AN EFFECTIVE APPROACH TO SOLVE A MULTI-TERM TIME FRACTIONAL DIFFERENTIAL EQUATION($M - TFDE$) WITH FUNCTION SPACE APPROXIMATION

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Abstract

This paper studies a B-spline algorithm for calculating the solution of the multi-term time-fractional diffusion equations(M-TT-FDEs). This model describes the diffusion prossing in the fluid mechanics and provides valuable predictions. The solution of the M-TT-FDEs is discretized by means of B-spline function based on the B-spline shape technique. It is verified that the proposed strategy is more efficient in terms of computational time and accuracy in domain.

Key words: Multi-term time fractional; Fractional B-spline functions; Differential equation; Function space approximation;

1. Introduction

A significant tool in various sciences the fractional differential equation(FDE) [1, 2, 3, 4] that with a discretization method the FDE are solved by computer [5]. Finite difference, finite volume, finite element, discrete element, boundary element, no mesh, or combination of these methods are the most common methods of discretization [6, 7, 8, 9, 10, 11]. Most methods offer the same solution to the original $PDEs$ in theory. In [14] Baleanu et al., the FDE's existence was studied using Caputo, and some analytical solutions were obtained for the hybrid differential equation [15].

Numerical methods presented to solve approximate answers to differential equations of mathematical samples of different problems [16, 17, 18, 19]. The collocation method solves a finite number of nodes by solving the differential equation. The easy and high speed is the biggest advantage of this method. The fractional B-spline function($fBSf$) is a smoothness to connect with the low calculating cost of collocation.Our goal in this manuscript is to seek the performance of $fBSf$ at the collocation method to solve initial and boundary value problems. Our goal in this manuscript is to seek the performance of $fBSf$ at collocation method to solve initial and boundary value problems.

$M - TFDE$ reduced of the problem to a system of the ordinary by Edwards et. al. [20]. Another method is meshless that was introduced by Hosseini et. al.for solving $M - TFDE$ in [2, 12, 13]. That left-side caputo fractional derivative persented by Lin and Lazarov et. al where they got the $\mathcal{O}(h^2 + \tau^{2-\alpha})$ [21]. On different intervals focus on the fractional predictor-corrector method $M - TFDE$ by Liu [22]. The other method, the space-time spectral scheme presented by Zheng et. al. was an impressive numerical method [24]. Assuming the norm to be L^2 the stability and convergence proved at finite-difference scheme leads to a lower accuracy order $\mathcal{O}(\tau^{\alpha})$. With spectral collocation method expanded an power accurate fractional for solving time-dependent fractional partial

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differential equations with help new fractional Lagrange interpolants by Zayernouri et. al [25]. A composition of finite difference and matrix transfer method presented by Zhao et. al. [26].

This manuscript is formed as follows: in section 2, some basic definitions and theorems of $fBSf$ are expressed. Section 3 is dedicated to the solution of $M-TDFE$ using the collocation technique with $fBSf$. Insection 4, five numerical examples are presented.

2. Basic function

In this section, the efficiency and usefulness of spline functions in computers, math and Box splines have been demonstrated in [27, 28, 29, 30]. We will provide several definitions and theorems of [31].

Definition 2.1. *Functions are called polynomial spline function of degree $n+1$. The conditions of functions is a piece of multinomial function with degree n on interval $[a,b]$ are as follows:*
1) The points interpolation are $a=t_1 \leq t_2 \leq t_3 \leq \ldots \leq t_d = b$ and in amongst any $[t_i, t_{(i+1)}]$ is one polynomials of degree n too conjunction $[t_{(i+1)}, t_{(i+2)}]$ to another polynomials:

$$S^n(t) = \begin{cases} s_1(t) & ; t_1 \leq t \leq t_2, \\ s_2(t) & ; t_2 \leq t \leq t_3, \\ \quad . & \\ \quad . & \\ \quad . & \\ s_{(d-1)}(t) & ; t_{(d-1)} \leq t \leq t_d. \end{cases} \tag{1}$$

Spline function presented $S^n(t)$ that on each partition $s_i(t), i=1,2,\ldots,d-1$ is a polynomial of n degree.
2)The characteristics of the nth derivative which are limited, displays several isolated case that it is not continuities in points, and they are continuities at knots among the polynomial piece where the continuous derivative of the order of $n-1$ is one of the properties of $s_i(t), i=1,2,\ldots,d-1$ functions at $[t_i, t_{(i+1)}]$.

B-Splines functions(BSf) polynomials were introduced by I. J. Schoenberg in [32, 33] in 1946. He formed the basic functions for terms BSf as follows:

$$S^n(t) = \sum_{j \in \mathbb{Z}} c_j \beta^n(t-j), \tag{2}$$

$$\beta^n(t) = \frac{1}{n!} \sum_{j=0}^{n+1} (-1)^j \binom{n+1}{j} (t-j)_+^n. \tag{3}$$

Where

$$(t-j)_+^n = \begin{cases} (t-j)^n & t > j, \\ t > j & t \leq j. \end{cases} \tag{4}$$

The BSf with different powers:

In $Fig.1$, the power 0 for $\beta^0(t)$ if constant function, in $Fig.2$, $\beta^1(t)$ called Hat function that is a linear function, in $Fig.3$, $\beta^2(t)$ of degree two and in $Fig.4$, $\beta^3(t)$ called bell function that is degree tree. These functions play essential role in the theory of defense approximation and analysis. The reason for using these functions in a variety of applications and their widespread use is that they have desirable properties [34, 35, 36, 37].

The extension of constant's presented by Thierry Blu and Michael Unser of $fBSf$[38]. The favorable attributes of $fBSf$ showed to transfer to the fractional case.

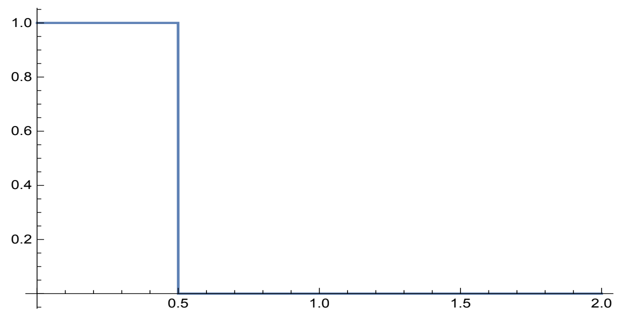

Figure 1: The BSf shapes with 0 degree is really $\beta^0(t)$.

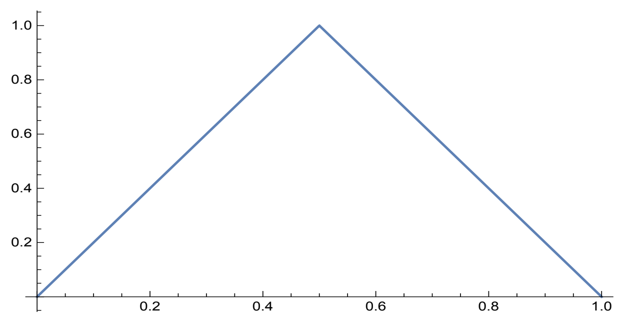

Figure 2: The BSf shapes with 1 degree is really $\beta^1(t)$.

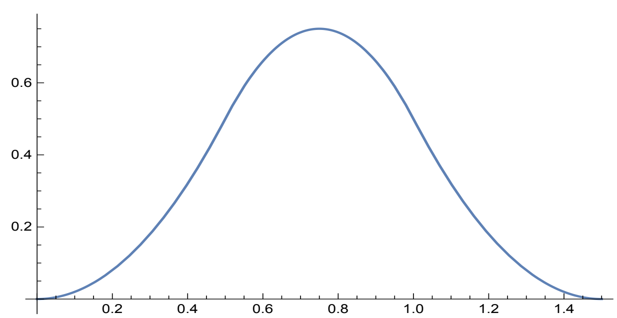

Figure 3: The BSf shapes with 2 degree is really $\beta^2(t)$.

Definition 2.2. *The $fBSf$ $\beta^{\alpha}(t)$ is:*

$$\beta^{\alpha}(t) = \frac{1}{\Gamma(\alpha+1)} \sum_{k \leq 0} (-1)^k \binom{\alpha+1}{k} (t-k)_+^{\alpha} \tag{5}$$

the $Eq.5$ is credible point to point for everyone $t \in \mathbb{R}$ and a well as into the $L^2(\mathbb{R})$.

In $Figs.5, 6, 7$, and $Fig.8$ several samples of $fBSf$ are introduced, it seems to be destroyed, only time the α be an integer then the $fBSf$ are compactly supported. in this sample, we have covered the classical BS. Generally, they

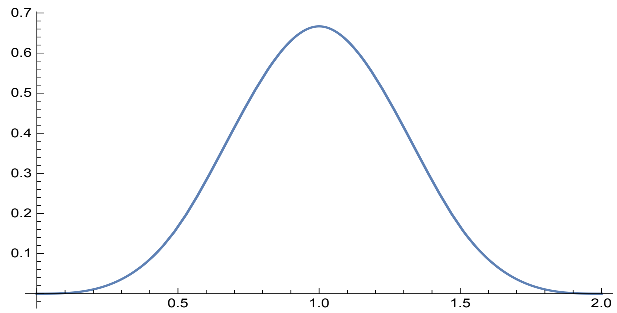

Figure 4: The BSf shapes with 3 degree is really $\beta^3(t)$.

have an axis of asymmetric.

Functions with fractional power are well approximated by the $fBSf$ because they have fractional power. They

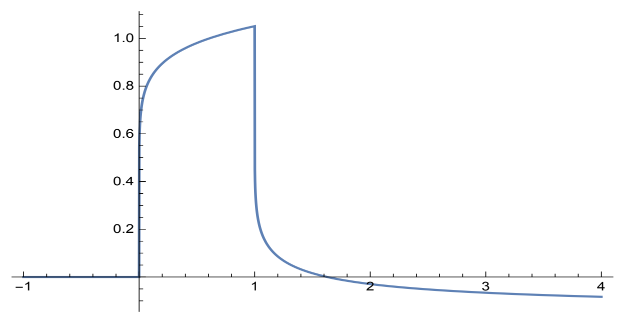

Figure 5: The $fBSf$ shapes with 0.1 degree is really $\beta^{0.1}(t)$.

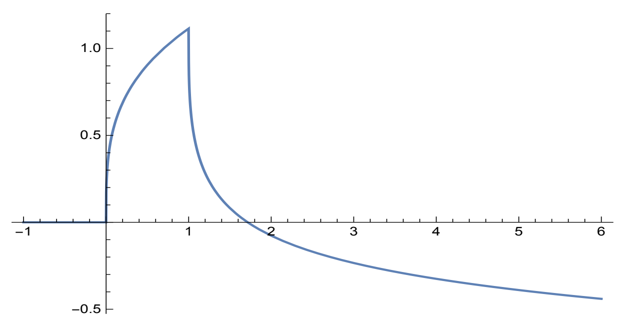

Figure 6: The $fBSf$ shapes with 0.3 degree is really $\beta^{0.3}(t)$.

have every continuous parameter $\alpha > -1$. If the α is an integer, this function interpolates the normal splines.

First of all, investigated a rather forced adjust univariate analysis with spaced points; for making multiresolution wavelet bases their monotonous net in special is needed. Second, these functions can be used in many numerical methods, and also the $fBSf$ have the characteristics of a type the BS such as the support domain of the BS for

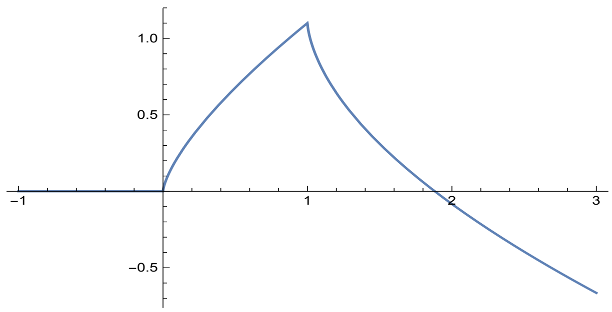

Figure 7: The $fBSf$ shapes with 0.3 degree is really $\beta^{0.7}(t)$.

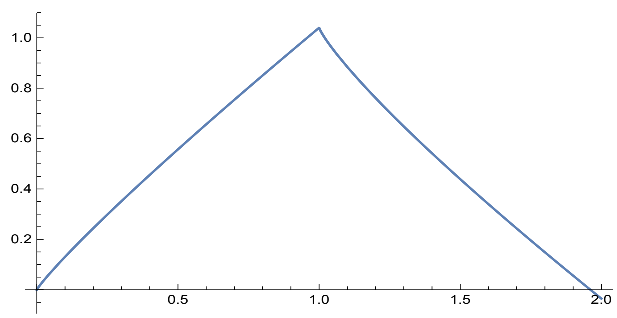

Figure 8: The $fBSf$ shapes with 1.3 degree is really $\beta^{1.3}(t)$.

nonintegral where α is no longer compact. Particulary, functions were dense in L^2 with condition $\alpha > \frac{-1}{2}$. The definition of $fBSf$ spaces on the a scale is as follows:

$$S_a^{\alpha} = \{s_a : \exists c \in l^2, s_a(x) = \sum_{k\in\mathbb{Z}} c_k \beta^{\alpha}(\frac{x}{a} - k)\} \tag{6}$$

We assess its least squares approximation in S_a^{α} for an arbitrary function $f \in L^2(\mathbb{R})$.

