Journal of Mathematical Extension
Vol. 15, SI-NTFCA, (2021) (15)1-54
URL: https://doi.org/10.30495/JME.SI.2021.2070
ISSN: 1735-8299
Original Research Paper

Investigation of a Class of the Singular Fractional Integro-differential Quantum Equations with Multi-Step Methods

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Abstract. The objective of this paper is to investigate, by applying the standard Caputo fractional $q-$derivative of order α, the existence of solutions for a class of the singular fractional $q-$integro-differential equation under some boundary conditions on a time scale. We consider the compact map and the Lebesgue dominated theorem for finding solutions of the problem. Our attention is concentrated on fractional multi-step methods of both implicit and explicit type, for which sufficient existence conditions are investigated. Lastly, we present some examples involving graphs, tables and algorithms to illustrate the validity of our theoretical findings.

AMS Subject Classification: MSC 34A08; 34B16; 39A13
Keywords and Phrases: Singularity, multi-step methods, quantum equation

Received: June 2021; Published: August 2021
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1 Introduction

The field of fractional calculus plays a fundamental role in mathematical analysis. It provides efficient techniques to solve fractional differential equations and inclusions ([2–4, 7, 11, 13, 16, 18, 23, 29, 34]). On the other hand, one of the most interesting topic is $q-$difference equations which were introduced by Jackson ([17]).

In 2007, Atici *et al.* studied discrete fractional calculus and considered a family of finite fractional linear difference equations ([9]). They developed the theory of linear finite fractional difference equations analogously to the theory of finite difference equations ([9]). The fractional problem

$$\mathcal{D}^{\sigma}[k](r) + w\big(r, k(r), \mathcal{D}^{\xi}[k](r)\big) = 0,$$

with boundary conditions $k(0) = k(1) = 0$ was investigated, where $r \in (0,1)$, $\sigma \in (1,2)$, $0 < \xi \leq \sigma - 1$, $\mathcal{D}^{\sigma}$ is the standard Riemann–Liouville fractional derivative, w satisfies the Carathéodory conditions on $[0,1] \times (0,\infty) \times \mathbb{R}$, w is positive and $w(t,k,l)$ is singular at $t = 0$ ([5]). Also, the fractional differential equation $\mathcal{D}^{\sigma}[k](r) + w(r, k(r)) = 0$ with boundary conditions $k(0) = k''(0) = 0$ and $k(1) = \lambda \int_0^1 k(s)\,\mathrm{d}s$ was studied by Ahmad *et al.* and Makhlouf *et al.* where $r \in (0,1)$, $\sigma \in (2,3)$, $\lambda \in (0,2)$, $\mathcal{D}^{\sigma}$ is the Caputo fractional derivative and $w : [0,1] \times [0,\infty) \to [0,\infty)$ is a continuous function ([6, 21]). The singular fractional problem

$${}^{c}\mathcal{D}^{q}_{0^+}[k](r) + w\big(r, k(r), {}^{c}\mathcal{D}^{\sigma}_{0^+}[k](r)\big) = 0,$$

with boundary conditions $k(0) = k'(0) = 0$ and $k'(1) = {}^{c}\mathcal{D}^{\sigma}_{0^+}[k](1)$ was considered, where $r, \sigma \in (0,1)$, $q \in (2,3)$, $w : (0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous with $w(t,k,l)$ may be singular at $t = 0$ and ${}^{c}\mathcal{D}^{q}_{0^+}$ is the Caputo derivative ([19]). Zhang *et al.* and through the spectral analysis and fixed point index theorem obtained the existence of positive solutions of the singular nonlinear fractional differential equation

$$\mathcal{D}^{\alpha}_{t}[u](t) = w\big(t, [u](t), \mathcal{D}^{\beta}_{t}[u](t)\big),$$

for $0 < t < 1$, with integral boundary value conditions $\mathcal{D}^{\beta}_{t}[u](0) = 0$ and

$$\mathcal{D}^{\beta}_{t}[u](1) = \int_0^1 \mathcal{D}^{\beta}_{t}[u](r)\,\mathrm{d}N(r),$$

where $1 < \alpha \leq 2$, $0 < \beta \leq 1$, $w(t, u, v)$ may be singular at both $t = 0$, 1 and $u = v = 0$, $\int_0^1 u(r)\,\mathrm{d}N(r)$ denotes the Riemann–Stieltjes integral with a signed measure, in which $N : [0, 1] \to \mathbb{R}$ is a function of bounded variation ([36]). Then Ahmad *et al.* investigated the existence of solutions for a $q-$antiperiodic boundary value problem of fractional $q-$difference inclusions given by

$$\mathcal{D}_q^{\alpha}[k](t) \in F\left(t, k(t), \mathcal{D}_q[k](t), \mathcal{D}_q^2[k](t)\right),$$

for $t \in [0, 1]$, $q \in (0, 1)$, $\alpha \in (2, 3]$ and $\beta \in (0, 3]$ under conditions $k(0) + k(1) = 0$ and

$$\mathcal{D}_q[k](0) + \mathcal{D}_q[k](1) = 0, \qquad \mathcal{D}_q^2[k](0) + \mathcal{D}_q^2[k](1) = 0,$$

where ${}^C\mathcal{D}_q^{\alpha}$ denotes Caputo fractional $q-$derivative of order α and $F : [0, 1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map with $\mathcal{P}(\mathbb{R})$ a class of all subsets of $\mathbb{R}$ ([6]). In 2019, Ntouyas *et al.* by applying definition of the fractional $q-$derivative of the Caputo type and the fractional $q-$integral of the Riemann–Liouville type, studied the existence and uniqueness of solutions for a multi-term nonlinear fractional $q-$integro-differential equations under some boundary conditions ([22])

$${}^C\mathcal{D}_q^{\sigma}[k](r) = w\big(r, k(r), (\varphi_1 k)(r), (\varphi_2 k)(r), {}^C\mathcal{D}_q^{\beta_1}[k](r), {}^C\mathcal{D}_q^{\beta_2}[k](r), \ldots, {}^C\mathcal{D}_q^{\beta_n}[k](t)\big).$$

