coefficients, in $\cos n$, $n = 1, 2, 3, \cdots$ with degree less than or equal to three. Also, when the degree is four, the mentioned set is found in some special cases.

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1. Introduction

Finding the limit points of a sequences or, at least, finding some topological properties of the limit points of a sequence is one of the remarkable problems in analysis. For instance, in [2], the authors have found some necessary and sufficient conditions for the connectedness of the set of all limit points of a sequence in a metric space. Some other results on the limit points of certain sequences is obtained, for example, in [3] and [4].

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Our claim in this article is to find the set of all limit points of sequence of polynomials with real coefficients, in $\cos n, n = 1, 2, 3, \cdots$ with degree less than or equal to three. Also, when the degree is four, the mentioned set is found in some cases.

2. Main Results

Theorem 1. *Suppose f is a real valued continuous, periodic function on the real numbers $\mathbb{R}$ and its period is an irrational number α. Then the set of all limit points of the sequence $\{f(n)\}_{-\infty}^{+\infty}$ is the closed interval $[m, M]$ where $m = Min\{f(x) : x \in \mathbb{R}\}$ and $M = Max\{f(x) : x \in \mathbb{R}\}$.*

Proof. Since f is continous and periodic, it is uniformly continuous. So for $\varepsilon > 0$, there exists a $\delta > 0$ such that for every x, y in $\mathbb{R}$, if $|x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$. But the set $\mathbb{Z} + \alpha\mathbb{Z} = \{m + \alpha n : m \in \mathbb{Z}, n \in \mathbb{Z}\}$ is a countable dense subset of $\mathbb{R}$ where $\mathbb{Z}$ denotes the set of all integers. Therefore, for each $x \in \mathbb{R}$, integers m and n can be found so that

$$|m - (n\alpha + x)| < \delta,$$

and consequently, $|f(m) - f(x)| < \varepsilon$. Now, considering the fact that $f(\mathbb{R})$ is a connected subset of $\mathbb{R}$, the result follows. $\square$

We remark that for an irrational number α, $\mathbb{N} + \alpha\mathbb{Z}$ is not dense in $\mathbb{R}$ where $\mathbb{N}$ denotes the natural numbers and so this proof can not be used when replacing $\{f(n)\}_{-\infty}^{+\infty}$ by $\{f(n)\}_{n=1}^{\infty}$. Nevertheless, a direct conclusion of the above theorem runs as follows:

Theorem 2. *Let the function f satisfy the hypotheses of the preceding theorem. Suppose, furthermore, that f is an even function. Then the set of limit points of the sequence $\{f(n)\}_{n=1}^{\infty}$ is the range of f.*

In all that follows, for a sequence $\{p(n)\}_{n=1}^{\infty}$ let L_p be the set of all limit points of this sequence.

Theorem 3. *Let $q(x) = a_0 + a_1x + a_2x^2 + a_3x^3$, where $a_3 \neq 0$, and take $p(n) = q(\cos n)$. If $a_2^2 - 3a_1a_3 < 0$ then $L_p = [m, M]$ where m and M are, respectively, the minimum and maximum of the set $\{q(1), q(-1)\}$. If $a_2^2 - 3a_1a_3 \geqslant 0$ then $L_p = [m, M]$ where m and M are, respectively, the minimum and maximum of the set*

$$\{q(1), q(-1), q(\frac{-a_2 + \sqrt{a_2^2 - 3a_1a_3}}{3a_3}), q(\frac{-a_2 - \sqrt{a_2^2 - 3a_1a_3}}{3a_3})\}$$

Proof. Consider the function p defined on $[0, 2\pi]$ by $p(x) = q(\cos x)$. Then $p(x)$ is clearly even and periodic, allowing us to use Theorem 2; it remains to find the range of p. If $a_2^2 - 3a_1a_3 < 0$ then $p'(x) = 0$ implies that $x = 0, \pi, 2\pi$, and so the only values that $\cos x$ can take are 1 and -1. On the other hand, when $a_2^2 - 3a_1a_3 \geqslant 0$, an easy argument shows that if $p'(x) = 0$ then $\cos x$ can be

$$1, -1, \frac{-a_2 + \sqrt{a_2^2 - 3a_1a_3}}{3a_3}, \quad \text{or} \quad \frac{-a_2 - \sqrt{a_2^2 - 3a_1a_3}}{3a_3}. \quad \square$$

Theorem 4. *Let* $q(x) = a_0 + a_1x + a_2x^2$ *for* $a_2 \neq 0$, *and let* $p(n) = q(\cos n)$. *Then* $L_p = [m, M]$ *where* m *and* M *are, respectively, the minimum and maximum of the set*

$$\{a_0 + a_1 + a_2, a_0 - a_1 + a_2, a_0 - \frac{a_1^2}{4a_2}\}.$$

Proof. Considering $p(x) = a_0 + a_1 \cos x + a_2 \cos^2 x$, $x \in [0, 2\pi]$; it is sufficient to find x in the interval $[0, 2\pi]$ such that $p'(x) = 0$. Then apply Theorem 2.□

Theorem 5. *Let*

$$q(x) = a_0 + a_1a_3x + \frac{a_2a_3}{2}x^2 + \frac{a_1a_4}{3}x^3 + \frac{a_2a_4}{4}x^4,$$

where $a_2a_4 \neq 0$; *and for* $n \in \mathbb{N}$, *take* $p(n) = q(\cos n)$. *If* $a_3a_4 \leqslant 0$ *then* $L_p = [m, M]$ *where* m *and* M *are, respectively, the minimum and* *maximum of the set*

$$\{q(1), q(-1), q(\pm\sqrt{-\frac{a_3}{a_4}}), q(-\frac{a_1}{a_2})\}$$

and if $a_3a_4 > 0$ *then we use the set* $\{q(1), q(-1), q(-\frac{a_1}{a_2})\}$.

Proof. If $\frac{d}{dx}(q(\cos x)) = 0$ then $\sin x = 0$ or

$$\begin{aligned}(a_4\cos^2 x + a_3)(a_2\cos x + a_1) &= a_1a_3 + a_2a_3\cos x \\ &\quad + a_1a_4\cos^2 x + a_2a_4\cos^3 x = 0.\end{aligned}$$

Consequently, if the inequality $a_3a_4 \leqslant 0$ holds, we get $\sin x = 0$ or $\cos x = \pm\sqrt{-a_3/a_4}$ or $\cos x = -a_1/a_2$. Also, whenever $a_3a_4 > 0$ we get $\sin x = 0$ or $\cos x = -a_1/a_2$. In each case, the result holds from Theorem 2. $\square$

Remark 1. *An immediate consequence of Theorem 2, is that the set of limit points of the sequence* $\{\cos n\}_{n=1}^{\infty}$ *is* $[-1, 1]$. *This fact has been proved before, using more complicated techniques. For instance, one can see [1, Problem 3.15, p.14].*

Remark 2. *In Theorems 3, 4 and 5, substituting* $\cos n$ *by* $\sin n$, *one can show that the same results hold for the sequence* $\{\sin n\}_{-\infty}^{+\infty}$.

References

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