Theorem 2.3. *The $fBSf$ has a fractional order of approximation $\alpha + 1$. In particular, the least-squares approximation error is limited by*

$$\forall f \in W_2^{\alpha+1}, \|f - P_a f\|_{L^2} \leq a^{\alpha+1} \|\mathcal{D}^{\alpha+1} f\|_{L^2} \frac{\sqrt{2\xi(\alpha+2) - \frac{1}{2}}}{\prod^{\alpha+1}}; a \to 0 \tag{7}$$

Proof. *The proofs in [38], (Theorem 4.1).*

In this theorem, $P_a f$ is an interpolation function of function f. The $fBSf$ produces credible multiresolution analysis of L^2 for $\alpha > -\frac{1}{2}$. The $fBSf$ can be a scheme to have an optional order of smooth. These functions produce a sequence of space flow as:

$$0 \subset ... \subset \mathcal{X}_{-1} \subset \mathcal{X}_0 \subset \mathcal{X}_1 \subset ... \subset L^2(\mathbb{R}) \tag{8}$$

they have properties:
a) $\bigcap_{i\in\mathbb{Z}} \mathcal{X}_i = 0$ and $\overline{\bigcup_{i\in\mathbb{Z}} \mathcal{X}_i} = L^2(\mathbb{R})$.
b) $f(*) \in \mathcal{X}_i$ if and only if $f(2^{-i}*) \in \mathcal{X}_0$
c) $f(*) \in \mathcal{X}_0$ if and only if $f(* - k) \in \mathcal{X}_0$ for each $k \in \mathbb{Z}$ and there be a function $\varphi \in \mathcal{X}_0$, called a scale factor, such a way that $\varphi(* - k)_{k\in\mathbb{Z}}$ format an orthonormal foundations of $\mathcal{X}_0$. The spaces $fBSf$ produce $\mathcal{X}_n$ are of order $\mathbb{a} \in \mathbb{R}$ with points $k \times 2^n, k \in \mathbb{Z}$ where the forms spaces are:

$$\mathcal{X}_n = \overline{span\{\beta^{\mathbb{a}}(\frac{x - 2^n k}{2^n})^{L^2(\mathbb{R})}\}}; \mathbb{a} \geq -\frac{1}{2}, n \in \mathbb{Z}, \tag{9}$$

That $\beta^{\mathbb{a}}$ produces a multiresolution analysis. Let's take, $a = 2^i$, then several sample of multiresolution and shift $fBSf$ $\beta^{\mathbb{a}}$ as illustrated below:

$Figs.9, 10, 11,$ and $Fig.12$ are some shift $\beta^1(t-k)$,$\beta^2(t)$, $\beta^1(2t)$ and $\beta^2(2t)$,respectively. In our methods numerical analysis basic functions are those functions.

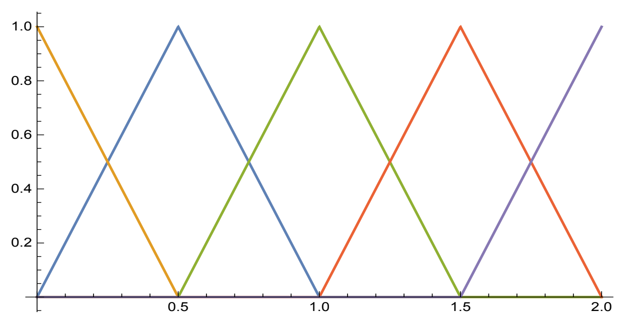

Figure 9: The one degree of BSf shape are by $i = 0$ i.e. $a = 1$ and several various k of $Eq.6$ really $\beta^1(t)$, $\beta^1(t-1)$, $\beta^1(t+1)$, $\beta^1(t+2)$.

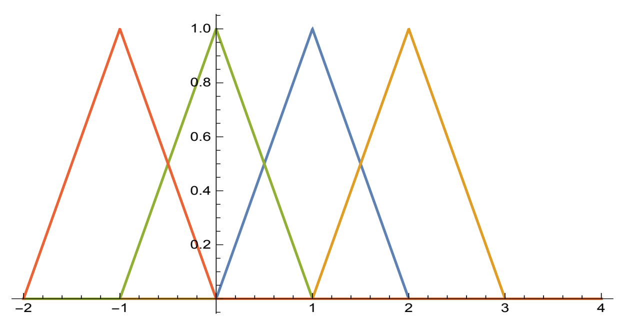

Figure 10: The two degree of BSf shape are by $i = 0$ i.e. $a = 1$ and several various k of $Eq.6$ really $\beta^2(t)$, $\beta^2(t-1)$, $\beta^2(t+1)$.

Several shift $fBSf$ of the $\mathbb{a} = 0.3$ with $a = 2^0$ and $a = 2^{-1}$ and several different k of conforming to $Eq.6$ in actuality $\beta^{0.3}(t)$ and $\beta^{0.3}(2t)$ are shown in $Fig.13$ and $Fig.14$.

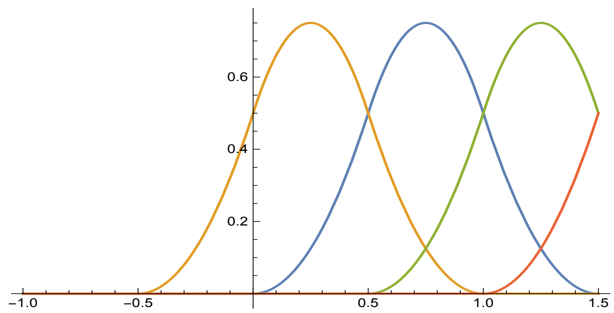

Figure 11: The one degree of BSf shape are by $i = -1$ i.e. $a = \frac{1}{2}$ and several various k of $Eq.6$ really $\beta^1(2t)$, $\beta^1(2t-1)$, $\beta^1(2t+1)$, $\beta^1(2t+2)$, $\beta^2(2t)$.

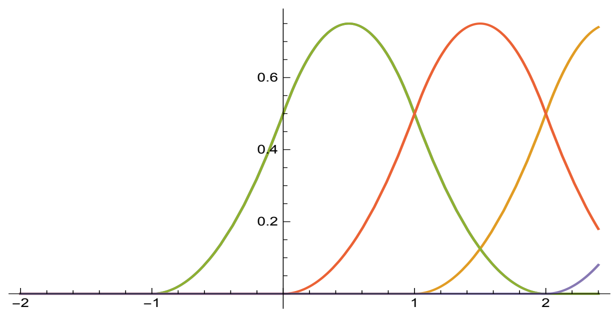

Figure 12: The two degree BSf shape are by $i = -1$ i.e. $a = \frac{1}{2}$ and several various k of$Eq.6$ really $\beta^2(2t-2)$, $\beta^2(2t-1)$, $\beta^2(2t+1)$.

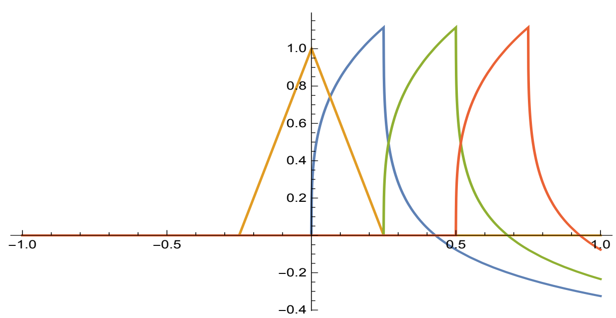

Figure 13: The diagram of the $\alpha = 0.3$ degree are by $i = 0$ i.e. $a = 1$ and several k of $Eq.6$ really $\beta^{0.3}(t)$, $\beta^{0.3}(t-1)$. , $\beta^{0.3}(t-2)$, $\beta^{0.3}(t-3)$ for $fBSf$.

3. $M - TFDE$

With $M - TFDE$ of diffusion-wave time equations a lot work extensions have been conducted. We are using base $fBSf$ in the collocation method on approximation. In this article, we discuss Caputo time derivative in one

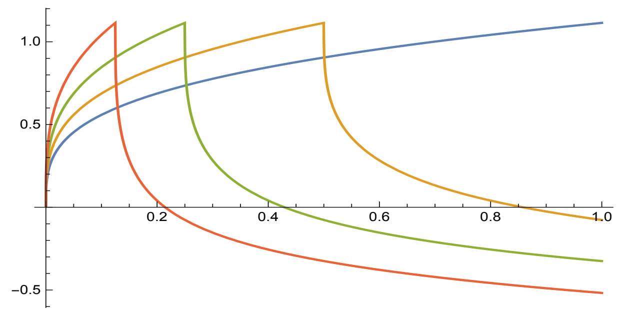

Figure 14: The diagram of the $\alpha = 0.3$ degree are by $a = -1$ and several k of $Eq.6$ really $\beta^{0.3}(2t)$, $\beta^{0.3}(2t-1)$, $\beta^{0.3}(2t+1)$, $\beta^{0.3}(2t-2)$ for $fBSf$.

and two dimensions:

$$\begin{cases} \mathbb{P}(\mathcal{D}_t)(\overline{\mathbf{X}},t) - \Delta\mathcal{U}(\overline{\mathbf{X}},t) = \mathbb{F}(\overline{\mathbf{X}},t) & (\overline{\mathbf{X}},t) \in \Omega \times (0,T], \\ \mathcal{U}(\overline{\mathbf{X}},0) = \psi_1(\overline{\mathbf{X}}), & \overline{\mathbf{X}} \in \Omega \\ \mathcal{U}(\overline{\mathbf{X}},t) = \Phi(\overline{\mathbf{X}},t), & \overline{\mathbf{X}} \in \partial\Omega, \end{cases} \tag{10}$$

where Ω is domain and $\partial\Omega$ is a boundary.
The $\mathbb{F}$ is the source term in equation above, issued to the suitable initial and boundary condition, respectively. Condition ψ_1 and Φ are presented functions on Ω.

Then, the $\mathbb{P}(\mathcal{D}_t)$ is fractional operator to form under:

$$\mathbb{P}(\mathcal{D}_t) = \mathcal{D}_t + \sum_{i=1}^{\mathbb{m}} r_i \mathcal{D}_t^{\alpha_i}, \tag{11}$$

where the $\mathbb{m} \in \mathbb{N}$ and $\mathcal{D}_t^{\alpha_i}$ represents the Caputo fractional derivative of order $\alpha_i \in (0,1)$, is defined by

$$\mathcal{D}_t^{\alpha_i}\mathcal{U}(t) = \begin{cases} \dfrac{1}{\Gamma(k-\alpha_i)} \displaystyle\int_0^t (t-\xi)^{k-\alpha_i-1}\mathcal{U}^k(\xi)d\xi & k-1 < \alpha_i < k, \quad t > 0, \\ \mathcal{U}^k(t) & \alpha_i = k. \end{cases} \tag{12}$$

the $\Gamma(.)$ is a usual Gamma function. The $fBSf$ does not have compact support but it decays toward infinity as:

$$\beta^{\alpha}(t) = \frac{1}{|t|^{-2-\alpha}},$$

moreover however, β^{α} is α-$H\ddot{o}lder$ continuous, belonging to $L^2(\mathbb{R})$ and reproducing polynomials up to degree $[\alpha]$.

