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Original Research Paper

On the Asymptotic Stability of Singular Systems with a Constant and Two Time-Varying Delays

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Abstract. This paper deals with the study of asymptotic stability of zero solution of a kind of delay singular system. Based on the delayed-decomposition approach, two new results are obtained on the asymptotic stability of the considered system by using some inequalities and Lyapunov- Krasovskiĭ functionals (LKFs). Two examples with their numerical simulations are provided to illustrate effectiveness of the proposed method by MATLAB software.

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1 Introduction

It is well known that investigation of qualitative properties of singular delay systems is a very attractive topic, but a difficult field of study.

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Up to now, in the relevant literature, numerous interesting results on stability of various singular systems have been obtained by using linear matrix inequalities (LMIs), the Lyapunov method, the Lyapunov-Krasovskii method (LKM) and so on (see, for example [5, 6, 12-14, 26-30] and references therein). Indeed, singular systems can be frequently appeared in many practical engineering systems such as electrical circuit networks, power systems, aerospace engineering, and network control and so on. It should be noted that the books of Dai [6] and Xu and Lam [26] are important reference sources, which include various qualitative results about singular systems. By these information, we would like to say that stability of delay singular systems deserves to the investigations.

As for the next step, we would like to summarize some works related to qualitative behaviors of singular systems.

Cong and Sheng [5] considered a class of singular systems with time-varying delay. By using the LKM and perturbation approach, the authors obtained some sufficient criteria to exponential stability.

Liu [12] took into consideration a linear singular system of first order with constant delay. By help of the LKM and the matrix integral inequalities, the author presented exponential stability criteria for the considered system.

Liu [13] considered the following linear singular system with time-varying delay:

$$E\dot{x}(t) = Ax(t) + Bx(t - h(t)).$$

By using the LKF, integral inequalities and delay decomposition approach, Liu [13] established some sufficient conditions to the asymptotic stability of this singular system.

In [14], Liu et al. derived asymptotic stability criteria for a linear singular time delay system by using the LKF, the LMI and the Wirtinger-based integral inequality.

Some similar results were also obtained on qualitative behaviors of solutions of certain singular system with delay by Xu et al. [27] and Tunç and Yiğit [28,29].

Furthermore, in recent years, very nice and interesting qualitative results have been obtained on various fractional and non-fractional models [2, 4 , 8, 10,11, 15].

The motivation this work comes, in particular, from the results of Liu [13] and those sources in the references of this paper [1, 3, 5-7, 9, 12, 14, 16-30]. Here, we consider the following singular system with a constant and two time-varying delays:

$$\begin{aligned} E\dot{x}(t) =& Ax(t) + B_h x(t-h(t)) + B_d x(t-d(t)) \\ &+ D\int_{t-\tau}^{t} x(s)ds, t \geq 0, \\ x(t) =& \phi(t), t \in [-\tau, 0], \tau = max\{h, d\} > 0, \end{aligned} \tag{1}$$

where $x(t) \in R^n$ is the state vector, $\phi(t)$ is a continuous initial function defined on $[-\tau, 0]$. $A \in R^{n\times n}$ is a negative definite real constant matrix and $E, B_h, B_d, D \in R^{n\times n}$ are real constant matrices such that the matrix $E \in R^{n\times n}$ is singular, and it is assumed that $rankE = r \leq n, n \geq 1$. The time varying delays $h(t)$ and $d(t)$ are continuous differentiable functions and satisfying

$$\begin{aligned} 0 \leq h(t) \leq h, \dot{h}(t) \leq h_k, \\ 0 \leq d(t) \leq d, \dot{d}(t) \leq d_k, \end{aligned}$$

where τ, h, h_k, d and d_k are positive constants.

It is worth to mention that when we compare the singular system (1) with that of Liu [13], which is given the above, the singular system (1) includes, extends and improve that of Liu [13]. Next, the results of this paper include and extend the results of [13], and they have contributions to the topic and the related literature. These are the novelty and contributions of this paper.

The remainder of this paper is organized as follows: Some basic definitions and lemmas are given in Section 2. In Section 3, the main results of this paper are given; two new theorems are proved on the asymptotic stability of singular system (1). In Section 4, two illustrative examples are presented. Section 5 includes a conclusion.

2 Preliminaries

In this section, we give some definitions and lemmas, which are needed in the proofs of the main results.

Definition 2.1. ([13]) The pair (E, A) is said to be regular if $det(sE-A) \neq 0$. The pair (E, A) is said to be impulse-free if $deg(det(sE-A)) = rank(E)$.

Definition 2.2. ([26]) System (1) is said to be regular and impulse-free if the pair (E, A) is regular and impulse-free. System (1) is said to be admissible if it is regular, impulse-free and stable.

Lemma 2.1. ([13]) For any positive semi-definite matrices

$$X = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{12}^T & X_{22} & X_{23} \\ X_{13}^T & X_{23}^T & X_{33} \end{bmatrix} \geq 0$$

the following integral inequality holds:

$$-\int_{t-h(t)}^{t} \dot{x}^T(s) X_{33} \dot{x}(s) ds \leq \int_{t-h(t)}^{t} \begin{bmatrix} x^T(t) & x^T(t-h(t)) & \dot{x}^T(s) \end{bmatrix} \times \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{12}^T & X_{22} & X_{23} \\ X_{13}^T & X_{23}^T & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h(t)) \\ \dot{x}(s) \end{bmatrix} ds.$$

Lemma 2.2 (Schur Complement). ([26]) Given any real matrices P_1, P_2 and P_3 with $P_1 = P_1^T$ and $P_3 > 0$. Then, we have

$$P_1 + P_2 P_3^{-1} P_2^T < 0$$

if and only if

$$\begin{bmatrix} P_1 & P_2 \\ P_2^T & -P_3 \end{bmatrix} < 0$$

or equivalently

$$\begin{bmatrix} -P_3 & P_2^T \\ P_2 & P_1 \end{bmatrix} < 0.$$

Lemma 2.3. **([30])** For any constant positive symmetric matrix $M \in R^{n\times n}$, scalar $r > 0$ and vector function $g : [0, r] \to R^n$, if the below integrations are well defined, then

$$r\int_0^r g^T(s)Mg(s)ds \geq [\int_0^r g(s)ds]^T M[\int_0^r g(s)ds].$$

3 Main Results

Throughout this paper, we assume that the following conditions are satisfied.

(A1) Let α, h, h_k, d, β and d_k be positive constants such that

$$0 \leq h(t) \leq \alpha h, 0 < \alpha < 1, 0 \leq d(t) \leq \beta d, 0 < \beta < 1,$$
$$0 \leq h(t) \leq h, \dot{h}(t) \leq h_k, 0 \leq d(t) \leq d, \dot{d}(t) \leq d_k.$$