Also, Liang *et al.* investigated the existence of solutions for a nonlinear problems regular and singular fractional $q-$differential equation

$${}^C\mathcal{D}_q^{\sigma}[k](t) = w\big(r, k(r), k'(r), {}^C\mathcal{D}_q^{\beta}[k](r)\big),$$

with conditions $k(0) = c_1 k(1)$, $k'(0) = c_2 {}^c\mathcal{D}_q^{\beta}[k](1)$ and $k^{(m)}(0) = 0$ for $2 \leq m \leq n-1$, here $n-1 < \sigma < n$ with $n \geq 3$, $\beta, q, c_1 \in (0, 1)$, $0 < c_2 < \Gamma_q(2-\beta)$, function w is a L^{κ}-Carathéodory, $w(r, k_1, k_2, k_3)$ may be singular and ${}^c\mathcal{D}_q^{\sigma}$ the fractional Caputo type $q-$derivative ([20]). Further, they discussed the existence of solutions for the fractional $q-$derivative inclusions

$${}^C\mathcal{D}_q^{\sigma}[k](r) \in \mathcal{F}\big(r, k(r), k'(r), {}^C\mathcal{D}_q^{\beta}[k](r)\big),$$

under conditions

$$k(0) + k'(0) + {}^C\mathcal{D}_q^{\beta}[k](0) = \int_0^{\eta_1} k(s)\,\mathrm{d}s$$

and

$$k(1) + k'(1) + {}^{C}\mathcal{D}_q^{\beta}[k](1) = \int_0^{\eta_2} k(s)\,\mathrm{d}s,$$

for any $t \in [0,1]$ and q, η_1, η_2, $\beta \in (0,1)$, where F maps $[0,1] \times \mathbb{R}^3$ into $2^{\mathbb{R}}$ is a compact valued multifunction and ${}^{C}\mathcal{D}_q^{\sigma}$ is the fractional Caputo type $q-$derivative operator of order $\sigma \in (1,2]$, and

$$\Gamma_q(2-\beta)(\eta^2\nu - \nu^2\eta - \eta^2 + \nu^2 + 4\eta - 2\nu - 2) + 2(1-\eta) \neq 0,$$

such that $\sigma - \beta > 1$ ([30]). Relevant results have been presented in other studies [1, 10, 12, 20, 22, 25–28, 30, 31, 37].

In the research, motivated by the above mentioned achievements, we investigate the singular fractional $q-$integro-differential equation of the form

$$ {}^{C}\mathcal{D}_q^{\sigma}[k](t) = \Omega\left(t, k(t), k'(t), {}^{C}\mathcal{D}_q^{\zeta}[k](t), \int_0^t f(r)k(r)\,\mathrm{d}r\right), \tag{1}$$

for $t \in J = (0,1)$ under boundary conditions $k(0) = 0$ and

$$k(1) = {}^{C}\mathcal{D}_q^{\eta}[k](\tau),$$

where $k \in C^1(\overline{J})$, $n = [\eta] + 1$, $1 \leq \sigma < 2$, $\zeta, \eta, \tau \in J$, $f \in L^1(\overline{J})$ is non-negative with $\|f\|_1 = m$, $\Omega(t, k_1, k_2, k_3, k_4)$ is singular at some points of $t \in \overline{J} := [0,1]$ and ${}^{C}\mathcal{D}_q^{\sigma}$ is the Caputo fractional $q-$derivative of order σ. Existence of solutions is studied via multi-step methods.

The rest of the paper is organized as follows: Sec. 2 recalls some preliminary concepts and fundamental results of $q-$calculus. Secs. 3 and 4 are devoted to the main results and examples illustrating the obtained results and some algorithms for the addressed problem, respectively.

2 Essential Preliminaries

This section is devoted to starting some notations and essential preliminaries that are acting as necessary prerequisites for the results of the subsequent sections.

2.1 $q-$Fractional derivative and integral

Throughout this article, we shall apply the time scales calculus notations ([11]). In fact, we consider the fractional $q-$calculus on the specific time scale

$$\mathbb{T}_{t_0} = \{0\} \cup \{t : t = t_0 q^n\},$$

for $n \in \mathbb{N}$, $t_0 \in \mathbb{R}$ and $q \in (0,1)$. If there is no confusion concerning t_0 we shall denote $\mathbb{T}_{t_0}$ by $\mathbb{T}$. Let $a \in \mathbb{R}$. Define $[a]_q = (1-q^a)/(1-q)$ ([17]). The $q-$factorial function $(x-y)_q^{(n)}$ with $n \in \mathbb{N}_0$ is defined by

$$(x-y)_q^{(n)} = \prod_{k=0}^{n-1} (x - yq^k), \tag{2}$$

and $(x-y)_q^{(0)} = 1$, where x and y are real numbers and $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$ ([3]).