3.1. Collocation technique $fBSf$ with one variable for unknown function

First, we want to explain the method with a variable one dimension for unknown function, from $Eq.10$

$$\overline{\mathbf{X}} \in \mathcal{X}_N \subseteq \mathcal{X}$$

concerning $Eq.9$ since $\mathcal{X}$ to $\mathcal{X}$. The $\tilde{\mathcal{U}}_N(\overline{\mathbf{X}},t)$ is approximate of $\mathcal{U}_N(\overline{\mathbf{X}},t)$ that we select a limited family of functions. The $\overline{\mathbf{X}}$ is single variable thus $\overline{\mathbf{X}} = x$, the $\mathcal{X}_N$ is a series of dimensional subspace that $\mathcal{X}_N \subset \mathcal{X}; N \geq 0$

that $\mathcal{X}_N$ have a basis $\beta^r(\frac{x-2^N k}{2^N})$ and $\beta^p(\frac{t-2^N l}{2^N})$. We search a function $\tilde{\mathcal{U}}_N(x,t) \in \mathcal{X}_N \times \mathcal{X}_N$ that it can be written as:

$$\tilde{\mathcal{U}}_N(x,t) = \sum_{k,l=1}^{d,d} \mathbb{c}_{kl} \beta^r(\frac{x-2^N k}{2^N}) \beta^p(\frac{t-2^N l}{2^N}). \tag{13}$$

We sub $\tilde{\mathcal{U}}_N(x,t)$ to $\mathcal{U}_N(x,t)$ in the $Eq.10$ and dissolve it. then, assume considerate $(x,t) \in [a,b] \times [c,d]$, which the numbers k,l in $Eq.13$ is confined on $[a,b]$. We search knots $(x_i,t_i), i = 1,...,d$, so that $(x,t) \in [a,b] \times [c,d]$ and $\mathbb{c}_{11},...,\mathbb{c}_{dd}$ are assess by dissolving linear system:

$$\begin{aligned} R_N(x_i,t_j) &= \sum_{i=1}^{\mathbb{m}} r_i \mathcal{D}_t^{\alpha_i} \sum_{k,l=1}^{d,d} \mathbb{c}_{kl} \beta^r(\frac{x_i-2^N k}{2^N}) \beta^p(\frac{t_j-2^N l}{2^N}) \\ &- \sum_{k,l=1}^{d,d} \mathbb{c}_{kl} \Delta \beta^r(\frac{x_i-2^N k}{2^N}) \beta^p(\frac{t_j-2^N l}{2^N}) - \sum_{j,i=1}^{d,d} \mathbb{F}(x_i,t_j) = 0, \end{aligned} \tag{14}$$

next we utilization of $Eq.5$ at up equation, which is obtained:

$$\begin{aligned} & R_N(x_i,t_j) = \\ & \sum_{k,l=1}^{d,d} \mathbb{c}_{kl} \left(\sum_{s\geq 0} (-1)^s \binom{r+1}{s} \frac{(\frac{x_i-2^N k}{2^N} - s)_t^r}{\Gamma(r+1)} \right) \left(\sum_{i=1}^{\mathbb{m}} r_i \mathcal{D}_t^{\alpha_i} \sum_{h\geq 0} (-1)^s \binom{p+1}{h} \frac{(\frac{t_j-2^N l}{2^N} - s)_t^p}{\Gamma(p+1)} \right) \\ & - \sum_{k,l=1}^{d,d} \mathbb{c}_{kl} \Delta \left(\sum_{s\geq 0} (-1)^s \binom{r+1}{s} \frac{(\frac{x_i-2^N k}{2^N} - s)_t^r}{\Gamma(r+1)} \right) \left(\sum_{h\geq 0} (-1)^s \binom{p+1}{h} \frac{(\frac{t_j-2^N l}{2^N} - s)_t^p}{\Gamma(p+1)} \right) \\ = & \sum_{j,i=1}^{d,d} \mathbb{F}(x_i,t_j), i,j = 0,...,d-1. \end{aligned} \tag{15}$$

3.2. Collocation method $fBSf$ with two variable for unknown function

In the second case, we tend to explain the method with a variable two dimension for unknown function, from $Eq.10$, we assume $\overline{\mathbf{X}} \in \mathbb{R}^2$ i.e. $(\overline{\mathbf{X}}, t) = (x,y,t)$ then like the mode of a variable we select a series of dimensional subspace $\mathcal{X}_N \subset \mathcal{X}; N \geq 0$ that $\mathcal{X}_N$ have a basis $\beta^r(\frac{x-2^N i}{2^N}), \beta^q(\frac{y-2^N j}{2^N})$ and $\beta^p(\frac{t-2^N k}{2^N})$. We seek a function $\tilde{\mathcal{U}}_N(x,y,t) \in \mathcal{X}_N \times \mathcal{X}_N \times \mathcal{X}_N$ that can be written as:

$$\tilde{\mathcal{U}}_N(x,y,t) = \sum_{i,j,k \in \mathbb{N}} \mathbb{c}_{ijk} \beta^r(\frac{x-2^N i}{2^N}) \beta^q(\frac{y-2^N j}{2^N}) \beta^p(\frac{t-2^N k}{2^N}). \tag{16}$$

next change $\tilde{\mathcal{U}}_N(x,y,t)$ with $\mathcal{U}(x,y,t)$ in the $Eq.10$ and dissolving it. Next, we assume by considering $(x,y,t) \in [c,d] \times [e,f] \times [a,b]$, with this i,j,k in $Eq.16$ is limited on $[a,b]$.
Now we search knots $(x_i,y_j,t_k), i,j,k = 1,...,d$ where $(x,y,t) \in [a,b] \times [c,d] \times [e,f]$ and $\mathbb{c}_{111}$, $\mathbb{c}_{211}$, ..., $\mathbb{c}_{ddd}$

are assess by dissolve linear system below:

$$\begin{aligned}R_N(x_w, y_v, t_z) &= \sum_{i=1}^{\mathbb{m}} r_i \mathcal{D}_t^{\alpha_i} \sum_{i,j,k=1}^{d,d,d} \mathbb{c}_{ijk} \beta^r(\frac{x_w - 2^N i}{2^N}) \beta^p(\frac{y_v - 2^N j}{2^N}) \beta^q(\frac{t_z - 2^N k}{2^N}) \\ &- \Delta \sum_{i,j,k=1}^{d,d,d} \mathbb{c}_{ijk} \beta^r(\frac{x_w - 2^N i}{2^N}) \beta^p(\frac{y_v - 2^N j}{2^N}) \beta^q(\frac{t_z - 2^N k}{2^N}) \\ &- \sum_{i,j,k=1}^{d,d,d} \mathbb{F}(x_w, y_v, t_z) = 0, w, v, z = 0, ..., d-1. \end{aligned} \tag{17}$$

Similar previous case, putting $Eq.5$ can obtain the unknown factors. With Placement points in two modes are mentioned, two matrices are created. we solve $Eq.10$ with collocation technique by usage of $fBSf$. we assume P_n that maps $\mathcal{X}$ onto $\mathcal{X}_n$, define $P_n \mathcal{U}(\overline{\mathbf{x}}, t)$ to be that atom of $\mathcal{X}_n$ that approximate $\mathcal{X}$ at the knots used at $Eq.13$ and $Eq.16$. We can found following relation:

$$P_n \mathcal{U}(\overline{\mathbf{X}}, t) = \tilde{\mathcal{U}}_N(\overline{\mathbf{X}}, t)$$

with the factors $\mathbb{c}_{ij}$ with one variable and $\mathbb{c}_{ijk}$ with two variable specified dissolving the linear system $Eq.15$ and $Eq.5$ next our problem has a alone one answer if

$$det(R_N(x_i, t_j)) \neq 0$$

or

$$det(R_N(x_w, y_v, t_z)) \neq 0.$$

The convergence of this method is guaranteed by means of Theorem 2.3.

4. Applications and Results

Now, we present the conclusions made for several samples using our method with $fBSf$ for $Eq.10$ explained previously. At samples, the precision of the methods, and we compare with the suggested technique two types of error measures, ε_∞ that is a maximum absolute error and RMS ε_R:

$$Error = \left\| \widetilde{\mathcal{U}}_N(\overline{\mathbf{X}}_i, t) - \mathcal{U}(\overline{\mathbf{X}}_i, t) \right\|_\infty, \quad 0 \le t \le T \tag{18}$$

$$RMS = \sqrt{\frac{\sum_{i=1}^n \left(\widetilde{\mathcal{U}}_N(\overline{\mathbf{X}}_i, t) - \mathcal{U}(\overline{\mathbf{X}}_i, t) \right)^2}{n}}, \tag{19}$$

are employed, which the $\mathcal{U}(\overline{\mathbf{X}}_i, t)$ is exact answers and $\widetilde{\mathcal{U}}_N(\overline{\mathbf{X}}_i, t)$ is approximate answers, N is dimension of $fBSf$ and n is number knots for plot shape and compute error between exact and approximate answers in order. At every example, we are assume regular node be regular partition next by solve $Eq.15$ or (18) and obtain $\mathbb{c}_{kl}$ or $\mathbb{c}_{ijk}$ for $Eq.13$ and $Eq.16$ that it is approximate answers then we divide to n of the equal part the scope of the answer and by using $Eq.$

Example 1.

First example, we discuss the $Eq.$10 with different α_1, α_2 and$t \in [0,1]$ and $\triangle t^i = t^i - t^{i-1} = 0.01$ in partition $\Omega = [0, 0.5]$. The $\mathcal{U}(x,t) = x^3(t^{1+\alpha_1+\alpha_2})$ is exact solution too

$$\begin{aligned} \mathbb{F}(x,t) &= -6t^{2+\alpha_1+\alpha_2}x \\ &+ x^3\Gamma(1+\alpha_1+\alpha_2)(1+\alpha_1+\alpha_2)\left[\frac{(t^{1+\alpha_1})\Gamma(2-\alpha_1)}{\Gamma(3+\alpha_1)\Gamma(1-\alpha_2)} + \frac{(t^{1+\alpha_2})\Gamma(2-\alpha_2)}{\Gamma(3+\alpha_2)\Gamma(1-\alpha_1)}\right] \end{aligned} \tag{20}$$

and tree term fractal $\alpha_i, i = 1, 2, 3,$

$$\mathcal{U}(x,t) = x^3(t^{1+\alpha_1+\alpha_2+\alpha_3})$$

also

$$\begin{aligned} \mathbb{F}(x,t) &= -6t^{2+\alpha_1+\alpha_2+\alpha_3}x \\ &+ x^3\Gamma(1+\alpha_1+\alpha_2+\alpha_3)(1+\alpha_1+\alpha_2+\alpha_3) \\ &\left[\frac{(t^{1+\alpha_1+\alpha_2})\Gamma(2-\alpha_3)}{\Gamma(3+\alpha_1+\alpha_2)\Gamma(1-\alpha_3)} + \frac{(t^{1+\alpha_1+\alpha_3})\Gamma(2-\alpha_2)}{\Gamma(3+\alpha_2+\alpha_3)\Gamma(1-\alpha_2)} + \frac{(t^{1+\alpha_2+\alpha_3})\Gamma(2-\alpha_1)}{\Gamma(3+\alpha_2+\alpha_3)\Gamma(1-\alpha_1)}\right] \end{aligned} \tag{21}$$

Table 1: Sample of $Eq.$10 and RMS $Eq.$19 and the α_1, α_2 have tree variable t, x, n .

	RMS_j^0	RMS_j^1	RMS_j^2
$\alpha_1 = 0.1, \alpha_2 = 0.4$	$1.37691715 \times 10^{-4}$	$1.36817007 \times 10^{-4}$	$1.36784227 \times 10^{-4}$
$\alpha_1 = 0.3, \alpha_2 = 0.4$	$1.31697622 \times 10^{-4}$	$1.31062956 \times 10^{-4}$	$1.31000040 \times 10^{-4}$
$\alpha_1 = 0.2, \alpha_2 = 0.6$	$1.28816508 \times 10^{-4}$	$1.28369975 \times 10^{-4}$	$1.27977642 \times 10^{-4}$
$\alpha_1 = 0.1, \alpha_2 = 0.9$	$2.44772992 \times 10^{-4}$	$2.12264571 \times 10^{-5}$	$4.87391324 \times 10^{-6}$
$\alpha_1 = 0.3, \alpha_2 = 0.8$	$3.03165220 \times 10^{-5}$	$1.34647287 \times 10^{-5}$	$4.79382664 \times 10^{-6}$

Table 2: Sample of $Eq.$10 and RMS $Eq.$19 and the $\alpha_1, \alpha_2, \alpha_3$ have tree variable t, x, n.

	RMS_j^0	RMS_j^1	RMS_j^2
$\alpha_1 = 0.1, \alpha_2 = 0.2, \alpha_3 = 0.3$	$1.35596505 \times 10^{-4}$	$1.34395454 \times 10^{-4}$	$1.34377414 \times 10^{-4}$
$\alpha_1 = 0.2, \alpha_2 = 0.3, \alpha_3 = 0.4$	$1.27265808 \times 10^{-4}$	$1.26561905 \times 10^{-4}$	$1.25629737 \times 10^{-4}$
$\alpha_1 = 0.3, \alpha_2 = 0.4, \alpha_3 = 0.5$	$1.20259940 \times 10^{-5}$	$1.19793031 \times 10^{-4}$	$1.16883116 \times 10^{-4}$
$\alpha_1 = 0.1, \alpha_2 = 0.3, \alpha_3 = 0.8$	$2.99362980 \times 10^{-5}$	$1.32748298 \times 10^{-5}$	$4.65066226 \times 10^{-6}$
$\alpha_1 = 0.2, \alpha_2 = 0.3, \alpha_3 = 0.9$	$2.88319590 \times 10^{-5}$	$1.27763920 \times 10^{-5}$	$4.65066226 \times 10^{-6}$

At our tables, we obtain RMS of $Eq.$19 for several α's. The RMS solutions is not much more than 10^{-4}. The table 1 with α_1, α_2 and the table 2 with $\alpha_1, \alpha_2, \alpha_3$, shows the RMS produced using with $n = 500$ and several of α and Δt. When the N grow, the RMS is reducing slowly and decreasing the error by grow the X to little by little in $Fig.$15, and $Fig.$16.