(A2) There exist a singular matrix E with $rankE = r \leq n$, positive definite symmetric matrices $H, P, R_i, Q_i, (i = 1, 2, ..., 6)$, any matrix S with appropriate dimension and positive semi definite matrices

$$X = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{12}^T & X_{22} & X_{23} \\ X_{13}^T & X_{23}^T & X_{33} \end{bmatrix} \geq 0, Y = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{12}^T & Y_{22} & Y_{23} \\ Y_{13}^T & Y_{23}^T & Y_{33} \end{bmatrix} \geq 0,$$
$$Z = \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{12}^T & Z_{22} & Z_{23} \\ Z_{13}^T & Z_{23}^T & Z_{33} \end{bmatrix} \geq 0, K = \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{12}^T & K_{22} & K_{23} \\ K_{13}^T & K_{23}^T & K_{33} \end{bmatrix} \geq 0,$$
$$T = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{12}^T & T_{22} & T_{23} \\ T_{13}^T & T_{23}^T & T_{33} \end{bmatrix} \geq 0, W = \begin{bmatrix} W_{11} & W_{12} & W_{13} \\ W_{12}^T & W_{22} & W_{23} \\ W_{13}^T & W_{23}^T & W_{33} \end{bmatrix} \geq 0,$$

such that the following LMIs hold:

$$P^TE = E^TP \geq 0, \tag{2}$$

$$M = \begin{bmatrix} M_{11} & M_{12} & 0 & 0 & M_{15} & 0 & 0 & M_{18} & M_{19} \\ * & M_{22} & M_{23} & 0 & 0 & 0 & 0 & 0 & M_{29} \\ * & * & M_{33} & M_{34} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & M_{44} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & M_{55} & M_{56} & 0 & 0 & M_{59} \\ * & * & * & * & * & M_{66} & M_{67} & 0 & 0 \\ * & * & * & * & * & * & M_{77} & 0 & 0 \\ * & * & * & * & * & * & * & M_{88} & M_{89} \\ * & * & * & * & * & * & * & * & M_{99} \end{bmatrix} < 0, \tag{3}$$

$$\begin{aligned} &E^T(R_1 - X_{33})E \geq 0, E^T(R_2 - Y_{33})E \geq 0, \\ &E^T(R_1 + (1-h_k)R_3 - Z_{33})E \geq 0, E^T(R_4 - T_{33})E \geq 0, \\ &E^T(R_5 - K_{33})E \geq 0, E^T(R_4 + (1-d_k)R_6 - W_{33})E \geq 0, \end{aligned} \tag{4}$$

where $U \in R^{n\times(n-r)}$ is any matrix satisfying $E^TU = 0$ and

$$\begin{aligned} M_{11} =& A^TP + PA + A^TUS^T + SU^TA + \tau H + Q_1 + Q_3 + Q_4 + Q_6 \\ &+ E^T(\alpha h Z_{11} + Z_{13} + Z_{13}^T)E + E^T(\beta d W_{11} + W_{13} + W_{13}^T)E, \\ M_{12} =& PB_h + SU^TB_h + E^T(\alpha h Z_{12} - Z_{13} + Z_{23}^T)E, \\ M_{15} =& PB_d + SU^TB_d + E^T(\beta d W_{12} - W_{13} + W_{23}^T)E, M_{18} = PD, \\ M_{19} =& A^T[\alpha h R_1 + (1-\alpha)hR_2 + \alpha h R_3 + \beta d R_4 + (1-\beta)dR_5 + \beta d R_6], \\ M_{22} =& -(1-h_k)Q_3 + E^T(\alpha h X_{11} + X_{13} + X_{13}^T + \alpha h Z_{22} - Z_{23} - Z_{23}^T)E, \\ M_{23} =& E^T(\alpha h X_{12} - X_{13} + X_{23}^T)E, \\ M_{29} =& B_h^T[\alpha h R_1 + (1-\alpha)hR_2 + \alpha h R_3 + \beta d R_4 + (1-\beta)dR_5 + \beta d R_6], \\ M_{33} =& Q_2 - Q_1 + E^T(\alpha h X_{22} - X_{23} - X_{23}^T + (1-\alpha)hY_{11} + Y_{13}^T + Y_{13})E, \\ M_{34} =& E^T((1-\alpha)hY_{12} - Y_{13} + Y_{23}^T)E, \\ M_{44} =& -Q_2 + E^T((1-\alpha)hY_{22} - Y_{23} - Y_{23}^T)E, \end{aligned}$$

$$M_{55} =-(1-d_k)Q_6+E^T(\beta dT_{11}+T_{13}+T_{13}^T+\beta dW_{22}-W_{23}-W_{23}^T)E,$$
$$M_{56} =E^T(\beta dT_{12}-T_{13}+T_{23}^T)E,$$
$$M_{59} =B_d^T[\alpha hR_1+(1-\alpha)hR_2+\alpha hR_3+\beta dR_4+(1-\beta)dR_5+\beta dR_6],$$
$$M_{66} =Q_5-Q_4+E^T(\beta dT_{22}-T_{23}-T_{23}^T+(1-\beta)dK_{11}+K_{13}^T+K_{13})E,$$
$$M_{67} =E^T((1-\beta)dK_{12}-K_{13}+K_{23}^T)E,$$
$$M_{77} =-Q_5+E^T((1-\beta)dK_{22}-K_{23}-K_{23}^T)E, M_{88}=-\tau^{-1}H,$$
$$M_{89} =D^T[\alpha hR_1+(1-\alpha)hR_2+\alpha hR_3+\beta dR_4+(1-\beta)dR_5+\beta dR_6],$$
$$M_{99} =-[\alpha hR_1+(1-\alpha)hR_2+\alpha hR_3+\beta dR_4+(1-\beta)dR_5+\beta dR_6].$$

Theorem 3.1. If conditions (A1) and (A2) hold, then system (1) is asymptotically stable.

proof. we define an LKF by

$$V(t,x)=\sum_{i=1}^{4}V_i(t,x), \tag{5}$$

where

$$\begin{aligned}
V_1(t,x) =&x^T(t)PEx(t),\\
V_2(t,x) =&\int_{t-\alpha h}^{t}x^T(s)Q_1x(s)ds+\int_{t-h}^{t-\alpha h}x^T(s)Q_2x(s)ds\\
&+\int_{t-h(t)}^{t}x^T(s)Q_3x(s)ds+\int_{t-\beta d}^{t}x^T(s)Q_4x(s)ds\\
&+\int_{t-d}^{t-\beta d}x^T(s)Q_5x(s)ds+\int_{t-d(t)}^{t}x^T(s)Q_6x(s)ds,\\
V_3(t,x) =&\int_{-\alpha h}^{0}\int_{t+\theta}^{t}\dot{x}^T(\alpha)E^TR_1E\dot{x}(\alpha)d\alpha d\theta\\
&+\int_{-h}^{-\alpha h}\int_{t+\theta}^{t}\dot{x}^T(\alpha)E^TR_2E\dot{x}(\alpha)d\alpha d\theta\\
&+\int_{-h(t)}^{0}\int_{t+\theta}^{t}\dot{x}^T(\alpha)E^TR_3E\dot{x}(\alpha)d\alpha d\theta\\
&+\int_{-\beta d}^{0}\int_{t+\theta}^{t}\dot{x}^T(\alpha)E^TR_4E\dot{x}(\alpha)d\alpha d\theta
\end{aligned}$$

$$
\begin{aligned}
&+\int_{-d}^{-\beta d}\int_{t+\theta}^{t}\dot{x}^T(\alpha)E^TR_5E\dot{x}(\alpha)d\alpha d\theta\\
&+\int_{-d(t)}^{0}\int_{t+\theta}^{t}\dot{x}^T(\alpha)E^TR_6E\dot{x}(\alpha)d\alpha d\theta,\\
V_4(t,x)=&\int_{-\tau}^{0}\int_{t+\theta}^{t}x^T(\alpha)Hx(\alpha)d\alpha d\theta.
\end{aligned}
$$