Algorithm 1: MATLAB lines for calculation $q-$factorial function $(x-y)_q^{(n)}$

```
function p = qfunction(x, y, q, n)
if n==0
    s=1;
else
    s=1;
    for k=0:n-1
        s = s*(x-y*q^k);
    end;
    p=s;
end;
end
```

Also, for $\sigma \in \mathbb{R}$ and $a \neq 0$, we have

$$(x-y)_q^{(\sigma)} = x^\sigma \prod_{k=0}^{\infty} \frac{x - yq^k}{x - yq^{\sigma+k}}. \tag{3}$$

Algorithms 1 and 2 simplify $q-$factorial functions $(x-y)_q^{(n)}$ and $(x-y)_q^{(\sigma)}$ respectively. In the paper [9], the authors proved

$$(x-y)_q^{(\sigma+\nu)} = (x-y)_q^{(\sigma)} (x - q^\sigma y)_q^{(\nu)},$$

and $(ax-ay)_q^{(\sigma)} = a^\sigma (x-y)_q^{(\sigma)}$.

Algorithm 2: MATLAB lines for calculation $q-$factorial function $(x-y)_q^{(\sigma)}$.

```
function p = qfunctionreal(x,y,q,sigma,n)
if n==0
    p=1;
else
    s=1;
    for k=0:n-1
        s = s*(x-y*q^k)/(x-y*q^(sigma+k));
    end;
    p=s*x^sigma;
end;
end
```

If $y=0$, then it is clear that $x^{(\sigma)} = x^\sigma$. The $q-$Gamma function is given by

$$\Gamma_q(z) = (1-q)^{1-z}(1-q)_q^{(z-1)},$$

where $z \in \mathbb{R}\backslash\{\cdots,-2,-1,0\}$ ([17]). In fact, by using (3), we have

$$\Gamma_q(z) = (1-q)^{1-z} \prod_{k=0}^{\infty} \frac{1-q^{k+1}}{1-q^{z+k-1}}. \tag{4}$$

Algorithm 3: MATLAB lines for calculation $\Gamma_q(x)$.

```
function p = qGamma(q,x,n)
s=1;
for k=0:n
    s=s*(1-q^(k+1))/(1-q^(x+k-1));
end;
p = s*(1-q)^(1-x);
end
```

Algorithm 3 shows the MATLAB lines for calculation of $\Gamma_q(x)$ which we tend n to infinity in it. Note that, $\Gamma_q(z+1) = [z]_q \Gamma_q(z)$ ([9, Lemma 1]). For a function $w : \mathbb{T} \to \mathbb{R}$, the $q-$derivative of w, is

$$\mathcal{D}_q[w](x) = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)_q w(x) = \frac{w(x)-w(qx)}{(1-q)x}, \tag{5}$$

for all $t \in \mathbb{T} \setminus \{0\}$, and ([3])

$$\mathcal{D}_q[w](0) = \lim_{x \to 0} \mathcal{D}_q[w](x).$$

Also, the higher order $q-$derivative of the function w is defined by

$$\mathcal{D}_q^n[w](x) = \mathcal{D}_q\left[\mathcal{D}_q^{n-1}[w]\right](x),$$

for all $n \geq 1$, where $\mathcal{D}_q^0[w](x) = w(x)$ ([3]). In fact

$$\mathcal{D}_q^n[w](x) = \frac{1}{x^n(1-q)^n} \sum_{k=0}^{n} \frac{(1-q^{-n})_q^{(k)}}{(1-q)_q^{(k)}} q^k w(xq^k), \tag{6}$$

for $x \in \mathbb{T} \setminus \{0\}$ ([8]).

Remark 2.1. By using Eq. (2), we can change Eq. (6) as follows:

$$\begin{aligned}
\mathcal{D}_q^n[w](x) &= \frac{1}{x^n(1-q)^n} \sum_{k=0}^{n} \frac{\prod_{i=0}^{k-1}\left(1-q^{i-n}\right)}{\prod_{i=0}^{k-1}\left(1-q^{i+1}\right)} q^k w(xq^k) \\
&= \frac{1}{x^n(1-q)^n} \sum_{k=0}^{n} \prod_{i=0}^{k-1} \frac{\left(1-q^{i-n}\right)}{\left(1-q^{i+1}\right)} q^k w(xq^k).
\end{aligned} \tag{7}$$

Algorithms 4 and 5 show the MATLAB codes for calculation of Eqs. (5) and (7) respectively.

Algorithm 4: MATLAB lines for calculation $\mathcal{D}_q[w](x)$.

```
function p = Dq(q,x,fun)
if x==0
    p=limit((subs(fun,x)-subs(fun,q*x))/((1-q)*x),x,0);
else
    p=(eval(subs(fun,x))-eval(subs(fun,q*x)))/((1-q)*x);
end;
end
```

Algorithm 5: MATLAB lines for calculation $\mathcal{D}_q^n[w](x)$.

```
function g = Dqnatural(q,x,n,fun)
s=0;
for k=0:n
    p=1;
    for i=0:k-1
        p=p*(1-q^(i-n))/(1-q^(i+1));
    end;
    p=p*q^k*eval(subs(fun,x*q^k));
    s=s+p;
end;
g=s/(x^n*(1-q)^n);
end
```

The $q-$integral of the function w is defined by

$$\mathcal{I}_q[w](x) = \int_0^x w(s)\,\mathrm{d}_q s = x(1-q)\sum_{k=0}^{\infty} q^k w(xq^k), \tag{8}$$

for $0 \leq x \leq b$, provided the series is absolutely converges ([3]).