We are displaying the $Error$ of $Eq.$18 that estimate answers with $\alpha_1 = 0.1$, $\alpha_2 = 0.4$ and $\alpha_1 = 0.1$, $\alpha_2 = 0.2$, $\alpha_3 = 0.3$, the N is number of variable of $fBSf$ at $Fig.$15 and $Fig.$16. We view in the $Fig.$15 and $Fig.$16, $Error$ in axis X is not decrease until 10^{-3} by attention to that in $N = 2$ it is 10^{-4}, it is manner is not fast, it is not t

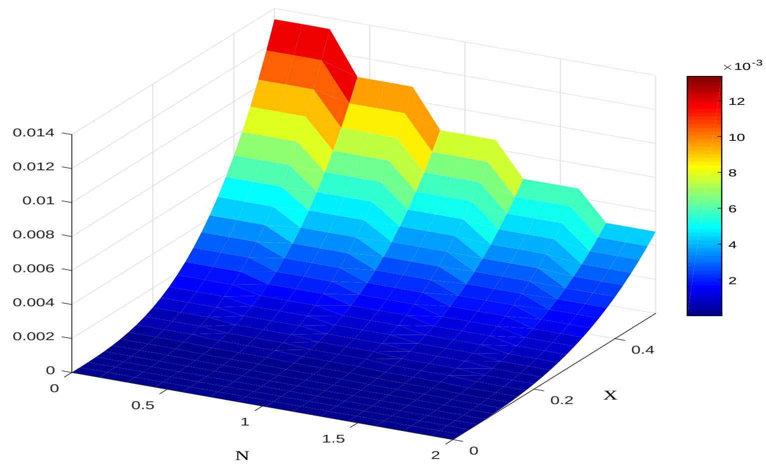

Figure 15: The shape RMS for α_1, α_2 that are $\alpha_1 = 0.1$, $\alpha_2 = 0.4$ of $Eq.10$ and error $Eq.18$.

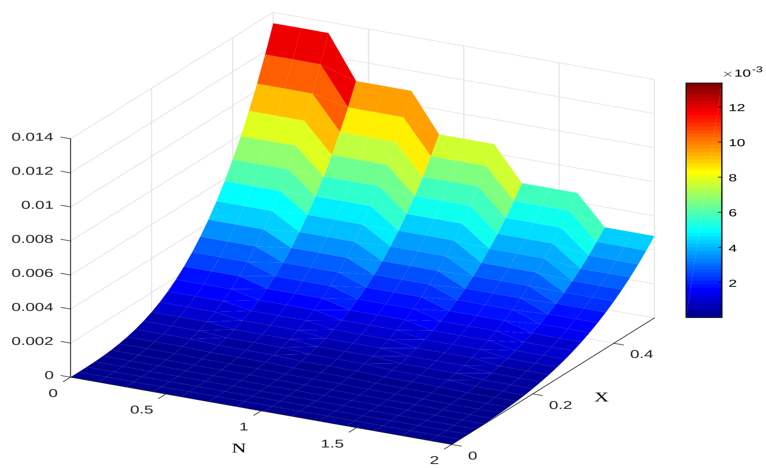

Figure 16: The shape RMS for $\alpha_1, \alpha_2, \alpha_3$ that are $\alpha_1 = 0.1$, $\alpha_2 = 0.2$, $\alpha_3 = 0.3$ of $Eq.10$ and error $Eq.18$.

rapidity increase tangible .

Example 2

We discuss the $Eq.10$ with two variable x, y that is mean $\overline{\mathbf{X}} \in \mathbb{R}^2$ and several amounts for α and $\triangle t^i = 0.01$ and $t \in [0, 1]$ in partition $\Omega = [0, 0.5] \times [0, 0.5]$. The $\mathcal{U}(x, y, t) = t^{1+\alpha_1+\alpha_2} x^2 y^2$ is solution, and force term can expreesed as follows

$$\mathbb{F}(x,y,t) = -2t^{2+\alpha_1+\alpha_2}(x^2+y^2)x^2y^2 + \Gamma(1+\alpha_1+\alpha_2)(1+\alpha_1+\alpha_2) \left[\frac{(t^{2+\alpha_1})\Gamma(2-\alpha_1)}{\Gamma(3+\alpha_1)\Gamma(1-\alpha_2)} + \frac{(t^{2+\alpha_2})\Gamma(2-\alpha_2)}{\Gamma(3+\alpha_2)\Gamma(1-\alpha_1)}\right]$$

and tree term fractional $\alpha_i, i = 1, 2, 3$

$$\mathcal{U}(x,y,t) = t^{1+\alpha_1+\alpha_2+\alpha_3}x^2y^2$$

also

$$\mathbb{F}(x,y,t) = -2t^{2+\alpha_1+\alpha_2+\alpha_3}(x^2+y^2) + x^2y^2\Gamma(1+\alpha_1+\alpha_2+\alpha_3)(1+\alpha_1+\alpha_2) \left[\frac{(t^{1+\alpha_1+\alpha_2})\Gamma(2-\alpha_3)}{\Gamma(3+\alpha_1+\alpha_2)\Gamma(1-\alpha_3)} + \frac{(t^{1+\alpha_1+\alpha_3})\Gamma(2-\alpha_2)}{\Gamma(3+\alpha_2+\alpha_3)\Gamma(1-\alpha_2)} + \frac{(t^{1+\alpha_2+\alpha_3})\Gamma(2-\alpha_1)}{\Gamma(3+\alpha_2+\alpha_3)\Gamma(1-\alpha_1)}\right]$$

In this sample plotting the error of obtained answers by amounts of Degree of fraction, assume one of the variables the variable X or Y to be constant then we calculate the RMS.We assume amounts fixed away from knots primary. Anew the N is dimension of $fBSf$ and the N is grow $Error$ isn't increase. The $Fig.17$, $Fig.18$, $Fig.19$ and $Fig.20$ are answers at several time surfaces for α have been presented.

Table 3: Sample of $Eq.10$ and RMS $Eq.19$ and the α_1, α_2 have tree variable t, x, n, that y is fixed.

	RMS_j^0	RMS_j^1	RMS_j^2
$\alpha_1 = 0.1, \alpha_2 = 0.2$	$3.94497585 \times 10^{-4}$	$9.15524676 \times 10^{-5}$	$1.59141638 \times 10^{-5}$
$\alpha_1 = 0.1, \alpha_2 = 0.8$	$2.48475179 \times 10^{-4}$	$4.72961107 \times 10^{-5}$	$1.25629737 \times 10^{-5}$
$\alpha_1 = 0.5, \alpha_2 = 0.6$	$2.17263429 \times 10^{-4}$	$3.81143002 \times 10^{-5}$	$3.81143002 \times 10^{-5}$
$\alpha_1 = 0.2, \alpha_2 = 0.6$	$3.17518103 \times 10^{-5}$	$1.93898497 \times 10^{-6}$	$1.41841301 \times 10^{-7}$
$\alpha_1 = 0.3, \alpha_2 = 0.7$	$2.85753808 \times 10^{-5}$	$1.56945742 \times 10^{-6}$	$1.13979494 \times 10^{-7}$

Table 4: Sample of $Eq.10$ and RMS $Eq.19$ and the $\alpha_1, \alpha_2, \alpha_3$ have tree variable t, x, n, that y is fixed.

	RMS_j^0	RMS_j^1	RMS_j^2
$\alpha_1 = 0.1, \alpha_2 = 0.2, \alpha_3 = 0.3$	$1.35596505 \times 10^{-4}$	$1.34395454 \times 10^{-4}$	$1.34377414 \times 10^{-4}$
$\alpha_1 = 0.2, \alpha_2 = 0.4, \alpha_3 = 0.6$	$1.27265808 \times 10^{-4}$	$1.26561905 \times 10^{-4}$	$1.25629737 \times 10^{-4}$
$\alpha_1 = 0.3, \alpha_2 = 0.6, \alpha_3 = 0.7$	$1.20259940 \times 10^{-4}$	$1.19793031 \times 10^{-4}$	$1.16883116 \times 10^{-4}$
$\alpha_1 = 0.1, \alpha_2 = 0.5, \alpha_3 = 0.8$	$2.99362980 \times 10^{-5}$	$1.32748298 \times 10^{-5}$	$4.65066226 \times 10^{-6}$
$\alpha_1 = 0.4, \alpha_2 = 0.5, \alpha_3 = 0.6$	$2.88319590 \times 10^{-5}$	$1.27763920 \times 10^{-5}$	$4.65066226 \times 10^{-6}$

In our tables, we obtain RMS of $Eq.19$ for several α's. The RMS solutions isn't much more than 10^{-4}. With $n = 500$, several amounts α_1, α_2 and Δt with $y = 0.5$, Beginning The RMS is of 10^{-4} until to 10^{-7} that the outcomes and the answers are accord and variable time at has nearly effectless when it is tiny enough at tables 3

Table 5: Sample of $Eq.$10 and $RMS\ Eq.$19 and the α_1, α_2 have tree variable t, y, n, that x is fixed.

	RMS_j^0	RMS_j^1	RMS_j^2
$\alpha_1 = 0.1, \alpha_2 = 0.2$	$6.54169632 \times 10^{-4}$	$1.382539696 \times 10^{-4}$	$3.93536798 \times 10^{-5}$
$\alpha_1 = 0.1, \alpha_2 = 0.8$	$4.82846136 \times 10^{-4}$	$1.999813782 \times 10^{-5}$	$7.21156527 \times 10^{-5}$
$\alpha_1 = 0.5, \alpha_2 = 0.6$	$4.55836128 \times 10^{-4}$	$5.821545927 \times 10^{-5}$	$1.60713243 \times 10^{-5}$
$\alpha_1 = 0.2, \alpha_2 = 0.6$	$5.75138282 \times 10^{-5}$	$2.944033108 \times 10^{-6}$	$1.78113445 \times 10^{-7}$
$\alpha_1 = 0.3, \alpha_2 = 0.7$	$5.68271757 \times 10^{-5}$	$2.393451379 \times 10^{-6}$	$1.43214173 \times 10^{-7}$

Table 6: Sample of $Eq.$10 and $RMS\ Eq.$19 and the $\alpha_1, \alpha_2, \alpha_3$ have tree variable t, y, n, that x is fixed.

	RMS_j^0	RMS_j^1	RMS_j^2
$\alpha_1 = 0.1, \alpha_2 = 0.2, \alpha_3 = 0.3$	$2.33138317 \times 10^{-3}$	$1.56846535 \times 10^{-4}$	$3.18440163 \times 10^{-5}$
$\alpha_1 = 0.2, \alpha_2 = 0.4, \alpha_3 = 0.6$	$1.92621300 \times 10^{-3}$	$8.62977077 \times 10^{-5}$	$1.32199024 \times 10^{-5}$
$\alpha_1 = 0.3, \alpha_2 = 0.6, \alpha_3 = 0.7$	$1.28240167 \times 10^{-3}$	$5.23971866 \times 10^{-5}$	$1.01573409 \times 10^{-5}$
$\alpha_1 = 0.1, \alpha_2 = 0.5, \alpha_3 = 0.8$	$1.46864232 \times 10^{-4}$	$2.23977676 \times 10^{-6}$	$9.71019231 \times 10^{-8}$
$\alpha_1 = 0.4, \alpha_2 = 0.5, \alpha_3 = 0.6$	$1.79950021 \times 10^{-4}$	$2.21143135 \times 10^{-6}$	$9.21224652 \times 10^{-8}$

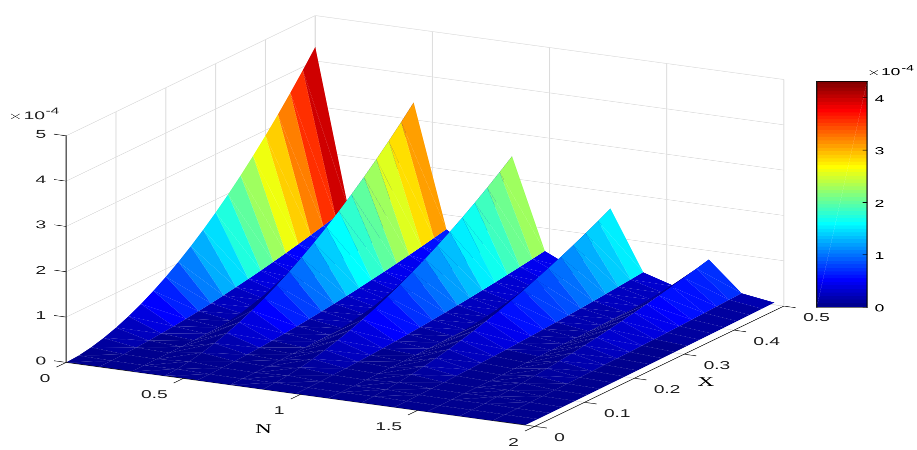

Figure 17: The shape RMS for $u(x, 0.5, t)$ with α_1, α_2 that are $\alpha_1 = 0.2$, $\alpha_2 = 0.6$ of $Eq.$10 and error $Eq.$18.

and the table 4 we have tree fractional the $\alpha_i, i = 1, 2, 3$ that have been illustrated for two term α_1, α_2 and tree term α_1, α_2, α_3 with $x = 0.5$, the RMS is among 10^{-4} until 10^{-6} and 10^{-3} to 10^{-8} respectively. When the N grow, the RMS is reducing slowly and decreasing the error by grow the X to little by little in $Fig.$15 and $Fig.$16.