It is clear that the LKF (5) is positive definite. Calculating the derivative of the LKF $V(t,x)$ along the solutions of the system (1), we obtain

$$\dot{V}(t,x)=\sum_{i=1}^{4}\dot{V}_i(t,x), \tag{6}$$

where

$$
\begin{aligned}
\dot{V}_1(t,x)=&x^T(t)A^TPx(t)+x^T(t)PAx(t)+x^T(t-h(t))B_h^TPx(t)\\
&+x^T(t-d(t))B_d^TPx(t)+x^T(t)PB_hx(t-h(t))\\
&+x^T(t)PB_dx(t-d(t))+(\int_{t-\tau}^{t}x(s)ds)^TD^TPx(t)\\
&+x^T(t)PD\int_{t-\tau}^{t}x(s)ds, \qquad (7)\\
\dot{V}_2(t,x)=&x^T(t)[Q_1+Q_3+Q_4+Q_6]x(t)\\
&-x^T(t-h(t))(1-\dot{h}(t))Q_3x(t-h(t))\\
&+x^T(t-\alpha h)[Q_2-Q_1]x(t-\alpha h)-x^T(t-h)Q_2x(t-h)\\
&-x^T(t-d(t))(1-\dot{d}(t))Q_6x(t-d(t))\\
&+x^T(t-\beta d)[Q_5-Q_4]x(t-\beta d)-x^T(t-d)Q_5x(t-d)\\
\leq&x^T(t)[Q_1+Q_3+Q_4+Q_6]x(t)\\
&-x^T(t-h(t))(1-h_k)Q_3x(t-h(t))\\
&+x^T(t-\alpha h)[Q_2-Q_1]x(t-\alpha h)-x^T(t-h)Q_2x(t-h)\\
&-x^T(t-d(t))(1-d_k)Q_6x(t-d(t))\\
&+x^T(t-\beta d)[Q_5-Q_4]x(t-\beta d)-x^T(t-d)Q_5x(t-d), \qquad (8)
\end{aligned}
$$

$$
\begin{aligned}
\dot{V}_3(t,x) =& \int_{-\alpha h}^{0} \dot{x}^T(t)E^T R_1 E\dot{x}(t)d\theta - \int_{t-\alpha h}^{t} \dot{x}^T(s)E^T R_1 E\dot{x}(s)ds \\
&+ \int_{-h}^{-\alpha h} \dot{x}^T(t)E^T R_2 E\dot{x}(t)d\theta - \int_{t-h}^{t-\alpha h} \dot{x}^T(s)E^T R_2 E\dot{x}(s)ds \\
&+ \int_{-h(t)}^{0} \dot{x}^T(t)E^T R_3 E\dot{x}(t)d\theta - \int_{t-h(t)}^{t} \dot{x}^T(s)E^T R_3 E\dot{x}(s)ds \\
&+ \int_{-\beta d}^{0} \dot{x}^T(t)E^T R_4 E\dot{x}(t)d\theta - \int_{t-\beta d}^{t} \dot{x}^T(s)E^T R_4 E\dot{x}(s)ds \\
&+ \int_{-d}^{-\beta d} \dot{x}^T(t)E^T R_5 E\dot{x}(t)d\theta - \int_{t-d}^{t-\beta d} \dot{x}^T(s)E^T R_5 E\dot{x}(s)ds \\
&+ \int_{-d(t)}^{0} \dot{x}^T(t)E^T R_6 E\dot{x}(t)d\theta - \int_{t-d(t)}^{t} \dot{x}^T(s)E^T R_6 E\dot{x}(s)ds.
\end{aligned}
$$

Using the Newton-Leibnitz formula ([6]), we find

$$
\begin{aligned}
\dot{V}_3(t,x) =& \dot{x}^T(t)E^T[\alpha h R_1 + (1-\alpha)hR_2 + h(t)R_3]E\dot{x}(t) \\
&- \int_{t-\alpha h}^{t} \dot{x}^T(s)E^T R_1 E\dot{x}(s)ds - \int_{t-h}^{t-\alpha h} \dot{x}^T(s)E^T R_2 E\dot{x}(s)ds \\
&- (1-\dot{h}(t)) \int_{t-h(t)}^{t} \dot{x}^T(s)E^T R_3 E\dot{x}(s)ds \\
&+ \dot{x}^T(t)E^T[\beta d R_4 + (1-\beta)dR_5 + d(t)R_6]E\dot{x}(t) \\
&- \int_{t-\beta d}^{t} \dot{x}^T(s)E^T R_4 E\dot{x}(s)ds - \int_{t-d}^{t-\beta d} \dot{x}^T(s)E^T R_5 E\dot{x}(s)ds \\
&- (1-\dot{d}(t)) \int_{t-d(t)}^{t} \dot{x}^T(s)E^T R_6 E\dot{x}(s)ds \\
\leq& \dot{x}^T(t)E^T[\alpha h R_1 + (1-\alpha)hR_2 + \alpha h R_3]E\dot{x}(t) \\
&- \int_{t-\alpha h}^{t} \dot{x}^T(s)E^T R_1 E\dot{x}(s)ds - \int_{t-h}^{t-\alpha h} \dot{x}^T(s)E^T R_2 E\dot{x}(s)ds \\
&- (1-h_k) \int_{t-h(t)}^{t} \dot{x}^T(s)E^T R_3 E\dot{x}(s)ds \\
&+ \dot{x}^T(t)E^T[\beta d R_4 + (1-\beta)dR_5 + \beta d R_6]E\dot{x}(t)
\end{aligned}
$$

$$
\begin{aligned}
&- \int_{t-\beta d}^{t} \dot{x}^T(s) E^T R_4 E \dot{x}(s) ds - \int_{t-d}^{t-\beta d} \dot{x}^T(s) E^T R_5 E \dot{x}(s) ds \\
&- (1-d_k) \int_{t-d(t)}^{t} \dot{x}^T(s) E^T R_6 E \dot{x}(s) ds. \qquad (9)
\end{aligned}
$$

As for the next step, if the upper bounds for the integral terms of (9) are arranged, then

$$
\begin{aligned}
&- \int_{t-\alpha h}^{t} \dot{x}^T(s) E^T R_1 E \dot{x}(s) ds - \int_{t-h}^{t-\alpha h} \dot{x}^T(s) E^T R_2 E \dot{x}(s) ds \\
&- (1-h_k) \int_{t-h(t)}^{t} \dot{x}^T(s) E^T R_3 E \dot{x}(s) ds \\
&- \int_{t-\beta d}^{t} \dot{x}^T(s) E^T R_4 E \dot{x}(s) ds - \int_{t-d}^{t-\beta d} \dot{x}^T(s) E^T R_5 E \dot{x}(s) ds \\
&- (1-d_k) \int_{t-d(t)}^{t} \dot{x}^T(s) E^T R_6 E \dot{x}(s) ds \\
=& - \int_{t-\alpha h}^{t-h(t)} \dot{x}^T(s) E^T R_1 E \dot{x}(s) ds - \int_{t-h}^{t-\alpha h} \dot{x}^T(s) E^T R_2 E \dot{x}(s) ds \\
&- \int_{t-h(t)}^{t} \dot{x}^T(s) E^T [R_1 + (1-h_k) R_3] E \dot{x}(s) ds \\
&- \int_{t-\beta d}^{t-d(t)} \dot{x}^T(s) E^T R_4 E \dot{x}(s) ds - \int_{t-d}^{t-\beta d} \dot{x}^T(s) E^T R_5 E \dot{x}(s) ds \\
&- \int_{t-d(t)}^{t} \dot{x}^T(s) E^T [R_4 + (1-d_k) R_6] E \dot{x}(s) ds \\
=& - \int_{t-\alpha h}^{t-h(t)} \dot{x}^T(s) E^T (R_1 - X_{33}) E \dot{x}(s) ds \\
&- \int_{t-h}^{t-\alpha h} \dot{x}^T(s) E^T (R_2 - Y_{33}) E \dot{x}(s) ds \\
&- \int_{t-h(t)}^{t} \dot{x}^T(s) E^T [R_1 + (1-h_k) R_3 - Z_{33}] E \dot{x}(s) ds
\end{aligned}
$$

$$
\begin{aligned}
&-\int_{t-\beta d}^{t-d(t)} \dot{x}^T(s)E^T(R_4-T_{33})E\dot{x}(s)ds\\
&-\int_{t-d}^{t-\beta d} \dot{x}^T(s)E^T(R_5-K_{33})E\dot{x}(s)ds\\
&-\int_{t-d(t)}^{t} \dot{x}^T(s)E^T[R_4+(1-d_k)R_6-W_{33}]E\dot{x}(s)ds\\
&-\int_{t-\alpha h}^{t-h(t)} \dot{x}^T(s)E^TX_{33}E\dot{x}(s)ds-\int_{t-h}^{t-\alpha h} \dot{x}^T(s)E^TY_{33}E\dot{x}(s)ds\\
&-\int_{t-h(t)}^{t} \dot{x}^T(s)E^TZ_{33}E\dot{x}(s)ds-\int_{t-\beta d}^{t-d(t)} \dot{x}^T(s)E^TT_{33}E\dot{x}(s)ds\\
&-\int_{t-d}^{t-\beta d} \dot{x}^T(s)E^TK_{33}E\dot{x}(s)ds-\int_{t-d(t)}^{t} \dot{x}^T(s)E^TW_{33}E\dot{x}(s)ds. \qquad (10)
\end{aligned}
$$