Algorithm 6: MATLAB lines for calculation $\mathcal{I}_q[w](t)$.

```
function p = Iq(q,x,n,fun)
s=1;
for k=0:n
    s=s+q^k*eval(subs(fun,x*q^k));
end;
p=x*(1-q)*s;
end
```

By using the Algorithm 6, we can obtain the numerical results of $\mathcal{I}_q[w](x)$ when $n \to \infty$. If a in $[0, b]$, then

$$\begin{aligned}\int_a^b w(s)\,\mathrm{d}_q s &= \mathcal{I}_q[w](b) - \mathcal{I}_q[w](a)\\ &= (1-q)\sum_{k=0}^{\infty} q^k \left[bw(bq^k) - aw(aq^k)\right], \end{aligned}\tag{9}$$

whenever the series exists. The operator $\mathcal{I}_q^n$ is given by $\mathcal{I}_q^0[w](x) = w(x)$ and

$$\mathcal{I}_q^n[w](x) = \mathcal{I}_q\left[\mathcal{I}_q^{n-1}[w]\right](x),$$

for $n \geq 1$ and $g \in C([0,b])$ ([3]). It has been proved that

$$\mathcal{D}_q\left[\mathcal{I}_q[w]\right](x) = w(x), \qquad \mathcal{I}_q\left[\mathcal{D}_q[w]\right](x) = w(x) - w(0),$$

whenever the function w is continuous at $x = 0$ ([3]). The fractional Riemann–Liouville type $q-$integral of the function w is defined by

$$\mathcal{I}_q^\sigma[w](t) = \frac{1}{\Gamma_q(\sigma)}\int_0^t (t-s)_q^{(\sigma-1)} w(s)\,\mathrm{d}_q s, \qquad \mathcal{I}_q^0[w](t) = w(t), \tag{10}$$

for $t \in [0,1]$ and $\sigma > 0$ ([8, 13]).

Remark 2.2. By using Eqs. (3), (4) and (8), we obtain

$$\begin{aligned}
&\frac{1}{\Gamma_q(\sigma)}\int_0^t (t-s)_q^{(\sigma-1)} w(s)\,\mathrm{d}_q s\\
&\quad= \frac{1}{\Gamma_q(\sigma)}\int_0^t t^{\sigma-1}\prod_{i=0}^{\infty}\frac{t-sq^i}{t-sq^{\sigma+i-1}} w(s)\,\mathrm{d}_q s\\
&\quad= \frac{t^{\sigma-1}}{\Gamma_q(\sigma)}\int_0^t \prod_{i=0}^{\infty}\frac{t-sq^i}{t-sq^{\sigma+i-1}} w(s)\,\mathrm{d}_q s\\
&\quad= \frac{t^\sigma(1-q)}{\Gamma_q(\sigma)}\sum_{k=0}^{\infty} q^k \prod_{i=0}^{\infty}\frac{t-(tq^k)q^i}{t-(tq^k)q^{\sigma+i-1}} w(tq^k)\\
&\quad= \frac{t^\sigma(1-q)}{\Gamma_q(\sigma)}\sum_{k=0}^{\infty} q^k \prod_{i=0}^{\infty}\frac{1-q^{k+i}}{1-q^{\sigma+k+i-1}} w(tq^k)\\
&\quad= t^\sigma(1-q)^\sigma \prod_{i=0}^{\infty}\frac{1-q^{\sigma+i-1}}{1-q^{i+1}}\sum_{k=0}^{\infty} q^k \prod_{i=0}^{\infty}\frac{1-q^{k+i}}{1-q^{\sigma+k+i-1}} w(tq^k).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathcal{I}_q^\sigma[w](t) &= t^\sigma(1-q)^\sigma\\
&\quad\times \lim_{n\to\infty}\sum_{k=0}^{n} q^k \prod_{i=0}^{n}\frac{\left(1-q^{\sigma+i-1}\right)\left(1-q^{k+i}\right)}{\left(1-q^{i+1}\right)\left(1-q^{\sigma+k+i-1}\right)} w(tq^k),
\end{aligned} \tag{11}$$

Algorithm 7 shows the MATLAB codes of numerical technique.

Algorithm 7: MATLAB lines for calculation $I_q^\sigma[w](x)$

```
function g = Iq_sigma(q,sigma,t,n,fun)
p=0;
for k=0:n
    s=1;
    for i=0:n
        s=s*(1-q^(sigma+i-1))*(1-q^(k+i))/((1-q^(i+1)) ...
         *(1-q^(sigma+k+i-1)));
    end
    p=p+q^k*s*eval(subs(fun,t*q^k));
end;
g=round(p*(t^sigma)*(1-q)^sigma,6);
end
```