It is in the above figures$\Delta t = 0.01$ and $n = 500$. For approximate answers with $y = 0.5$ that in $Fig.$17 in fact displays the $Error$ of $Eq.$18 and we considered $\alpha_1 = 0.2$, $\alpha_2 = 0.6$ in $Fig.$18 we considered $\alpha_1 = 0.1$, $\alpha_2 = 0.5$, $\alpha_3 = 0.8$, the N is dimensions of $fBSf$. we look in the shapes RMS in axis X isn't decrease than 10^{-3} by notice with $N = 2$ it is 10^{-4}, at in $Fig.$19 and $Fig.$20 the powers factional are look to $Fig.$17 and $Fig.$18 in order only $x = 0.5$ instead $y = 0.5$. It is manner is not fast it is not rapidity increase tangible .

Example 3

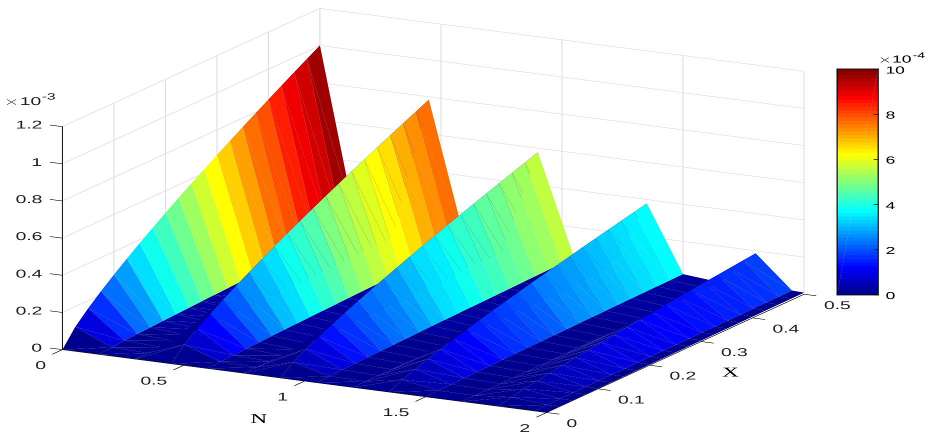

Figure 18: The shape RMS for $u(x, 0.5, t)$ with $\alpha_1, \alpha_2, \alpha_3$ that are $\alpha_1 = 0.1$, $\alpha_2 = 0.5$, $\alpha_3 = 0.8$ of $Eq.10$ and error $Eq.18$.

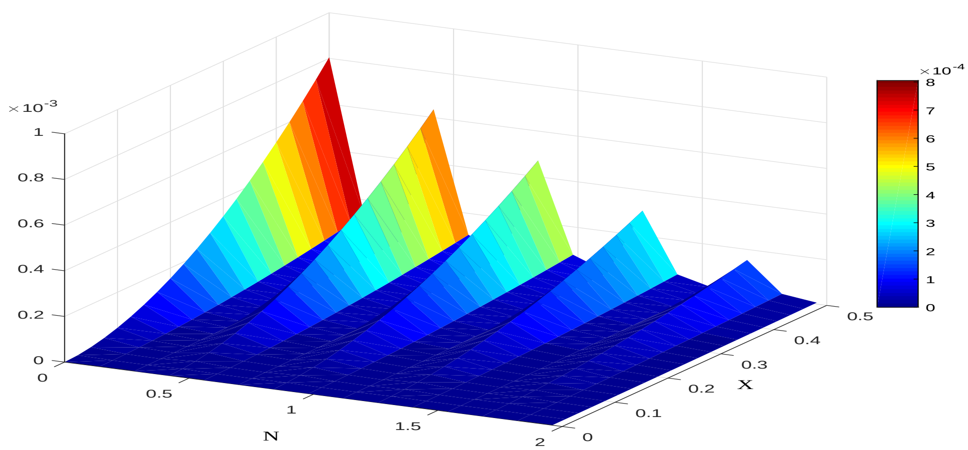

Figure 19: The shape RMS for $u(0.5, y, t)$ with α_1, α_2 that are $\alpha_1 = 0.5$, $\alpha_2 = 0.6$ of $Eq.10$ and error $Eq.18$.

The third example, we discuss the $Eq.10$ with two variable x, y that's mean $\overline{\mathbf{X}} \in \mathbb{R}^2$ and several amounts for α and $t \in [0, 1]$ and $\triangle t^i = 0.01$ in partition $\Omega = [0, 0.5] \times [0, 0.5]$. The $\mathcal{U}(x, y, t) = t^{1+\alpha_1+\alpha_2} x^2 e^y$ is solution

$$\mathbb{F}(x, y, t) \quad = \quad -2t^{1+\alpha_1+\alpha_2} e^y + x^2 e^y \Gamma(1 + \alpha_1 + \alpha_2)(1 + \alpha_1 + \alpha_2) \left[\frac{(t^{2+\alpha_1})\Gamma(2 - \alpha_1)}{\Gamma(3 + \alpha_1)\Gamma(1 - \alpha_2)} + \frac{(t^{2+\alpha_2})\Gamma(2 - \alpha_2)}{\Gamma(3 + \alpha_2)\Gamma(1 - \alpha_1)} \right]$$

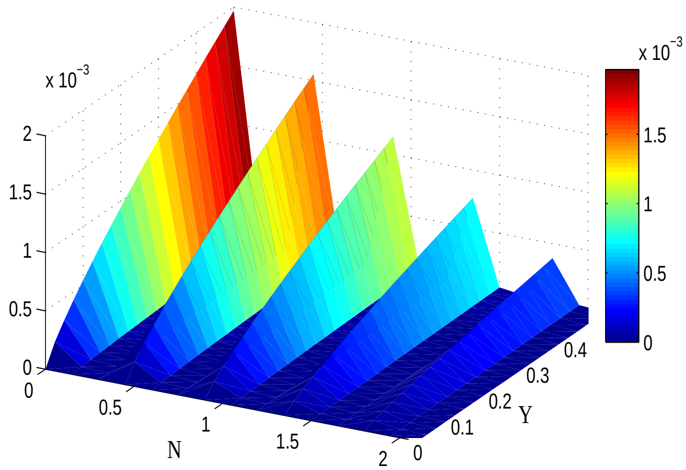

Figure 20: The shape RMS for $u(0.5, y, t)$ with $\alpha_1, \alpha_2, \alpha_3$ that are $\alpha_1 = 0.1, \alpha_2 = 0.5, \alpha_3 = 0.8$ of $Eq.10$ and error $Eq.18$.

and tree term fractional $\alpha_i, i = 1, 2, 3$

$$\mathcal{U}(x, y, t) = t^{1+\alpha_1+\alpha_2+\alpha_3} x^2 e^y$$

also

$$\mathbb{F}(x,y,t) = -2t^{2+\alpha_1+\alpha_2+\alpha_3}(x^2+y^2) + x^2 e^y \Gamma(1+\alpha_1+\alpha_2+\alpha_3)(1+\alpha_1+\alpha_2) \left[\frac{(t^{1+\alpha_1+\alpha_2})\Gamma(2-\alpha_3)}{\Gamma(3+\alpha_1+\alpha_2)\Gamma(1-\alpha_3)} + \frac{(t^{1+\alpha_1+\alpha_3})\Gamma(2-\alpha_2)}{\Gamma(3+\alpha_2+\alpha_3)\Gamma(1-\alpha_2)} + \frac{(t^{1+\alpha_2+\alpha_3})\Gamma(2-\alpha_1)}{\Gamma(3+\alpha_2+\alpha_3)\Gamma(1-\alpha_1)}\right]$$

In this sample the exact answers is one exponent function in x variable for plot the $Error$ of obtained answers by amounts of Degree of fraction, assume one of the variables the variable X or Y to be constant then we calculate the RMS.We assume amounts fixed away from knots primary. Anew the N is dimension of $fBSf$ and the N is grow $Error$ is not increase. The $Fig.21$, $Fig.22$, $Fig.23$ and $Fig.24$ are answers at several time surfaces for α have been presented.

Table 7: Sample of $Eq.10$ and RMS $Eq.19$ and the α_1, α_2 have tree variable t, x, n, that y is fixed.

	RMS_j^0	RMS_j^1	RMS_j^2
$\alpha_1 = 0.5, \alpha_2 = 0.6$	$9.04541182 \times 10^{-5}$	$1.41615859 \times 10^{-6}$	$6.03249119 \times 10^{-7}$
$\alpha_1 = 0.1, \alpha_2 = 0.7$	$4.16261408 \times 10^{-5}$	$1.93217574 \times 10^{-6}$	$8.35037092 \times 10^{-7}$
$\alpha_1 = 0.3, \alpha_2 = 0.6$	$8.58065467 \times 10^{-5}$	$1.73144761 \times 10^{-6}$	$7.46330818 \times 10^{-7}$
$\alpha_1 = 0.2, \alpha_2 = 0.4$	$4.56260027 \times 10^{-5}$	$3.62032205 \times 10^{-6}$	$6.53930371 \times 10^{-7}$
$\alpha_1 = 0.7, \alpha_2 = 0.8$	$1.80267851 \times 10^{-5}$	$1.36214067 \times 10^{-6}$	$2.43485256 \times 10^{-7}$

Table 8: Sample of $Eq.$10 and RMS $Eq.$19 and the $\alpha_1, \alpha_2, \alpha_3$ have tree variable t, x, n, that y is fixed.

	RMS_j^0	RMS_j^1	RMS_j^2
$\alpha_1 = 0.3, \alpha_2 = 0.5, \alpha_3 = 0.6$	$5.19353341 \times 10^{-4}$	$3.80155456 \times 10^{-5}$	$9.73121322 \times 10^{-6}$
$\alpha_1 = 0.2, \alpha_2 = 0.5, \alpha_3 = 0.7$	$4.80850444 \times 10^{-4}$	$3.78465263 \times 10^{-5}$	$9.69569172 \times 10^{-6}$
$\alpha_1 = 0.1, \alpha_2 = 0.3, \alpha_3 = 0.8$	$4.68682804 \times 10^{-4}$	$3.43935168 \times 10^{-5}$	$8.59295668 \times 10^{-6}$
$\alpha_1 = 0.2, \alpha_2 = 0.4, \alpha_3 = 0.6$	$4.04031852 \times 10^{-4}$	$2.32012072 \times 10^{-5}$	$1.42442171 \times 10^{-6}$
$\alpha_1 = 0.3, \alpha_2 = 0.4, \alpha_3 = 0.9$	$3.09153935 \times 10^{-4}$	$1.74275616 \times 10^{-5}$	$1.04006198 \times 10^{-6}$

Table 9: Sample of $Eq.$10 and RMS Eq19 and the α_1, α_2 have tree variable t, y, n, that x is fixed.

	RMS_j^0	RMS_j^1	RMS_j^2
$\alpha_1 = 0.5, \alpha_2 = 0.6$	$9.04541182 \times 10^{-5}$	$1.24974484 \times 10^{-6}$	$4.05615235 \times 10^{-7}$
$\alpha_1 = 0.7, \alpha_2 = 0.1$	$9.90638751 \times 10^{-5}$	$1.71281036 \times 10^{-6}$	$5.62713624 \times 10^{-7}$
$\alpha_1 = 0.6, \alpha_2 = 0.3$	$9.26941318 \times 10^{-5}$	$1.05348294 \times 10^{-6}$	$5.03312048 \times 10^{-7}$
$\alpha_1 = 0.2, \alpha_2 = 0.4$	$5.54470808 \times 10^{-5}$	$6.02212710 \times 10^{-6}$	$3.83331255 \times 10^{-7}$
$\alpha_1 = 0.7, \alpha_2 = 0.8$	$2.16824420 \times 10^{-5}$	$2.26147866 \times 10^{-6}$	$1.43730085 \times 10^{-7}$

Table 10: Sample of $Eq.$10 and RMS $Eq.$19 and the $\alpha_1, \alpha_2, \alpha_3$ have tree variable t, y, n, that x is fixed.