Using Lemma 2.1, some terms included in (10) satisfy the following inequalities:

$$
\begin{aligned}
-\int_{t-\alpha h}^{t-h(t)} \dot{x}^T(s)X_{33}\dot{x}(s)ds \le& \int_{t-\alpha h}^{t-h(t)} \begin{bmatrix} x^T(t-h(t)) & x^T(t-\alpha h) & \dot{x}^T(s) \end{bmatrix}\\
&\times \begin{bmatrix} X_{11} & X_{12} & X_{13}\\ X_{12}^T & X_{22} & X_{23}\\ X_{13}^T & X_{23}^T & 0 \end{bmatrix} \begin{bmatrix} x(t-h(t))\\ x(t-\alpha h)\\ \dot{x}(s) \end{bmatrix} ds\\
\le& x^T(t-h(t))[\alpha h X_{11}+X_{13}^T+X_{13}]x(t-h(t))\\
&+x^T(t-h(t))[\alpha h X_{12}-X_{13}+X_{23}^T]\\
&\times x(t-\alpha h)+x^T(t-\alpha h)[\alpha h X_{12}^T-X_{13}^T\\
&+X_{23}]x(t-h(t))+x^T(t-\alpha h)[\alpha h X_{22}\\
&-X_{23}-X_{23}^T]x(t-\alpha h), \qquad (11)
\end{aligned}
$$

$$
\begin{aligned}
-\int_{t-h}^{t-\alpha h} \dot{x}^T(s)Y_{33}\dot{x}(s)ds \le& x^T(t-\alpha h)[(1-\alpha)hY_{11}+Y_{13}^T+Y_{13}]x(t-\alpha h)\\
&+x^T(t-\alpha h)[(1-\alpha)hY_{12}-Y_{13}+Y_{23}^T]x(t-h)\\
&+x^T(t-h)[(1-\alpha)hY_{12}^T-Y_{13}^T+Y_{23}]x(t-\alpha h)\\
&+x^T(t-h)[(1-\alpha)hY_{22}-Y_{23}-Y_{23}^T]x(t-h), \qquad (12)
\end{aligned}
$$

$$
\begin{aligned}
-\int_{t-h(t)}^{t} \dot{x}^T(s) Z_{33} \dot{x}(s) ds \leq & x^T(t)[h(t)Z_{11} + Z_{13}^T + Z_{13}]x(t) \\
& + x^T(t)[h(t)Z_{12} - Z_{13} + Z_{23}^T]x(t-h(t)) \\
& + x^T(t-h(t))[h(t)Z_{12}^T - Z_{13}^T + Z_{23}]x(t) \\
& + x^T(t-h(t))[h(t)Z_{22} - Z_{23} - Z_{23}^T]x(t-h(t)) \\
\leq & x^T(t)[\alpha h Z_{11} + Z_{13}^T + Z_{13}]x(t) \\
& + x^T(t)[\alpha h Z_{12} - Z_{13} + Z_{23}^T]x(t-h(t)) \\
& + x^T(t-h(t))[\alpha h Z_{12}^T - Z_{13}^T + Z_{23}]x(t) \\
& + x^T(t-h(t))[\alpha h Z_{22} - Z_{23} - Z_{23}^T]x(t-h(t)),
\end{aligned} \tag{13}
$$

$$
\begin{aligned}
-\int_{t-\beta d}^{t-d(t)} \dot{x}^T(s) T_{33} \dot{x}(s) ds \leq & x^T(t-d(t))[\beta d T_{11} + T_{13}^T + T_{13}]x(t-d(t)) \\
& + x^T(t-d(t))[\beta d T_{12} - T_{13} + T_{23}^T]x(t-\beta d) \\
& + x^T(t-\beta d)[\beta d T_{12}^T - T_{13}^T + T_{23}]x(t-d(t)) \\
& + x^T(t-\beta d)[\beta d T_{22} - T_{23} - T_{23}^T]x(t-\beta d),
\end{aligned} \tag{14}
$$

$$
\begin{aligned}
-\int_{t-d}^{t-\beta d} \dot{x}^T(s) K_{33} \dot{x}(s) ds \leq & x^T(t-\beta d)[(1-\beta)d K_{11} + K_{13}^T + K_{13}] \\
& \times x(t-\beta d) + x^T(t-\beta d)[(1-\beta)d K_{12} \\
& - K_{13} + K_{23}^T]x(t-d) + x^T(t-d)[(1-\beta)d K_{12}^T \\
& - K_{13}^T + K_{23}]x(t-\beta d) + x^T(t-d)[(1-\beta)d K_{22} \\
& - K_{23} - K_{23}^T]x(t-d),
\end{aligned} \tag{15}
$$

$$
\begin{aligned}
-\int_{t-d(t)}^{t} \dot{x}^T(s) W_{33} \dot{x}(s) ds \leq & x^T(t)[d(t)W_{11} + W_{13}^T + W_{13}]x(t) \\
& + x^T(t)[d(t)W_{12} - W_{13} + W_{23}^T]x(t-d(t)) \\
& + x^T(t-d(t))[d(t)W_{12}^T - W_{13}^T + W_{23}]x(t) \\
& + x^T(t-d(t))[d(t)W_{22} - W_{23} - W_{23}^T]x(t-d(t)) \\
\leq & x^T(t)[\beta d W_{11} + W_{13}^T + W_{13}]x(t)
\end{aligned}
$$

$$
\begin{aligned}
&+ x^T(t)[\beta d W_{12} - W_{13} + W_{23}^T]x(t-d(t)) \\
&+ x^T(t-d(t))[\beta d W_{12}^T - W_{13}^T + W_{23}]x(t) \\
&+ x^T(t-d(t))[\beta d W_{22} - W_{23} - W_{23}^T]x(t-d(t)),
\end{aligned} \tag{16}
$$

Similarly, we can arrange some terms of (9) as the following, respectively:

$$
\begin{aligned}
\dot{x}^T(t)E^T[\alpha hR_1 + (1-\alpha)hR_2 + \alpha hR_3]E\dot{x}(t) = [&x^T(t)A^T \\
&+ x^T(t-h(t)B_h^T + x^T(t-d(t)B_d^T \\
&+ (\int_{t-\tau}^{t} x(s)ds)^T D^T][\alpha hR_1 + (1-\alpha)hR_2 + \alpha hR_3] \\
&\times [Ax(t) + B_h x(t-h(t)) + B_d x(t-d(t)) \\
&+ D\int_{t-\tau}^{t} x(s)ds]
\end{aligned}
$$