The Caputo fractional $q-$derivative of the function w is defined by

$$\begin{aligned} {}^C\mathcal{D}_q^\sigma[w](t) &= \mathcal{I}_q^{[\sigma]-\sigma}\left[\mathcal{D}_q^{[\sigma]}[w]\right](t) \\ &= \frac{1}{\Gamma_q([\sigma]-\sigma)}\int_0^t (t-s)_q^{([\sigma]-\sigma-1)}\,\mathcal{D}_q^{[\sigma]}[w](s)\,\mathrm{d}_q s \end{aligned} \tag{12}$$

for $t \in [0,1]$ and $\sigma > 0$ ([13, 24]). It has been proved that

$$\mathcal{I}_q^\nu\left[\mathcal{I}_q^\sigma[w]\right](t) = \mathcal{I}_q^{\sigma+\nu}[w](t), \qquad {}^c\mathcal{D}_q^\sigma\left[\mathcal{I}_q^\sigma[w]\right](t) = w(t),$$

where $\sigma, \nu \geq 0$ ([13]). Also,

$$\mathcal{I}_q^\sigma\left[\mathcal{D}_q^n[w]\right](t) = \mathcal{D}_q^n\left[\mathcal{I}_q^\sigma[w]\right](t) - \sum_{k=0}^{n-1}\frac{t^{\sigma+k-n}}{\Gamma_q(\sigma+k-n+1)}\mathcal{D}_q^k[w](0),$$

where $\sigma > 0$ and $n \geq 1$ ([13]).

Remark 2.3. From Eq.(4), Remark 2.1 and Eq. (11) in Remark 2.2, we obtain

$$
\begin{aligned}
&\frac{1}{\Gamma_q\left([\sigma]-\sigma\right)}\int_0^t (t-s)_q^{([\sigma]-\sigma-1)}\,\mathcal{D}_q^{[\sigma]}[w](s)\,\mathrm{d}_q s\\
&\quad= \frac{1}{\Gamma_q\left([\sigma]-\sigma\right)}\int_0^t t^{[\sigma]-\sigma-1}\left[\prod_{i=0}^{\infty}\frac{t-sq^i}{t-sq^{[\sigma]-\sigma-1+i}}\right]\\
&\qquad\times\left(\frac{1}{t^{[\sigma]}(1-q)^{[\sigma]}}\sum_{k=0}^{[\sigma]}\left[\prod_{i=0}^{k-1}\frac{\left(1-q^{i-[\sigma]}\right)}{(1-q^{i+1})}\right]q^k w(xq^k)\right)\mathrm{d}_q s\\
&\quad= \frac{t^{-\sigma-1}}{\Gamma_q\left([\sigma]-\sigma\right)}\int_0^t \left[\prod_{i=0}^{\infty}\frac{t-sq^i}{t-sq^{[\sigma]-\sigma-1+i}}\right]\\
&\qquad\times\left(\sum_{k=0}^{[\sigma]}\left[\prod_{i=0}^{k-1}\frac{\left(1-q^{i-[\sigma]}\right)}{(1-q^{i+1})}\right]q^k w(xq^k)\right)\mathrm{d}_q s\\
&\quad= \frac{t^{-\sigma}(1-q)}{\Gamma_q\left([\sigma]-\sigma\right)}\sum_{k=0}^{\infty}\left(\left[\prod_{i=0}^{\infty}\frac{t-(tq^k)q^i}{t-(tq^k)q^{[\sigma]-\sigma-1+i}}\right]\right.\\
&\qquad\times\left.\left(\sum_{k=0}^{[\sigma]}\left[\prod_{i=0}^{k-1}\frac{\left(1-q^{i-[\sigma]}\right)}{(1-q^{i+1})}\right]q^k w\left((tq^k)q^k\right)\right)\right)\\
&\quad= \frac{t^{-\sigma}(1-q)}{\Gamma_q\left([\sigma]-\sigma\right)}\sum_{k=0}^{\infty}\left(\left[\prod_{i=0}^{\infty}\frac{1-q^{k+i}}{1-1q^{[\sigma]-\sigma-1+k+i}}\right]\right.\\
&\qquad\times\left.\left(\sum_{m=0}^{[\sigma]}\left[\prod_{i=0}^{m-1}\frac{\left(1-q^{i-[\sigma]}\right)}{(1-q^{i+1})}\right]q^m w\left(tq^{k+m}\right)\right)\right)\\
&\quad= t^{-\sigma}(1-q)^{[\sigma]-\sigma}\left[\prod_{i=0}^{\infty}\frac{1-q^{[\sigma]-\sigma+i-1}}{1-q^{i+1}}\right]\\
&\qquad\times\sum_{k=0}^{\infty}\left(\left[\prod_{i=0}^{\infty}\frac{1-q^{k+i}}{1-1q^{[\sigma]-\sigma-1+k+i}}\right]\right.\\
&\qquad\times\left.\left(\sum_{m=0}^{[\sigma]}\left[\prod_{i=0}^{m-1}\frac{\left(1-q^{i-[\sigma]}\right)}{(1-q^{i+1})}\right]q^m w\left(tq^{k+m}\right)\right)\right)\\
&\quad= \frac{1}{t^{\sigma}(1-q)^{\sigma-[\sigma]}}\sum_{k=0}^{\infty}\left(\left[\prod_{i=0}^{\infty}\frac{(1-q^{[\sigma]-\sigma+i-1})(1-q^{k+i})}{(1-q^{i+1})(1-q^{[\sigma]-\sigma-1+k+i})}\right]\right.
\end{aligned}
$$

$$\times\left(\sum_{m=0}^{[\sigma]}\left[\prod_{i=0}^{m-1}\frac{(1-q^{i-[\sigma]})}{(1-q^{i+1})}\right]q^m w\left(tq^{k+m}\right)\right)\bigg).$$