	RMS_j^0	RMS_j^1	RMS_j^2
$\alpha_1 = 0.3, \alpha_2 = 0.5, \alpha_3 = 0.6$	$7.50950353e \times 10^{-5}$	$9.16838821 \times 10^{-6}$	$3.04125495 \times 10^{-7}$
$\alpha_1 = 0.2, \alpha_2 = 0.5, \alpha_3 = 0.7$	$6.99727485 \times 10^{-5}$	$9.13493187 \times 10^{-6}$	$3.03772247 \times 10^{-7}$
$\alpha_1 = 0.1, \alpha_2 = 0.3, \alpha_3 = 0.8$	$6.64170418 \times 10^{-5}$	$8.22103967 \times 10^{-6}$	$2.73865500 \times 10^{-7}$
$\alpha_1 = 0.2, \alpha_2 = 0.4, \alpha_3 = 0.6$	$5.59944023 \times 10^{-5}$	$4.34802764 \times 10^{-6}$	$3.44168282 \times 10^{-7}$
$\alpha_1 = 0.3, \alpha_2 = 0.4, \alpha_3 = 0.9$	$3.74528508 \times 10^{-5}$	$2.84022165 \times 10^{-6}$	$4.65066226 \times 10^{-6}$

At Our tables, we obtain RMS of $Eq.$19 for several α's. The RMS solutions is not much more than 10^{-4}. With $n = 1000$, several amounts α_1, α_2 and Δt with $y = 0.5$ at tables 7 and 8, Beginning The RMS is of 10^{-5} until to 10^{-7} that the outcomes and the answers are accord and variable time at has nearly effectless when it is tiny enough at tables 9 and 10 we have tree fractional the $\alpha_1, \alpha_2, \alpha_3$ that have been illustrated for two term α_1, α_2 and tree term $\alpha_1, \alpha_2, \alpha_3$ with $x = 0.5$, the RMS is among 10^{-4} until 10^{-6}. It is in the above figures $\Delta t = 0.01$ and $n = 500$. For approximate answers with $y = 0.5$ that in $Fig.$21 in fact displays the $Error$ of $Eq.$18 and we considered $\alpha_1 = 0.3, \alpha_2 = 0.6$ in $Fig.$22 we considered $\alpha_1 = 0.3, \alpha_2 = 0.4, \alpha_3 = 0.9$, the N is dimensions of $fBSf$. we look in the shapes RMS in axis X is not decrease than 10^{-3} by notice with $N = 2$ it is 10^{-4}, at in $Fig.$23 and $Fig.$24 the powers factional are look to $Fig.$21 and $Fig.$22 in order only $x = 0.5$ instead $y = 0.5$. It is manner is not fast it is not rapidity increase tangible.

Example 4 We discuss the $Eq.$(10) with two variable x, y that's mean $\overline{\mathbf{X}} \in \mathbb{R}^2$ and several amounts for α and $\triangle t^i = t^i - t^{i-1} = 0.01$ in partition $\Omega = [0, 0.5] \times [0, 0.5]$ and $t \in [0, 1]$. The $\mathcal{U}(x, y, t) = t^{1+\alpha_1+\alpha_2} x^2 \sin(\pi y)$ is solution

$$\begin{aligned}\mathbb{F}(x,y,t) &= (t^{1+\alpha_1+\alpha_2}\sin(\pi y))(-2+\pi^2x^2) + x^2\sin\pi y\Gamma(1+\alpha_1+\alpha_2)(1+\alpha_1+\alpha_2) \\ &\quad \left[\frac{(t^{2+\alpha_1})\Gamma(2-\alpha_1)}{\Gamma(3+\alpha_1)\Gamma(1-\alpha_2)} + \frac{(t^{2+\alpha_2})\Gamma(2-\alpha_2)}{\Gamma(3+\alpha_2)\Gamma(1-\alpha_1)}\right]\end{aligned}$$

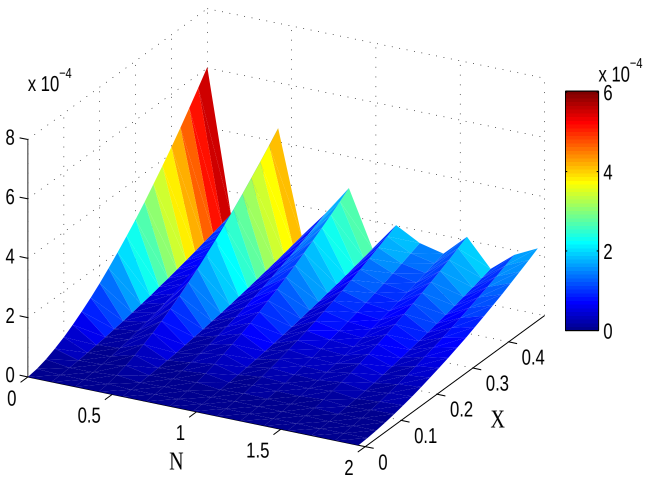

Figure 21: Example of $Eq.$10 and error $Eq.$18 and in diagram of absolute error of $u(x, 0.5, t)$ at with α_1, α_2 that are $\alpha_1 = 0.3, \alpha_2 = 0.6$.

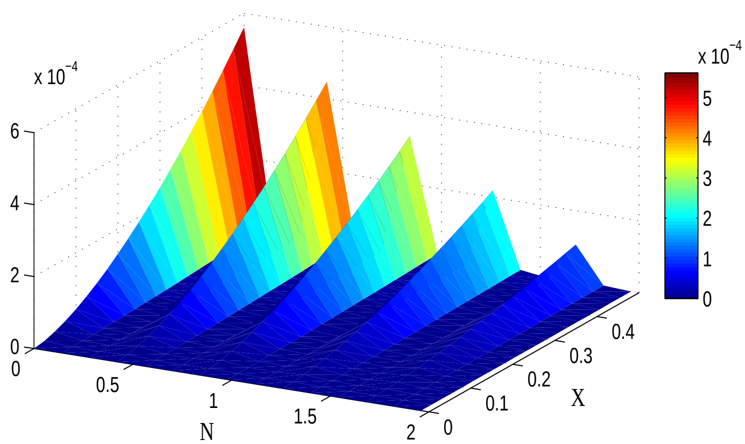

Figure 22: The shape RMS for $u(x, 0.5, t)$ with $\alpha_1, \alpha_2, \alpha_3$ that are $\alpha_1 = 0.3, \alpha_2 = 0.4, \alpha_3 = 0.9$. of $Eq.$10 and error $Eq.$18.

and tree term fractional $\alpha_i, i = 1, 2, 3$

$$\mathcal{U}(x, y, t) = t^{1+\alpha_1+\alpha_2+\alpha_3} x^2 \sin(\pi y)$$

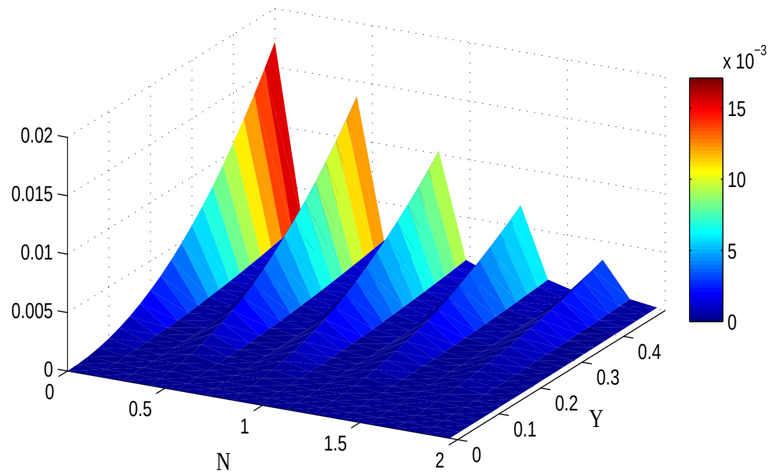

Figure 23: The shape RMS for $u(0.5, y, t)$ with α_1, α_2 that are $\alpha_1 = 0.3, \alpha_2 = 0.6$ of $Eq.$10 and error $Eq.$18.

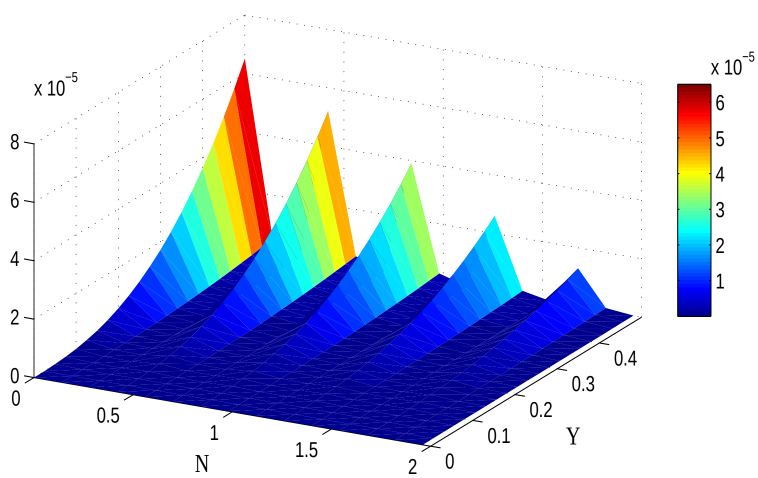

Figure 24: The shape RMS for $u(0.5, y, t)$ with $\alpha_1, \alpha_2, \alpha_3$ that are $\alpha_1 = 0.01, \alpha_2 = 0.4, \alpha_3 = 0.9.$ of $Eq.$10 and error $Eq.$18.

also

$$\begin{aligned}\mathbb{F}(x,y,t) &= (t^{2+\alpha_1+\alpha_2+\alpha_3})(-2+(x^2\sin(\pi y))+x^2\sin(\pi y)\Gamma(1+\alpha_1+\alpha_2+\alpha_3)(1+\alpha_1+\alpha_2+\alpha_3)\\ &\left[\frac{(t^{1+\alpha_1+\alpha_2})\Gamma(2-\alpha_3)}{\Gamma(3+\alpha_1+\alpha_2)\Gamma(1-\alpha_3)}+\frac{(t^{1+\alpha_1+\alpha_3})\Gamma(2-\alpha_2)}{\Gamma(3+\alpha_2+\alpha_3)\Gamma(1-\alpha_2)}+\frac{(t^{1+\alpha_2+\alpha_3})\Gamma(2-\alpha_1)}{\Gamma(3+\alpha_2+\alpha_3)\Gamma(1-\alpha_1)}\right]\end{aligned}$$

In this sample the exact answers is one $\sin(x)$ function in x variable for plot the $Error$ of obtained answers by amounts of Degree of fraction, assume one of the variables the variable X or Y to be constant then we calculate the RMS.We assume amounts fixed away from knots primary. Anew the N is dimension of $fBSf$ and the N is grow $Error$ is not increase. The $Fig.25$, $Fig.262$, $Fig.27$ and $Fig.28$ are answers at several time surfaces for α have been presented.

Table 11: Sample of $Eq.10$ and RMS $Eq.19$ and the α_1, α_2 have tree variable t, x, n, that y is fixed.

	RMS_j^0	RMS_j^1	RMS_j^2
$\alpha_1 = 0.1, \alpha_2 = 0.2$	$2.48704511 \times 10^{-5}$	$2.48680178 \times 10^{-5}$	$2.50683895 \times 10^{-6}$
$\alpha_1 = 0.1, \alpha_2 = 0.4$	$2.11915060 \times 10^{-6}$	$2.11905899 \times 10^{-6}$	$2.11839033 \times 10^{-6}$
$\alpha_1 = 0.3, \alpha_2 = 0.6$	$1.47744861 \times 10^{-6}$	$1.47738445 \times 10^{-6}$	$1.47691804 \times 10^{-6}$
$\alpha_1 = 0.5, \alpha_2 = 0.7$	$4.45767624 \times 10^{-8}$	$1.32072454 \times 10^{-8}$	$2.85215545 \times 10^{-9}$
$\alpha_1 = 0.4, \alpha_2 = 0.8$	$4.45767614 \times 10^{-8}$	$1.32072443 \times 10^{-8}$	$2.85215514 \times 10^{-9}$

Table 12: Sample of $Eq.10$ and RMS $Eq.19$ and the $\alpha_1, \alpha_2, \alpha_3$ have tree variable t, x, n, that y is fixed.

	RMS_j^0	RMS_j^1	RMS_j^2
$\alpha_1 = 0.1, \alpha_2 = 0.2, \alpha_3 = 0.3$	$1.93352892 \times 10^{-9}$	$1.93352789 \times 10^{-10}$	$1.93351972 \times 10^{-10}$
$\alpha_1 = 0.2, \alpha_2 = 0.4, \alpha_3 = 0.5$	$1.24062859 \times 10^{-9}$	$1.24062783 \times 10^{-10}$	$1.24062144 \times 10^{-10}$
$\alpha_1 = 0.5, \alpha_2 = 0.6, \alpha_3 = 0.7$	$6.87005350 \times 10^{-9}$	$6.87004855 \times 10^{-10}$	$6.87000483 \times 10^{-10}$
$\alpha_1 = 0.3, \alpha_2 = 0.5, \alpha_3 = 0.9$	$2.61481782 \times 10^{-9}$	$7.74760432 \times 10^{-10}$	$1.67347895 \times 10^{-10}$
$\alpha_1 = 0.7, \alpha_2 = 0.8, \alpha_3 = 0.9$	$2.61481782 \times 10^{-9}$	$7.74760433 \times 10^{-10}$	$1.67347895 \times 10^{-10}$

Table 13: Sample of $Eq.10$ and RMS $Eq.19$ and the α_1, α_2 have tree variable t, y, n, that x is fixed.