$$
\begin{aligned}
=&x^T(t)A^T[\alpha hR_1 + (1-\alpha)hR_2 + \alpha hR_3]Ax(t) \\
&+ x^T(t)A^T[\alpha hR_1 + (1-\alpha)hR_2 + \alpha hR_3]B_h x(t-h(t)) \\
&+ x^T(t)A^T[\alpha hR_1 + (1-\alpha)hR_2 + \alpha hR_3]B_d x(t-d(t)) \\
&+ x^T(t)A^T[\alpha hR_1 + (1-\alpha)hR_2 + \alpha hR_3]D\int_{t-\tau}^{t} x(s)ds \\
&+ x^T(t-h(t))B_h^T[\alpha hR_1 + (1-\alpha)hR_2 + \alpha hR_3]Ax(t) \\
&+ x^T(t-h(t))B_h^T[\alpha hR_1 + (1-\alpha)hR_2 + \alpha hR_3]B_h x(t-h(t)) \\
&+ x^T(t-h(t))B_h^T[\alpha hR_1 + (1-\alpha)hR_2 + \alpha hR_3]B_d x(t-d(t)) \\
&+ x^T(t-h(t))B_h^T[\alpha hR_1 + (1-\alpha)hR_2 + \alpha hR_3]D\int_{t-\tau}^{t} x(s)ds \\
&+ x^T(t-d(t))B_d^T[\alpha hR_1 + (1-\alpha)hR_2 + \alpha hR_3]Ax(t) \\
&+ x^T(t-d(t))B_d^T[\alpha hR_1 + (1-\alpha)hR_2 + \alpha hR_3]B_h x(t-h(t)) \\
&+ x^T(t-d(t))B_d^T[\alpha hR_1 + (1-\alpha)hR_2 + \alpha hR_3]B_d x(t-d(t)) \\
&+ x^T(t-d(t))B_d^T[\alpha hR_1 + (1-\alpha)hR_2 + \alpha hR_3]D\int_{t-\tau}^{t} x(s)ds \\
&+ (\int_{t-\tau}^{t} x(s)ds)^T D^T[\alpha hR_1 + (1-\alpha)hR_2 + \alpha hR_3]Ax(t)
\end{aligned}
$$

$$+ (\int_{t-\tau}^{t} x(s)ds)^T D^T[\alpha hR_1 + (1-\alpha)hR_2 + \alpha hR_3]B_h x(t-h(t))$$
$$+ (\int_{t-\tau}^{t} x(s)ds)^T D^T[\alpha hR_1 + (1-\alpha)hR_2 + \alpha hR_3]B_d x(t-d(t))$$
$$+ (\int_{t-\tau}^{t} x(s)ds)^T D^T[\alpha hR_1 + (1-\alpha)hR_2 + \alpha hR_3]D \int_{t-\tau}^{t} x(s)ds, \quad (17)$$

$$\begin{aligned}\dot{x}^T(t)E^T[\beta dR_4 + (1-\beta)dR_5 + \beta dR_6]E\dot{x}(t) = {} & [x^T(t)A^T \\ & + x^T(t-h(t)B_h^T + x^T(t-d(t)B_d^T \\ & + (\int_{t-\tau}^{t} x(s)ds)^T D^T][\beta dR_4 + (1-\beta)dR_5 + \beta dR_6] \\ & \times [Ax(t) + B_h x(t-h(t)) + B_d x(t-d(t)) \\ & + D\int_{t-\tau}^{t} x(s)ds]\end{aligned}$$

$$\begin{aligned}= & x^T(t)A^T[\beta dR_4 + (1-\beta)dR_5 + \beta dR_6]Ax(t) \\ & + x^T(t)A^T[\beta dR_4 + (1-\beta)dR_5 + \beta dR_6]B_h x(t-h(t)) \\ & + x^T(t)A^T[\beta dR_4 + (1-\beta)dR_5 + \beta dR_6]B_d x(t-d(t)) \\ & + x^T(t)A^T[\beta dR_4 + (1-\beta)dR_5 + \beta dR_6]D\int_{t-\tau}^{t} x(s)ds \\ & + x^T(t-h(t))B_h^T[\beta dR_4 + (1-\beta)dR_5 + \beta dR_6]Ax(t) \\ & + x^T(t-h(t))B_h^T[\beta dR_4 + (1-\beta)dR_5 + \beta dR_6]B_h x(t-h(t)) \\ & + x^T(t-h(t))B_h^T[\beta dR_4 + (1-\beta)dR_5 + \beta dR_6]B_d x(t-d(t)) \\ & + x^T(t-h(t))B_h^T[\beta dR_4 + (1-\beta)dR_5 + \beta dR_6]D\int_{t-\tau}^{t} x(s)ds \\ & + x^T(t-d(t))B_d^T[\beta dR_4 + (1-\beta)dR_5 + \beta dR_6]Ax(t) \\ & + x^T(t-d(t))B_d^T[\beta dR_4 + (1-\beta)dR_5 + \beta dR_6]B_h x(t-h(t)) \\ & + x^T(t-d(t))B_d^T[\beta dR_4 + (1-\beta)dR_5 + \beta dR_6]B_d x(t-d(t)) \\ & + x^T(t-d(t))B_d^T[\beta dR_4 + (1-\beta)dR_5 + \beta dR_6]D\int_{t-\tau}^{t} x(s)ds\end{aligned}$$

$$+ (\int_{t-\tau}^{t} x(s)ds)^T D^T[\beta dR_4 + (1-\beta)dR_5 + \beta dR_6]Ax(t)$$
$$+ (\int_{t-\tau}^{t} x(s)ds)^T D^T[\beta dR_4 + (1-\beta)dR_5 + \beta dR_6]B_h x(t-h(t))$$
$$+ (\int_{t-\tau}^{t} x(s)ds)^T D^T[\beta dR_4 + (1-\beta)dR_5 + \beta dR_6]B_d x(t-d(t))$$
$$+ (\int_{t-\tau}^{t} x(s)ds)^T D^T[\beta dR_4 + (1-\beta)dR_5 + \beta dR_6]D \int_{t-\tau}^{t} x(s)ds. \quad (18)$$

Differentiating $V_4(t,x)$ and using Lemma 2.3, we get

$$\dot{V}_4(t,x) \le \tau x^T(t)Hx(t) - \tau^{-1}(\int_{t-\tau}^{t} x(s)ds)^T H(\int_{t-\tau}^{t} x(s)ds). \quad (19)$$

As the next step, since $E^TU = 0$, we see that

$$0 = 2\dot{x}^T(t)E^TUS^Tx(t),$$
$$0 = x^T(t)[A^TUS^T + SU^TA]x(t) + x^T(t-h(t))B_h^TUS^Tx(t)$$
$$+ x^T(t-d(t))B_d^TUS^Tx(t) + x^T(t)SU^TB_hx(t-h(t))$$
$$+ x^T(t)SU^TB_dx(t-d(t)). \quad (20)$$

On gathering the estimates (6)-(20), we have

$$\dot{V}(t,x) \le \xi^T(t)\Xi\xi(t) - \int_{t-\alpha h}^{t-h(t)} \dot{x}^T(s)E^T(R_1 - X_{33})E\dot{x}(s)ds$$
$$- \int_{t-h}^{t-\alpha h} \dot{x}^T(s)E^T(R_2 - Y_{33})E\dot{x}(s)ds$$
$$- \int_{t-h(t)}^{t} \dot{x}^T(s)E^T[R_1 + (1-h_k)R_3 - Z_{33}]E\dot{x}(s)ds$$
$$- \int_{t-\beta d}^{t-d(t)} \dot{x}^T(s)E^T(R_4 - T_{33})E\dot{x}(s)ds$$
$$- \int_{t-d}^{t-\beta d} \dot{x}^T(s)E^T(R_5 - K_{33})E\dot{x}(s)ds$$
$$- \int_{t-d(t)}^{t} \dot{x}^T(s)E^T[R_4 + (1-d_k)R_6 - W_{33}]E\dot{x}(s)ds,$$

where

$$\xi^T(t) = [\; x^T(t) \quad x^T(t-h(t)) \quad x^T(t-\alpha h) \quad x^T(t-h) \quad x^T(t-d(t)) \quad x^T(t-\beta d) \quad x^T(t-d) \quad (\int_{t-\tau}^{t} x(s)ds)^T \;]$$

and

$$\Xi = \begin{bmatrix} \Xi_{11} & \Xi_{12} & 0 & 0 & \Xi_{15} & 0 & 0 & \Xi_{18} \\ * & \Xi_{22} & \Xi_{23} & 0 & \Xi_{25} & 0 & 0 & \Xi_{28} \\ * & * & \Xi_{33} & \Xi_{34} & 0 & 0 & 0 & 0 \\ * & * & * & \Xi_{44} & 0 & 0 & 0 & 0 \\ * & * & * & * & \Xi_{55} & \Xi_{56} & 0 & \Xi_{58} \\ * & * & * & * & * & \Xi_{66} & \Xi_{67} & 0 \\ * & * & * & * & * & * & \Xi_{77} & 0 \\ * & * & * & * & * & * & * & \Xi_{88} \end{bmatrix}$$