Thus, we have

$$\begin{aligned}{}^{c}\mathcal{D}_q^{\sigma}[w](t) &= \frac{1}{t^{\sigma}(1-q)^{\sigma-[\sigma]}} \\ &\times \lim_{n\to\infty}\sum_{k=0}^{n}\left(\left[\prod_{i=0}^{n}\frac{(1-q^{[\sigma]-\sigma+i-1})(1-q^{k+i})}{(1-q^{i+1})(1-q^{[\sigma]-\sigma-1+k+i})}\right]\right. \\ &\times\left.\left(\sum_{m=0}^{[\sigma]}\left[\prod_{i=0}^{m-1}\frac{(1-q^{i-[\sigma]})}{(1-q^{i+1})}\right]q^m w\left(tq^{k+m}\right)\right)\right). \end{aligned} \tag{13}$$

Algorithm 8 shows the MATLAB codes of numerical technique.

Algorithm 8: MATLAB lines for calculation ${}^{C}\mathcal{D}_q^{\sigma}[w](t)$.

```
function g = IqCaputo_sigma(q,sigma,t,n,fun)
S=0;
for k=0:n
    p1=1;
    for i=0:n
        p1=p1*(1-q^(floor(sigma)-sigma+i-1)) ...
        *(1-q^(k+i))/((1-q^(i+1)) ...
        *(1-q^(floor(sigma)-sigma+k+i-1)));
    end;
    s2=0;
    for m=0:floor(sigma)
        p2=1;
        for i=0:m-1
            p2=p2*(1-q^(i-floor(sigma)))/(1-q^(i+1));
        end;
        p2=p2*q^m*eval(subs(fun,t*q^(k+m)));
        s2=s2+p2;
    end;
    S=S+p1*s2;
end;
g=round(S/( t^sigma*(1-q)^(sigma-floor(sigma))),6);
end
```

Throughout this article, we consider

$$\|k\|_1 := \int_0^1 |k(t)|\,\mathrm{d}t,$$

$$\|k\| := \sup\Big\{|k(t)| : t \in J\Big\},$$

$\|k\|_* := \max\{\|k\|, \|k'\|\}$, as the norm of $\mathcal{L} = L^1(J)$, $\mathcal{A} = C(J)$ and $\mathcal{B} = C^1(J)$, respectively.
The following lemmas are used in the subsequent sections.

Lemma 2.4. *([32]) Suppose that* $0 < n - 1 \le \sigma < n$ *and* $k \in \mathcal{A} \cap \mathcal{L}$*. Then*

$$\mathcal{I}_q^{\sigma}\left[{}^{C}\mathcal{D}_q^{\sigma}[k]\right](t) = k(t) + \sum_{i=0}^{n-1} c_i t^i,$$

for some constants $c_0, \dots, c_{n-1} \in \mathbb{R}$*.*

Lemma 2.5. *([33]) If* $\mathcal{C}$ *is a closed, bounded and convex subset of a Banach space* $\mathcal{X}$ *and* $\Phi : \mathcal{C} \to \mathcal{C}$ *is completely continuous, then* Φ *has a fixed point in* $\mathcal{C}$*.*

Lemma 2.6. *([35]) Let* $\mathcal{X}$ *be a Banach space,* $\mathcal{C}$ *a closed and convex subset of* $\mathcal{X}$*,* $\mathcal{O}$ *a relatively open subset of* $\mathcal{C}$ *with* $0 \in \mathcal{O}$ *and* $\Omega : \mathcal{O} \to \mathcal{C}$ *a continuous and compact map. Then either* Ω *has a fixed point in* $\bar{\mathcal{O}}$ *or there exist* $a \in \partial\mathcal{O}$ *and* $\lambda \in (0, 1)$ *such that* $a = \lambda\Omega(a)$*.*

2.2 Linear multi-step methods

As in the case of ordinary differential equations, linear multi-step methods for fractional differential equations makes use of approximations of values of $k_1(t)$, $k_2(t)$, $k_3(t)$, $k_4(t)$ and $\Omega(t, k_1(t), k_2(t), k_3(t), k_4(t))$ on some points of a partition $s_0 < s_1 < \cdots < s_n$ ([14, 15]). We can therefore write linear multi-step methods for the solution of (1) in the form

$$\sum_{j=0}^{n} {}_1\alpha_j \left({}_{n-j}k_1, {}_{n-j}k_2, {}_{n-j}k_3, {}_{n-j}k_4\right) = h^{\tau} \sum_{j=0}^{n} {}_2\alpha_j \Omega\left(s_{n-j}, {}_{n-j}k_1, {}_{n-j}k_2, {}_{n-j}k_3, {}_{n-j}k_4\right), \qquad (14)$$

where ${}_1\alpha_j$ and ${}_2\alpha_j$ are real parameters and we will indicate with ${}_1\rho_n(\xi)$ and ${}_2\rho_n(\xi)$ the generating polynomials

$$ {}_k\rho_n(\xi) = \sum_{j=0}^{n} {}_k\alpha_j \xi^{n-j}. $$

Numerical methods (14) are requested to be consistent with the original problem (1), in the sense that, as $h \to 0$, the discretized problem is expected to tend asymptotically to the continuous one ([15]). In order to formally introduce the consistency concept and study order conditions, it is usually to introduce, associated to (14), the linear difference operator