	RMS_j^0	RMS_j^1	RMS_j^2
$\alpha_1 = 0.1, \alpha_2 = 0.2$	$8.31787593 \times 10^{-13}$	$7.27500489 \times 10^{-13}$	$3.77477483 \times 10^{-13}$
$\alpha_1 = 0.1, \alpha_2 = 0.4$	$6.89621726 \times 10^{-13}$	$6.02980391 \times 10^{-13}$	$3.12902522 \times 10^{-13}$
$\alpha_1 = 0.3, \alpha_2 = 0.6$	$4.80796722 \times 10^{-13}$	$4.20121940 \times 10^{-13}$	$2.18027408 \times 10^{-13}$
$\alpha_1 = 0.5, \alpha_2 = 0.7$	$2.49135913 \times 10^{-14}$	$6.70824908 \times 10^{-15}$	$8.76460781 \times 10^{-16}$
$\alpha_1 = 0.4, \alpha_2 = 0.8$	$2.49126107 \times 10^{-14}$	$6.70727405 \times 10^{-15}$	$8.76281172 \times 10^{-16}$

Table 14: Sample of $Eq.10$ and RMS $Eq.19$ and the $\alpha_1, \alpha_2, \alpha_3$ have tree variable t, y, n, that x is fixed.

	RMS_j^0	RMS_j^1	RMS_j^2
$\alpha_1 = 0.1, \alpha_2 = 0.2, \alpha_3 = 0.3$	$6.29148619 \times 10^{-14}$	$5.33007986 \times 10^{-15}$	$6.79864808 \times 10^{-16}$
$\alpha_1 = 0.2, \alpha_2 = 0.4, \alpha_3 = 0.5$	$4.03686627 \times 10^{-14}$	$3.32564899 \times 10^{-15}$	$1.51765516 \times 10^{-16}$
$\alpha_1 = 0.5, \alpha_2 = 0.6, \alpha_3 = 0.7$	$2.23543833 \times 10^{-14}$	$1.77788355 \times 10^{-15}$	$1.13809033 \times 10^{-16}$
$\alpha_1 = 0.3, \alpha_2 = 0.5, \alpha_3 = 0.9$	$1.46147562 \times 10^{-14}$	$3.93388530 \times 10^{-15}$	$5.13326847 \times 10^{-16}$
$\alpha_1 = 0.7, \alpha_2 = 0.8, \alpha_3 = 0.9$	$1.46149330 \times 10^{-14}$	$3.93429190 \times 10^{-15}$	$5.13422205 \times 10^{-16}$

In our tables, we obtain RMS of $Eq.19$ for several α's. The RMS solutions is not much more than 10^{-4}. With $n = 1000$, several amounts α_1, α_2 and Δt with $y = 0.5$ at tables 11 and 12, Beginning The RMS is of 10^{-5} until to 10^{-7} that the outcomes and the answers are accord and variable time at has nearly effectless when it is tiny enough at tables 13 and 14 we have tree fractional the $\alpha_1, \alpha_2, \alpha_3$ that have been illustrated for two term α_1, α_2 and tree term α_1, α_2, α_3 with $x = 0.5$, the RMS is among 10^{-4} until 10^{-6}.

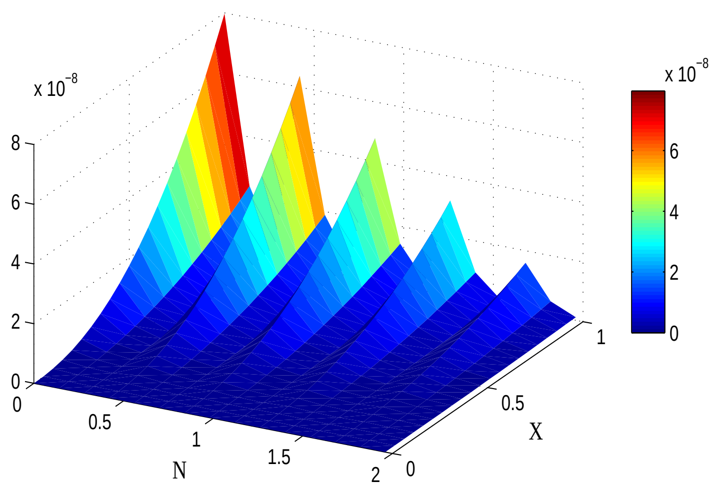

Figure 25: The shape RMS for $u(x, 0.5, t)$ with α_1, α_2 that are $\alpha_1 = 0.5, \alpha_2 = 0.7$ of $Eq.10$ and error $Eq.18$.

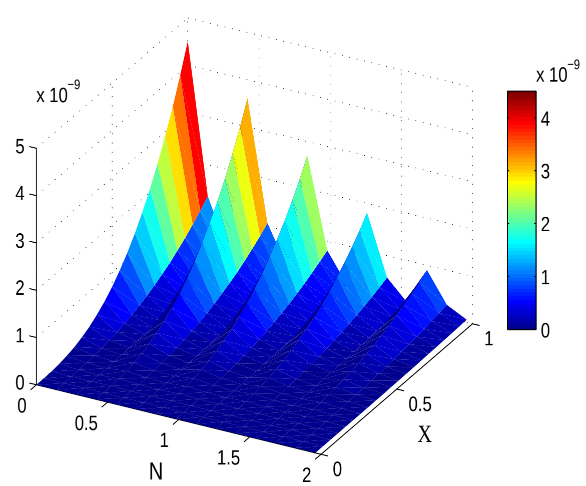

Figure 26: The shape RMS for $u(x, 0.5, t)$ with with $\alpha_1, \alpha_2, \alpha_3$ that are $\alpha_1 = 0.3$, $\alpha_2 = 0.5$, $\alpha_3 = 0.9$. of $Eq.10$ and error $Eq.18$.

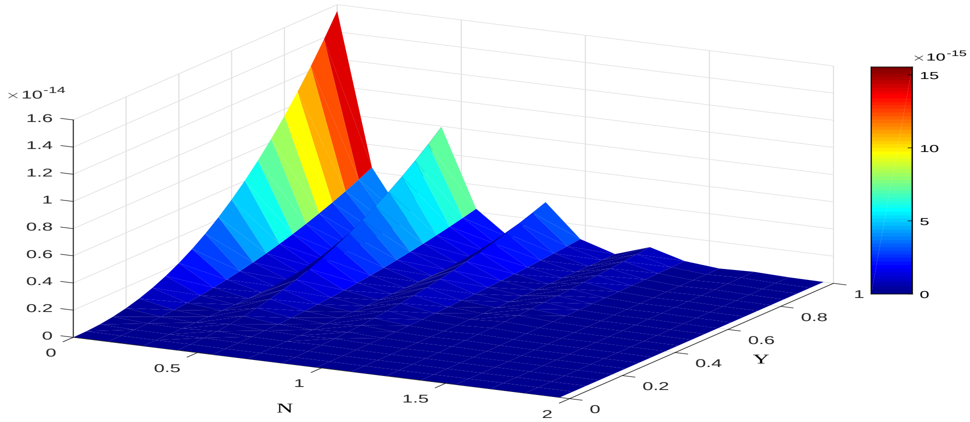

Figure 27: The shape RMS for $u(0.5, y, t)$ with α_1, α_2 that are $\alpha_1 = 0.5$, $\alpha_2 = 0.7$, $\alpha_3 = 0.9$. of $Eq.$10 and error $Eq.$18.

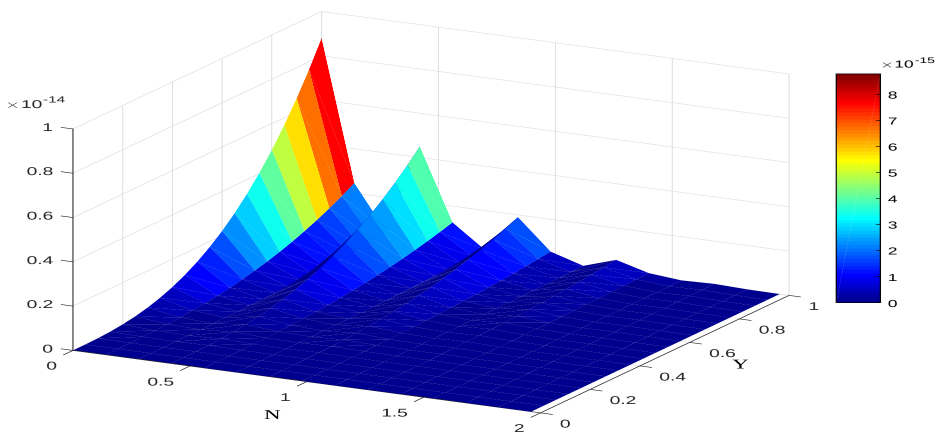

Figure 28: The shape RMS for $u(0.5, y, t)$ with $\alpha_1, \alpha_2, \alpha_3$ that are $\alpha_1 = 0.3$, $\alpha_2 = 0.5$, $\alpha_3 = 0.9$ of $Eq.$10 and error $Eq.$18.

From the above figures $\Delta t = 0.01$ and $n = 1000$. For approximate answers with $y = 0.5$ that in $Fig.$25 in fact displays the $Error$ of $Eq.$18 and we considered $\alpha_1 = 0.5, \alpha_2 = 0.7$ in $Fig.$26 we considered $\alpha_1 = 0.3, \alpha_2 = 0.5, \alpha_3 = 0.9$, the N is dimensions of $fBSf$. we look in the shapes RMS in axis X is not decrease than 10^{-3} by notice with $N = 2$ it is 10^{-4}, at in $Fig.$27 and $Fig.$28 the powers factional are look to $Fig.$25 and $Fig.$26 in order only $x = 0.5$ instead $y = 0.5$. It is manner is not fast it is not rapidity increase tangible.

Example 5

The fifth sample, we discuss the $Eq.10$ with two variable x, y that's mean $\overline{\mathbf{X}} \in \mathbb{R}^2$ and several amounts for α and $t \in [0,1]$ and $\triangle t^i = 0.01$ in partition $\Omega = [0,1] \times [0, 0.5]$. The $\mathcal{U}(x,y,t) = t^{1+\alpha 1+\alpha 2}\cos(\pi x)\sin(\pi y)$ is solution $\mathcal{U}(x,y,t) = t^{1+\alpha 1+\alpha 2}\cos(\pi x)\sin(\pi y)$ also

$$\begin{aligned}\mathbb{F}(x,y,t) &= (\cos(\pi x)\sin(\pi y))[(2\pi^2)(t^{1+\alpha_1+\alpha_2} + \Gamma(1+\alpha_1+\alpha_2)(1+\alpha_1+\alpha_2)\\ &\left[\frac{(t^{2+\alpha_1})\Gamma(2-\alpha_1)}{\Gamma(3+\alpha_1)\Gamma(1-\alpha_2)} + \frac{(t^{2+\alpha_2})\Gamma(2-\alpha_2)}{\Gamma(3+\alpha_2)\Gamma(1-\alpha_1)}\right]\end{aligned}$$

and tree term fractional $\alpha_i, i = 1,2,3\; \mathcal{U}(x,y,t) = t^{1+\alpha_1+\alpha_2+\alpha_3}x^2\sin(\pi y)$ also

$$\begin{aligned}\mathbb{F}(x,y,t) &= (\cos(\pi x)\sin(\pi y))[(t^{2+\alpha_1+\alpha_2+\alpha_3})(2\pi^2) + \Gamma(1+\alpha_1+\alpha_2+\alpha_3)(1+\alpha_1+\alpha_2+\alpha_3)\\ &\left[\frac{(t^{1+\alpha_1+\alpha_2})\Gamma(2-\alpha_3)}{\Gamma(3+\alpha_1+\alpha_2)\Gamma(1-\alpha_3)} + \frac{(t^{1+\alpha_1+\alpha_3})\Gamma(2-\alpha_2)}{\Gamma(3+\alpha_2+\alpha_3)\Gamma(1-\alpha_2)} + \frac{(t^{1+\alpha_2+\alpha_3})\Gamma(2-\alpha_1)}{\Gamma(3+\alpha_2+\alpha_3)\Gamma(1-\alpha_1)}\right]\end{aligned}$$

In this sample the exact answers is one $\cos(x)$ multiplied by $\sin(y)$ function in x variable and variable $y$for plot the $Error$ of obtained answers by amounts of Degree of fraction, assume one of the variables the variable X or $Y$to be constant then we calculate the RMS.We assume amounts fixed away from knots primary. Anew the N is dimension of $fBSf$ and the N is grow $Error$ is not increase. The $Fig.29$, $Fig.30$, $Fig.31$ and $Fig.28$ are answers at several time surfaces for α have been presented.

Table 15: Sample of $Eq.10$ and RMS $Eq.19$ and the α_1, α_2 have tree variable t, x, n, that y is fixed.