with

$$\begin{aligned}
\Xi_{11} =& A^TP + PA + A^TUS^T + SU^TA + \tau H + Q_1 + Q_3 + Q_4 + Q_6 \\
&+ E^T(\alpha h Z_{11} + Z_{13}^T + Z_{13})E + E^T(\beta d W_{11} + W_{13}^T + W_{13})E \\
&+ A^T[\alpha h R_1 + (1-\alpha)hR_2 + \alpha h R_3 + \beta d R_4 + (1-\beta)dR_5 + \beta d R_6]A, \\
\Xi_{12} =& PB_h + SU^TB_h + E^T(\alpha h Z_{12} - Z_{13} + Z_{23}^T)E \\
&+ A^T[\alpha h R_1 + (1-\alpha)hR_2 + \alpha h R_3 + \beta d R_4 + (1-\beta)dR_5 + \beta d R_6]B_h, \\
\Xi_{15} =& PB_d + SU^TB_d + E^T(\beta d W_{12} - W_{13} + W_{23}^T)E \\
&+ A^T[\alpha h R_1 + (1-\alpha)hR_2 + \alpha h R_3 + \beta d R_4 + (1-\beta)dR_5 + \beta d R_6]B_d, \\
\Xi_{18} =& PD + A^T[\alpha h R_1 + (1-\alpha)hR_2 + \alpha h R_3]D \\
&+ A^T[\beta d R_4 + (1-\beta)dR_5 + \beta d R_6]D, \\
\Xi_{22} =& -(1-h_k)Q_3 + E^T(\alpha h X_{11} + X_{13} + X_{13}^T + \alpha h Z_{22} - Z_{23} - Z_{23}^T)E \\
&+ B_h^T[\alpha h R_1 + (1-\alpha)hR_2 + \alpha h R_3 + \beta d R_4 + (1-\beta)dR_5 + \beta d R_6]B_h, \\
\Xi_{23} =& E^T(\alpha h X_{12} - X_{13} + X_{23}^T)E, \\
\Xi_{25} =& B_h^T[\alpha h R_1 + (1-\alpha)hR_2 + \alpha h R_3 + \beta d R_4 + (1-\beta)dR_5 + \beta d R_6]B_d, \\
\Xi_{28} =& B_h^T[\alpha h R_1 + (1-\alpha)hR_2 + \alpha h R_3 + \beta d R_4 + (1-\beta)dR_5 + \beta d R_6]D, \\
\Xi_{33} =& Q_2 - Q_1 + E^T(\alpha h X_{22} - X_{23} - X_{23}^T + (1-\alpha)hY_{11} + Y_{13} + Y_{13}^T)E, \\
\Xi_{34} =& E^T((1-\alpha)hY_{12} - Y_{13} + Y_{23}^T)E,
\end{aligned}$$

$$\begin{aligned}
\Xi_{44} =& -Q_2 + E^T((1-\alpha)hY_{22} - Y_{23} - Y_{23}^T)E,\\
\Xi_{55} =& -(1-d_k)Q_6 + E^T(\beta dT_{11} + T_{13} + T_{13}^T + \beta dW_{22} - W_{23} - W_{23}^T)E\\
&+ B_d^T[\alpha hR_1 + (1-\alpha)hR_2 + \alpha hR_3 + \beta dR_4 + (1-\beta)dR_5 + \beta dR_6]B_d,\\
\Xi_{56} =& E^T(\beta dT_{12} - T_{13} + T_{23}^T)E,\\
\Xi_{58} =& B_d^T[\alpha hR_1 + (1-\alpha)hR_2 + \alpha hR_3 + \beta dR_4 + (1-\beta)dR_5 + \beta dR_6]D,\\
\Xi_{66} =& Q_5 - Q_4 + E^T(\beta dT_{22} - T_{23} - T_{23}^T + (1-\beta)dK_{11} + K_{13} + K_{13}^T)E,\\
\Xi_{67} =& E^T((1-\beta)dK_{12} - K_{13} + K_{23}^T)E,\\
\Xi_{77} =& -Q_5 + E^T((1-\beta)dK_{22} - K_{23} - K_{23}^T)E,\\
\Xi_{88} =& D^T[\alpha hR_1 + (1-\alpha)hR_2 + \alpha hR_3 + \beta dR_4 + (1-\beta)dR_5 + \beta dR_6]D\\
&- \tau^{-1}H.
\end{aligned}$$

Since $\Xi < 0$, then $\dot{V}(t,x) < 0$. Hence, we conclude that system (1) is asymptotically stable.

Now, we give the below conditions for the next theorem.

B. Assumptions

(A3) Let α, h, h_k, d, β and d_k be positive constants such that the following inequalities hold:

$$\begin{aligned}
&\alpha h \le h(t) \le h, \beta d \le d(t) \le d, 0 \le h(t) \le h, \dot{h}(t) \le h_k,\\
&0 < \alpha < 1, 0 \le d(t) \le d, \dot{d}(t) \le d_k, 0 < \beta < 1.
\end{aligned}$$

(A4) There exist a singular matrix E with $rankE = r \le n$, positive definite symmetric matrices $H, P, R_i, Q_i, (i = 1, 2, ..., 6)$, any matrix S with appropriate dimension and positive semi definite matrices

$$X = \begin{bmatrix} X_{11} & X_{12} & X_{13}\\ X_{12}^T & X_{22} & X_{23}\\ X_{13}^T & X_{23}^T & X_{33}\end{bmatrix} \ge 0, Y = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13}\\ Y_{12}^T & Y_{22} & Y_{23}\\ Y_{13}^T & Y_{23}^T & Y_{33}\end{bmatrix} \ge 0,$$

$$Z = \begin{bmatrix} Z_{11} & Z_{12} & Z_{13}\\ Z_{12}^T & Z_{22} & Z_{23}\\ Z_{13}^T & Z_{23}^T & Z_{33}\end{bmatrix} \ge 0, K = \begin{bmatrix} K_{11} & K_{12} & K_{13}\\ K_{12}^T & K_{22} & K_{23}\\ K_{13}^T & K_{23}^T & K_{33}\end{bmatrix} \ge 0,$$

$$T=\begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{12}^T & T_{22} & T_{23} \\ T_{13}^T & T_{23}^T & T_{33} \end{bmatrix}\geq 0, W=\begin{bmatrix} W_{11} & W_{12} & W_{13} \\ W_{12}^T & W_{22} & W_{23} \\ W_{13}^T & W_{23}^T & W_{33} \end{bmatrix}\geq 0$$

such that the following LMIs hold:

$$P^TE=E^TP\geq 0, \tag{21}$$

$$\overline{M}=\begin{bmatrix} \overline{M}_{11} & \overline{M}_{12} & \overline{M}_{13} & 0 & \overline{M}_{15} & \overline{M}_{16} & 0 & \overline{M}_{18} & \overline{M}_{19} \\ * & \overline{M}_{22} & \overline{M}_{23} & \overline{M}_{24} & 0 & 0 & 0 & 0 & \overline{M}_{29} \\ * & * & \overline{M}_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \overline{M}_{44} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & \overline{M}_{55} & \overline{M}_{56} & 0 & 0 & \overline{M}_{59} \\ * & * & * & * & * & \overline{M}_{66} & 0 & 0 & 0 \\ * & * & * & * & * & * & \overline{M}_{77} & 0 & 0 \\ * & * & * & * & * & * & * & \overline{M}_{88} & \overline{M}_{89} \\ * & * & * & * & * & * & * & * & \overline{M}_{99} \end{bmatrix}<0, \tag{22}$$