$$ \begin{aligned} &\mathscr{L}_h\Big[\left(z_1(t), z_2(t), z_3(t), z_4(t)\right), t, \tau\Big] \\ &\quad = \sum_{j=0}^{n} {}_1\alpha_j \Big({}_{n-j}z_1(t-hj), {}_{n-j}z_2(t-hj), {}_{n-j}z_3(t-hj), {}_{n-j}z_4(t-hj)\Big) \\ &\qquad - h^{\tau} \sum_{j=0}^{n} {}_2\alpha_j {}^{C}\mathcal{D}_q^{\tau}\Big[{}_{n-j}z_1, {}_{n-j}z_2, {}_{n-j}z_3, {}_{n-j}z_4\Big](t-hj), \end{aligned} $$

where $(z_1(t), z_2(t), z_3(t), z_4(t))$ is a sufficiently smooth function ([15]). The linear multi-step method (14) is said to be consistent if, for any initial value problem (1), with exact solution $(k_1(t), k_2(t), k_3(t), k_4(t))$, it holds

$$ \lim_{h\to 0} \frac{1}{h^{\tau}} \mathscr{L}_h\Big[\left({}_{n-j}k_1(t), {}_{n-j}k_2(t), {}_{n-j}k_3(t), {}_{n-j}k_4(t)\right), t, \tau\Big] = (0,0,0,0), $$

with h and n related by $t = s_0 + hn$. Moreover the method is said to be of order ℓ if

$$ \frac{1}{h^{\tau}} \mathscr{L}_h\Big[\left({}_{n-j}k_1(t), {}_{n-j}k_2(t), {}_{n-j}k_3(t), {}_{n-j}k_4(t)\right), t, \tau\Big] = O(h^{\ell}), $$

as h tend to zero. Under the assumption that

$$ \left(k_1(t), k_2(t), k_3(t), k_4(t)\right), $$

is $(m+1)$–times differentiable, $t = s_n$, we can expand the true solution

$$ \begin{aligned} \big(k_1(t-jh), &k_2(t-jh), k_3(t-jh), k_4(t-jh)\big) \\ &= \big(k_1(s_0+(n-j)h), k_2(s_0+(n-j)h), k_3(s_0+(n-j)h), \\ &\quad k_4(s_0+(n-j)h)\big), \end{aligned} $$

of (1) as

$$\begin{aligned}\big(k_1(t-jh),&k_2(t-jh),k_3(t-jh),k_4(t-jh)\big)\\ &=\big(k_1(s_0),k_2(s_0),k_3(s_0),k_4(s_0)\big)\\ &\quad+\sum_{d=1}^{m}\frac{(n-j)^d h^d}{d!}\Big(k_1^d(s_0),k_2^d(s_0),k_3^d(s_0),k_4^d(s_0)\Big)\\ &\quad+\frac{h^{m+1}}{d!}\int_0^{n-j}(n-j-\xi)_q^{(m)}\Big(k_1^{m+1}(s_0+h\xi),\\ &\quad k_2^{m+1}(s_0+h\xi),k_3^{m+1}(s_0+h\xi),k_4^{m+1}(s_0+h\xi)\Big)\,\mathrm{d}_q\xi,\end{aligned}$$

and its τ-fractional $q-$derivative as

$$\begin{aligned}&{}^{C}\mathcal{D}_q^{\tau}\big[{}_{n-j}z_1,{}_{n-j}z_2,{}_{n-j}z_3,{}_{n-j}z_4\big](t-hj)\\ &=\sum_{d=1}^{m}\frac{h^{d-\tau}(n-j)^{d-\tau}}{\Gamma_q(d+1-\tau)}\Big(k_1^d(s_0),k_2^d(s_0),k_3^d(s_0),k_4^d(s_0)\Big)\\ &\quad+\frac{h^{m+1-\tau}}{\Gamma_q(m+1-\tau)}\int_0^{n-j}(n-j-\xi)_q^{(m-\tau)}\big(k_1^{m+1}(s_0+h\xi),\\ &\quad k_2^{m+1}(s_0+h\xi),k_3^{m+1}(s_0+h\xi),k_4^{m+1}(s_0+h\xi)\big)\,\mathrm{d}_q\xi.\end{aligned}$$

In this way, we can write the difference operator

$$\mathscr{L}_h\Big[\,(k_1(t),k_2(t),z_3(t),z_4(t))\,,t,\tau\Big],$$

as

$$\begin{aligned}&\mathscr{L}_h\Big[\big(k_1(t),k_2(t),z_3(t),z_4(t)\big),t,\tau\Big]\\ &\quad=C_0(n,\tau)+\sum_{d=1}^{m}h^dC_d(n,\tau)\Big(k_1^d(s_0),k_2^d(s_0),z_3^d(s_0),z_4^d(s_0)\Big)\\ &\qquad+h^{m+1}R_{m+1},\end{aligned}$$

where the remainder R_{m+1} is obtained from the Taylor's expansions and

$$\begin{aligned}C_0(n,\tau)&=\sum_{j=0}^{n}{}_1\alpha_j\\ C_d(n,\tau)&=\frac{1}{d!}\sum_{j=0}^{n}(n-j)^d{}_1\alpha_j-\frac{1}{\Gamma_q(d+1-\tau)}\sum_{j=0}^{n}{}_2\alpha_j(n-j)^{d-\tau},\end{aligned}$$

for $d = 1, 2, \cdots, m$.

3 Main results

We employ the multi-step methods to prove the main results in this section. First, we adopt the following lemma.