	RMS_j^0	RMS_j^1	RMS_j^2
$\alpha_1 = 0.2, \alpha_2 = 0.4$	$2.66410382 \times 10^{-5}$	$8.11472163 \times 10^{-6}$	$1.84662960 \times 10^{-6}$
$\alpha_1 = 0.1, \alpha_2 = 0.7$	$2.12768140 \times 10^{-5}$	$6.48025816 \times 10^{-6}$	$1.47455616 \times 10^{-6}$
$\alpha_1 = 0.3, \alpha_2 = 0.6$	$1.90424748 \times 10^{-5}$	$5.79995106 \times 10^{-6}$	$1.31984354 \times 10^{-6}$
$\alpha_1 = 0.5, \alpha_2 = 0.9$	$1.10666715 \times 10^{-5}$	$3.36960185 \times 10^{-6}$	$7.66820560 \times 10^{-7}$
$\alpha_1 = 0.6, \alpha_2 = 0.8$	$1.10663554 \times 10^{-5}$	$3.36976093 \times 10^{-6}$	$7.66936795 \times 10^{-7}$

Table 16: Sample of $Eq.10$ and RMS $Eq.19$ and the $\alpha_i, i = 1,2,3.$ have tree variable t, x, n, that y is fixed.

	RMS_j^0	RMS_j^1	RMS_j^2
$\alpha_1 = 0.1, \alpha_2 = 0.2, \alpha_3 = 0.3$	$2.64417141 \times 10^{-5}$	$7.83458201 \times 10^{-6}$	$1.692269728 \times 10^{-6}$
$\alpha_1 = 0.2, \alpha_2 = 0.4, \alpha_3 = 0.6$	$1.36523566 \times 10^{-5}$	$4.08011152 \times 10^{-6}$	$8.827629491 \times 10^{-7}$
$\alpha_1 = 0.3, \alpha_2 = 0.6, \alpha_3 = 0.9$	$7.20941288 \times 10^{-6}$	$2.15433676 \times 10^{-6}$	$4.661227410 \times 10^{-7}$
$\alpha_1 = 0.1, \alpha_2 = 0.5, \alpha_3 = 0.9$	$9.89595187 \times 10^{-6}$	$2.95725439 \times 10^{-6}$	$4.6.3983023 \times 10^{-7}$
$\alpha_1 = 0.6, \alpha_2 = 0.7, \alpha_3 = 0.8$	$5.27428503 \times 10^{-6}$	$1.57597159 \times 10^{-6}$	$3.409935412 \times 10^{-7}$

In our tables, we obtain RMS of $Eq.19$ for several α's. The RMS solutions is not much more than 10^{-4}. With $n = 1000$, several amounts α_1, α_2 and Δt with $y = 0.5$ at tables 15 and 16, Beginning The RMS is of 10^{-6} until to 10^{-7} that the outcomes and the answers are accord and variable time at has nearly effectless when it is tiny

Table 17: Sample of $Eq.$10 and RMS $Eq.$19 and the α_1, α_2 have tree variable t, y, n, that x is fixed.

	RMS_j^0	RMS_j^1	RMS_j^2
$\alpha_1 = 0.2, \alpha_2 = 0.4$	$1.12350796 \times 10^{-4}$	$3.70391510 \times 10^{-5}$	$8.11254912 \times 10^{-6}$
$\alpha_1 = 0.1, \alpha_2 = 0.7$	$5.82415382 \times 10^{-5}$	$1.98429677 \times 10^{-5}$	$4.56905448 \times 10^{-6}$
$\alpha_1 = 0.3, \alpha_2 = 0.6$	$3.08286709 \times 10^{-5}$	$1.04770384 \times 10^{-5}$	$2.41259187 \times 10^{-6}$
$\alpha_1 = 0.5, \alpha_2 = 0.9$	$4.22480211 \times 10^{-5}$	$1.43728151 \times 10^{-5}$	$3.30905635 \times 10^{-6}$
$\alpha_1 = 0.6, \alpha_2 = 0.8$	$2.26190713 \times 10^{-5}$	$7.67361570 \times 10^{-6}$	$1.76739514 \times 10^{-6}$

Table 18: Sample of $Eq.$10 and RMS $Eq.$19 and the $\alpha_i, i = 1, 2, 3$ have tree variable t, y, n, that x is fixed.

	RMS_j^0	RMS_j^1	RMS_j^2
$\alpha_1 = 0.1, \alpha_2 = 0.2, \alpha_3 = 0.3$	$1.35596506 \times 10^{-4}$	$1.34395454 \times 10^{-4}$	$1.34377414 \times 10^{-4}$
$\alpha_1 = 0.2, \alpha_2 = 0.4, \alpha_3 = 0.6$	$1.27265809 \times 10^{-4}$	$1.26561905 \times 10^{-4}$	$1.25629738 \times 10^{-4}$
$\alpha_1 = 0.3, \alpha_2 = 0.6, \alpha_3 = 0.9$	$1.20259941 \times 10^{-4}$	$1.19793031 \times 10^{-4}$	$1.16883116 \times 10^{-4}$
$\alpha_1 = 0.1, \alpha_2 = 0.5, \alpha_3 = 0.9$	$2.99362980 \times 10^{-5}$	$1.32748298 \times 10^{-5}$	$4.65066226 \times 10^{-6}$
$\alpha_1 = 0.6, \alpha_2 = 0.7, \alpha_3 = 0.8$	$2.88319590 \times 10^{-5}$	$1.27763920 \times 10^{-5}$	$4.65066226 \times 10^{-6}$

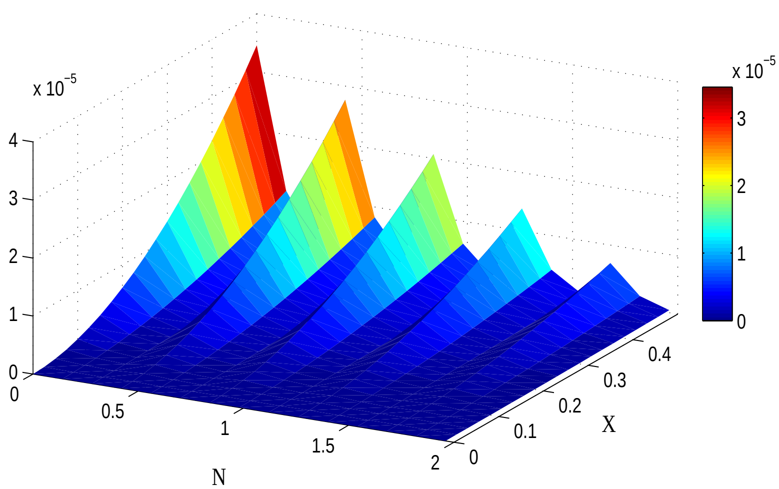

Figure 29: The shape RMS for $u(x, 0.5, t)$ with α_1, α_2 that are $\alpha_1 = 0.3, \alpha_2 = 0.6$ of $Eq.$10 and error $Eq.$18.

enough at tables 17 and 18 we have tree fractional the $\alpha_i, i = 1, 2, 3$ that have been illustrated for two term α_1, α_2 and tree term α_1, α_2, α_3 with $x = 0.5$, the RMS is among 10^{-4} until 10^{-6}.

From the above figures $\Delta t = 0.01$ and $n = 1000$. For approximate answers with $y = 0.5$ that in $Fig.$29 in fact displays the $Error$ of $Eq.$18 and we considered $\alpha_1 = 0.3, \alpha_2 = 0.6$ in $Fig.$30 we considered $\alpha_1 = 0.1, \alpha_2 = 0.5, \alpha_3 = 0.9$, the N is dimensions of $fBSf$. we look in the shapes RMS in axis X is not decrease than 10^{-4} by notice with $N = 2$ it is 10^{-5}, at in $Fig.$31 and $Fig.$32 the powers factional are look to $Fig.$29 and $Fig.$30 in order only $x = 0.5$ instead $y = 0.5$. It is manner is not fast to it is not rapidity increase tangible.

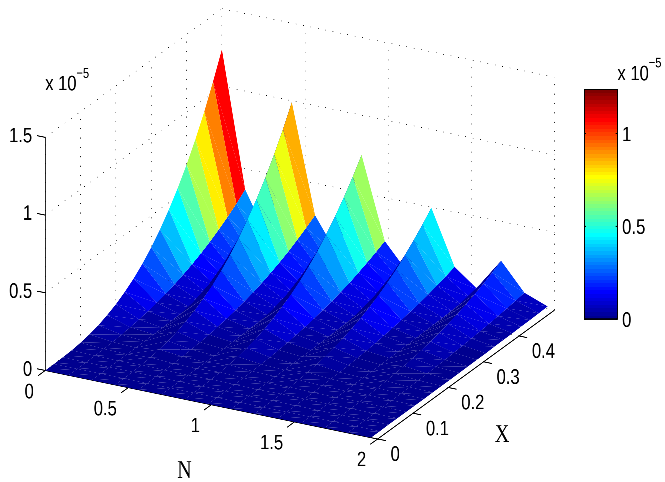

Figure 30: The shape RMS for $u(x, 0.5, t)$ with $\alpha_i, i = 1, 2, 3$ that are $\alpha_1 = 0.1, \alpha_2 = 0.5, \alpha_3 = 0.9$ of $Eq.10$ and error $Eq.18$.

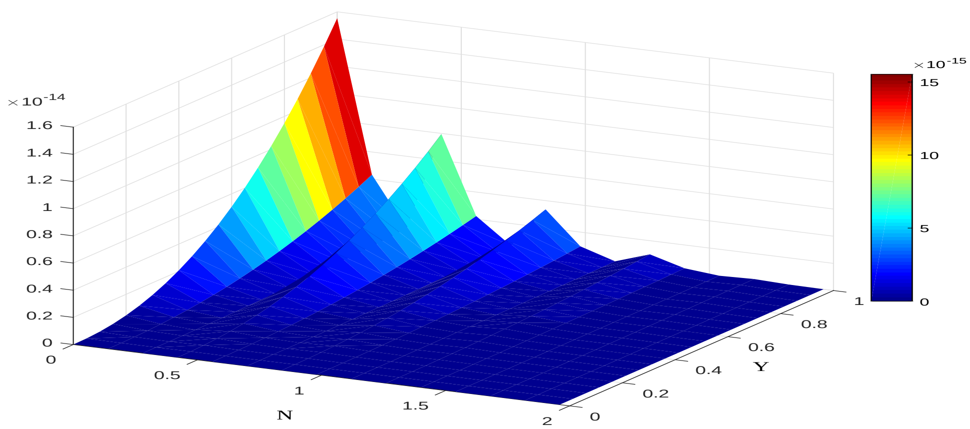

Figure 31: The shape RMS for $u(0.5, y, t)$ with α_1, α_2 that are $\alpha_1 = 0.3, \alpha_2 = 0.6$ of $Eq.10$ and error $Eq.18$.

5. conclusions

In our manuscript, we have solved multi-term time fractional diffusion-wave equation by Collocation Method where the $\mathcal{D}_t)$ in this is Caputo concept for $(0 < \alpha < 1)$. We have considered an arbitrary one- and two-dimentional. Of $fBSf$ used at collocation method.

We have examined two issues here, the first Simplicity and ease of applying this method to multi-term time frac-

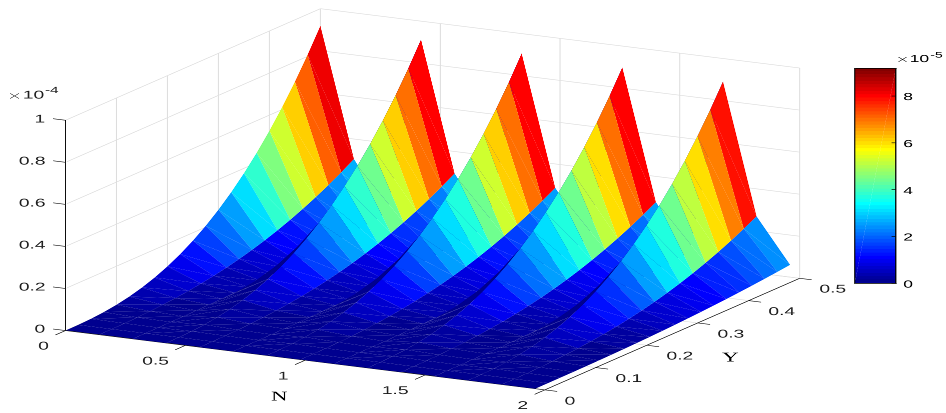

Figure 32: The shape RMS for $u(0.5, y, t)$ with $\alpha_i, i = 1, 2, 3$ that are $\alpha_1 = 0.1, \alpha_2 = 0.5, \alpha_3 = 0.9$ of $Eq.10$ and error $Eq.18$.

tional diffusion-wave equation. Our second goal was to apply these basic functions to these types of equations. The effectiveness and high accuracy of the proposed numerical approximate scheme provided numerical results and figures demonstrate. To test the correctness of the method, we provided several examples with different exact answers in the powers. Numerical simulations were performed using Mathlab.

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