$$\begin{aligned} &E^T[R_1+(1-h_k)R_3-X_{33}]E\geq 0,\\ &E^T[R_2+(1-h_k)R_3-Y_{33}]E\geq 0,\\ &E^T(R_2-Z_{33})E\geq 0, E^T(R_5-W_{33})E\geq 0,\\ &E^T[R_4+(1-d_k)R_6-T_{33}]E\geq 0,\\ &E^T[R_5+(1-d_k)R_6-K_{33}]E\geq 0, \end{aligned} \tag{23}$$

where $U\in R^{n\times(n-r)}$ is any matrix satisfying $E^TU=0$ and

$$\begin{aligned} \overline{M}_{11}=&A^TP+PA+A^TUS^T+SU^TA+\tau H+Q_1+Q_3+Q_4+Q_6\\ &+E^T(\alpha hX_{11}+X_{13}+X_{13}^T)E+E^T(\beta dT_{11}+T_{13}+T_{13}^T)E,\\ \overline{M}_{12}=&PB_h+SU^TB_h, \overline{M}_{13}=E^T(\alpha hX_{12}-X_{13}+X_{23}^T)E,\\ \overline{M}_{15}=&PB_d+SU^TB_d, \overline{M}_{16}=E^T(\beta dT_{12}-T_{13}+T_{23}^T)E, \overline{M}_{18}=PD,\\ \overline{M}_{19}=&A^T[\alpha hR_1+(1-\alpha)hR_2+\alpha hR_3+\beta dR_4+(1-\beta)dR_5+\beta dR_6], \end{aligned}$$

$$\begin{aligned}
\overline{M}_{22} =& -(1-h_k)Q_3 + E^T((1-\alpha)hZ_{11} + Z_{13} + Z_{13}^T)E \\
& + E^T((1-\alpha)hY_{22} - Y_{23} - Y_{23}^T)E, \\
\overline{M}_{23} =& E^T((1-\alpha)hY_{12}^T - Y_{13}^T + Y_{23})E, \\
\overline{M}_{24} =& E^T((1-\alpha)hZ_{12} - Z_{13} + Z_{23}^T)E, \\
\overline{M}_{29} =& B_h^T[\alpha hR_1 + (1-\alpha)hR_2 + \alpha hR_3 + \beta dR_4 + (1-\beta)dR_5 + \beta dR_6], \\
\overline{M}_{33} =& Q_2 - Q_1 + E^T(\alpha hX_{22} - X_{23} - X_{23}^T + (1-\alpha)hY_{11} + Y_{13}^T + Y_{13})E, \\
\overline{M}_{34} =& E^T((1-\alpha)hY_{12} - Y_{13} + Y_{23}^T)E, \\
\overline{M}_{44} =& -Q_2 + E^T((1-\alpha)hZ_{22} - Z_{23} - Z_{23}^T)E, \\
\overline{M}_{55} =& -(1-d_k)Q_6 + E^T((1-\beta)dW_{11} + W_{13} + W_{13}^T)E \\
& + E^T((1-\beta)dK_{22} - K_{23} - K_{23}^T)E, \\
\overline{M}_{56} =& E^T((1-\beta)dK_{12}^T - K_{13}^T + K_{23})E, \\
\overline{M}_{59} =& B_d^T[\alpha hR_1 + (1-\alpha)hR_2 + \alpha hR_3 + \beta dR_4 + (1-\beta)dR_5 + \beta dR_6], \\
\overline{M}_{66} =& Q_5 - Q_4 + E^T(\beta dT_{22} - T_{23} - T_{23}^T + (1-\beta)dK_{11} + K_{13}^T + K_{13})E, \\
\overline{M}_{77} =& -Q_5 + E^T((1-\beta)dW_{22} - W_{23} - W_{23}^T)E, \overline{M}_{88} = -\tau^{-1}H, \\
\overline{M}_{89} =& D^T[\alpha hR_1 + (1-\alpha)hR_2 + \alpha hR_3 + \beta dR_4 + (1-\beta)dR_5 + \beta dR_6], \\
\overline{M}_{99} =& -[\alpha hR_1 + (1-\alpha)hR_2 + \alpha hR_3 + \beta dR_4 + (1-\beta)dR_5 + \beta dR_6].
\end{aligned}$$

Theorem 3.2. If assumptions (A3) and (A4) are satisfied, then system (1) is asymptotically stable.

proof. By following the way as in the proof of Theorem 3.1, one can easily complete the proof of this theorem. Therefore, we ignore the details of the proof.

4 Numerical Applications

Example 4.1. As a particular case of singular system (1), we consider the following singular system with mixed delays:

$$\frac{d}{dt}\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}\right) = \begin{bmatrix} -6.885 & 0 \\ -2.95 & -4.125 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$+\begin{bmatrix} -0.4085 & 0 \\ -0.2045 & -0.5095 \end{bmatrix}\begin{bmatrix} x_1(t-0.25sin^2t) \\ x_2(t-0.25sin^2t) \end{bmatrix}$$
$$+\begin{bmatrix} -0.3025 & -0.1035 \\ 0 & -0.6035 \end{bmatrix}\begin{bmatrix} x_1(t-0.35sin^2t) \\ x_2(t-0.35sin^2t) \end{bmatrix}$$
$$+\begin{bmatrix} -1.25 & -2.135 \\ 0 & -0.35 \end{bmatrix}\begin{bmatrix} \int_{t-0.35}^{t} x_1(s)ds \\ \int_{t-0.35}^{t} x_2(s)ds \end{bmatrix},$$

where

$$E=\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, A=\begin{bmatrix} -6.885 & 0 \\ -2.95 & -4.125 \end{bmatrix}, B_h=\begin{bmatrix} -0.4085 & 0 \\ -0.2045 & -0.5095 \end{bmatrix},$$
$$B_d=\begin{bmatrix} -0.3025 & -0.1035 \\ 0 & -0.6035 \end{bmatrix}, D=\begin{bmatrix} -1.25 & -2.135 \\ 0 & -0.35 \end{bmatrix},$$

and

$$0\le h(t)=0.25sin^2(t)\le 0.25=h, \alpha=0.3, \alpha h=0.075, \tau=0.35,$$
$$\dot{h}(t)=0.25sin2t\le 0.25=h_k, 0\le d(t)=0.35sin^2(t)\le 0.35=d,$$
$$\beta=0.4, \beta d=0.14, \dot{d}(t)=0.35sin2t\le 0.35=d_k.$$

Let

$$P=\begin{bmatrix} 9.65 & 0 \\ 0 & 9.85 \end{bmatrix}, Q_1=\begin{bmatrix} 4.1 & 1.85 \\ 1.85 & 4.2 \end{bmatrix}, Q_2=\begin{bmatrix} 2.25 & 2.063 \\ 2.063 & 3.25 \end{bmatrix},$$
$$Q_3=\begin{bmatrix} 3.12 & 2.85 \\ 2.85 & 3.25 \end{bmatrix}, H=\begin{bmatrix} 8.65 & 1.15 \\ 1.15 & 8.95 \end{bmatrix}, Q_4=\begin{bmatrix} 4.2 & 3.063 \\ 3.063 & 5.5 \end{bmatrix},$$
$$Q_5=\begin{bmatrix} 4.1 & 3.85 \\ 3.85 & 4.2 \end{bmatrix}, Q_6=\begin{bmatrix} 3.2 & 3.063 \\ 3.063 & 3.5 \end{bmatrix}, R_1=\begin{bmatrix} 1.725 & 0.625 \\ 0.625 & 0.706 \end{bmatrix},$$
$$R_2=\begin{bmatrix} 0.825 & 0.724 \\ 0.724 & 0.725 \end{bmatrix}, R_3=\begin{bmatrix} 0.245 & 0.234 \\ 0.234 & 0.255 \end{bmatrix}, U=\begin{bmatrix} 0 \\ 2.45 \end{bmatrix},$$
$$R_4=\begin{bmatrix} 1.695 & 0.595 \\ 0.595 & 0.685 \end{bmatrix}, R_5=\begin{bmatrix} 2.655 & 0.015 \\ 0.015 & 0.106 \end{bmatrix}, S=\begin{bmatrix} -1.25 \\ -1.35 \end{bmatrix},$$
$$R_6=\begin{bmatrix} 0.205 & 0.025 \\ 0.025 & 0.103 \end{bmatrix}.$$

By some elementary calculations, in this example, for the corresponding special matrix M of (3), the eigenvalues of this constant matrix satisfy $\lambda_{max}(M) \leq -0.0149$. Then, the singular system of Example 4.1 is asymptotically stable.

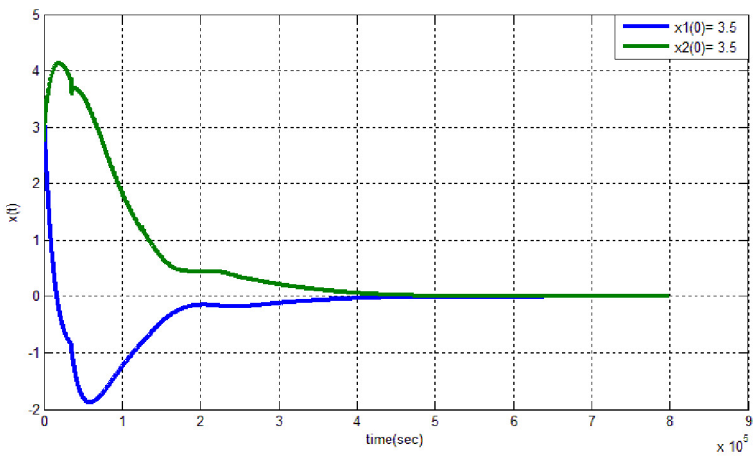

Figure 1: Trajectories of the solution $(x_1(t), x_2(t))$ of system of Example 4.1, when $\tau = 0.35$.

Example 4.2. As a special case of system (1), we consider the following singular system with mixed delays

$$\begin{aligned} \frac{d}{dt}\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}\right) &= \begin{bmatrix} -5.885 & 0 \\ -1.95 & -4.125 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \\ &+ \begin{bmatrix} -0.1085 & 0 \\ -0.2045 & -0.5095 \end{bmatrix}\begin{bmatrix} x_1(t-0.25sin^2t) \\ x_2(t-0.25sin^2t) \end{bmatrix} \\ &+ \begin{bmatrix} -0.1025 & -0.1035 \\ 0 & -0.6035 \end{bmatrix}\begin{bmatrix} x_1(t-0.35sin^2t) \\ x_2(t-0.35sin^2t) \end{bmatrix} \\ &+ \begin{bmatrix} -1.25 & -0.135 \\ 0 & 0.35 \end{bmatrix}\begin{bmatrix} \int_{t-0.35}^{t} x_1(s)ds \\ \int_{t-0.35}^{t} x_2(s)ds \end{bmatrix}, \end{aligned}$$

where

$$E=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, A=\begin{bmatrix}-5.885 & 0\\ -1.95 & -4.125\end{bmatrix}, B_h=\begin{bmatrix}-0.1085 & 0\\ -0.2045 & -0.5095\end{bmatrix},$$
$$B_d=\begin{bmatrix}-0.1025 & -0.1035\\ 0 & -0.6035\end{bmatrix}, D=\begin{bmatrix}-1.25 & -0.135\\ 0 & 0.35\end{bmatrix},$$

and

$$0\le h(t)=0.25sin^2(t)\le 0.25=h,\alpha=0.3,\alpha h=0.075,\tau=0.35,$$
$$\dot{h}(t)=0.25sin2t\le 0.25=h_k, 0\le d(t)=0.35sin^2(t)\le 0.35=d,$$
$$\beta=0.4,\beta d=0.14,\dot{d}(t)=0.35sin2t\le 0.35=d_k.$$

Let

$$P=\begin{bmatrix}10 & 0\\ 0 & 10.5\end{bmatrix}, Q_1=\begin{bmatrix}5.1 & 1.85\\ 1.85 & 4.2\end{bmatrix}, Q_2=\begin{bmatrix}2.25 & 2.063\\ 2.063 & 3.25\end{bmatrix},$$
$$Q_3=\begin{bmatrix}5.412 & 1.463\\ 1.463 & 8.725\end{bmatrix}, H=\begin{bmatrix}4.65 & -2.15\\ -2.15 & 5.95\end{bmatrix}, Q_4=\begin{bmatrix}3.673 & 3.063\\ 3.063 & 5.853\end{bmatrix},$$
$$Q_5=\begin{bmatrix}4.1 & 1.85\\ 1.85 & 3.2\end{bmatrix}, Q_6=\begin{bmatrix}8.2 & 1.063\\ 1.063 & 12.505\end{bmatrix}, R_1=\begin{bmatrix}1.7255 & 0.625\\ 0.625 & 0.706\end{bmatrix},$$
$$R_2=\begin{bmatrix}1.265 & 0.724\\ 0.724 & 0.725\end{bmatrix}, R_3=\begin{bmatrix}0.245 & 0.234\\ 0.234 & 0.255\end{bmatrix}, U=\begin{bmatrix}0\\ 2.45\end{bmatrix},$$
$$R_4=\begin{bmatrix}1.695 & 0.595\\ 0.595 & 0.685\end{bmatrix}, R_5=\begin{bmatrix}2.655 & 0.015\\ 0.015 & 0.106\end{bmatrix}, S=\begin{bmatrix}-1.25\\ -1.35\end{bmatrix},$$
$$R_6=\begin{bmatrix}0.205 & 0.025\\ 0.025 & 0.103\end{bmatrix}.$$

Next, in the light of the above information, by some elementary calculations, for the corresponding special matrix $\overline{M}$ of (22), all eigenvalues of the matrix $\overline{M}$ satisfy $\lambda_{max}(\overline{M})\le -0.1482$. Consequently, it is clear that all conditions of Theorem 3.2 are satisfied. Thus, the singular system of Example 4.2 is asymptotically stable.

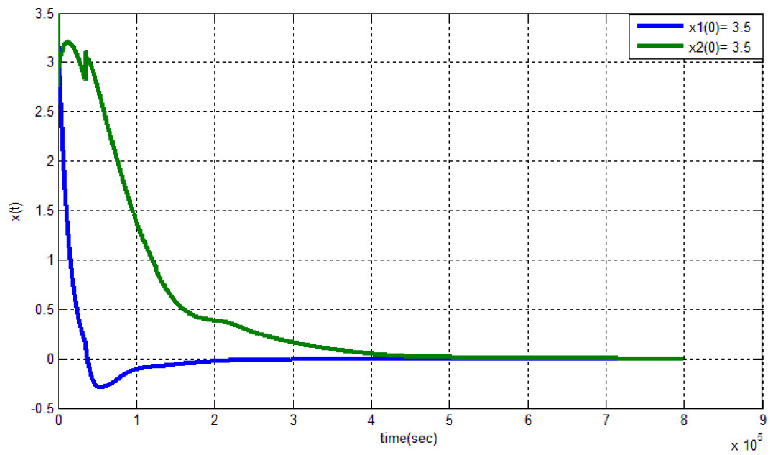

Figure 2: Trajectories of the solution of $(x_1(t), x_2(t))$ of Example 4.2, when $\tau = 0.35$.

5 Conclusion

In this paper, we consider a kind of delay singular systems. We establish new sufficient criteria for the asymptotic stability of the considered singular system with multiple delays. By using the delay decomposition and the LKF approaches, two new theorems including new asymptotic stability criteria are proved in terms of LMIs. Finally, two numerical examples are given with their simulations to demonstrate the applicability of the proposed results. The results of this paper include and generalize some recent and related results in the literature. As for suitable future studies, fractional forms of singular system (1) can be considered and their qualitative behaviors can be investigated.

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