Lemma 3.1. *Let $z \in \mathcal{L}$. The unique solution of problem*

$$ {}^{C}\mathcal{D}_q^{\sigma}[k](t) + z(t) = 0, \tag{15} $$

with boundary conditions $k(0) = 0$ and $k(1) = {}^{C}\mathcal{D}_q^{\eta}[k](\tau)$, here $1 \leq \sigma < 2$ and $\eta, \tau \in J$, is

$$ k_0(t) = \int_0^1 G_q(t, s) z(s) \, \mathrm{d}_q s, $$

where

$$ G_q(t,s) = \begin{cases} \dfrac{t(1-s)_q^{(\sigma-1)}}{\lambda \Gamma_q(\sigma)}, & t \leq s, \tau \leq s, \\ \dfrac{t(1-s)_q^{(\sigma-1)}}{\lambda \Gamma_q(\sigma)} - \dfrac{(t-s)_q^{(\sigma-1)}}{\Gamma_q(\sigma)}, & \tau \leq s \leq t, \\ \dfrac{t(1-s)_q^{(\sigma-1)}}{\lambda \Gamma_q(\sigma)} - \dfrac{t(\tau-s)_q^{(\sigma-\eta-1)}}{\lambda \Gamma_q(\sigma-\eta)}, & t \leq s \leq \tau, \\ \dfrac{t(1-s)_q^{(\sigma-1)}}{\lambda \Gamma_q(\sigma)} - \dfrac{t(\tau-s)_q^{(\sigma-\eta-1)}}{\lambda \Gamma_q(\sigma-\eta)} - \dfrac{\lambda(t-s)_q^{(\sigma-1)}}{\lambda \Gamma_q(\sigma)}, & s \leq t, s \leq \tau, \end{cases} \tag{16} $$

for $t, s \in \overline{J}$ where

$$ \lambda = 1 - \frac{\tau^{1-\eta}}{\Gamma_q(2-\eta)} \neq 0. \tag{17} $$

Proof. Let k be a solution for the problem (15). By using Lemma 2.4, we get $k(t) = -\mathcal{I}_q^{\sigma}[z](t) + d_1 t + d_0$. By utilizing the boundary conditions, we conclude $d_0 = 0$. Hence,

$$ {}^{C}\mathcal{D}_q^{\eta}[k](\tau) = -\mathcal{I}_q^{\sigma-\eta}[z](\tau) + d_1 \frac{\tau^{1-\eta}}{\Gamma_q(2-\eta)}, $$

and $k(1) = -\mathcal{I}_q^{\sigma}[z](1) + d_1$. Since $k(1) = {}^{C}\mathcal{D}_q^{\eta}[k](\tau)$, we conclude that

$$ d_1 \left[1 - \frac{\tau^{1-\eta}}{\Gamma_q(2-\eta)} \right] = \mathcal{I}_q^{\sigma}[z](1) - \mathcal{I}_q^{\sigma-\eta}[z](\tau), $$

and so

$$d_1 = \frac{1}{\lambda}\big[\mathcal{I}_q^{\sigma}[z](1) - \mathcal{I}_q^{\sigma-\eta}[(\tau)]\big].$$

Thus, we have

$$k(t) = -\mathcal{I}_q^{\sigma}[k](t) + \frac{t}{\lambda}\Big[\mathcal{I}_q^{\sigma}[z](1) - \mathcal{I}_q^{\sigma-\eta}[z](\tau)\Big].$$

Therefore, we have two cases.

1) If $t \leq \tau$, then we can see that

$$\begin{aligned}
k(t) &= -\mathcal{I}_q^{\sigma}[z](t) + \frac{t}{\lambda}\mathcal{I}_q^{\sigma}[z](t) + \frac{t}{\lambda\Gamma_q(\sigma)}\bigg[\int_t^{\tau}(1-s)_q^{(\sigma-1)}z(s)\,\mathrm{d}_q s \\
&\quad + \int_{\tau}^{1}(1-s)_q^{\sigma-1}z(s)\,\mathrm{d}_q s\bigg] - \frac{t}{\lambda}\bigg[\mathcal{I}_q^{\sigma-\eta}[z](t) \\
&\quad + \frac{1}{\Gamma_q(\sigma-\eta)}\int_t^{\tau}(\tau-s)_q^{(\sigma-\eta-1)}z(s)\,\mathrm{d}_q s\bigg] \\
&= \int_0^t\left[\frac{t(1-s)_q^{(\sigma-1)}}{\lambda\Gamma_q(\sigma)} - \frac{(t-s)_q^{(\sigma-1)}}{\Gamma_q(\sigma)} - \frac{t(\tau-s)_q^{(\sigma-\eta-1)}}{\Gamma_q(\sigma-\mu)}\right]z(s)\,\mathrm{d}_q s \\
&\quad + \int_t^{\tau}\left[\frac{t(1-s)_q^{(\sigma-1)}}{\lambda\Gamma_q(\sigma)} - \frac{t(\tau-s)_q^{(\sigma-\eta-1)}}{\lambda\Gamma_q(\sigma-\eta)}\right]z(s)\,\mathrm{d}_q s \\
&\quad + \int_{\tau}^{1}\frac{t(1-s)_q^{(\sigma-1)}}{\lambda\Gamma_q(\sigma)}z(s)\,\mathrm{d}_q s.
\end{aligned}$$

2) If $t \geq \tau$, then we can see that

$$\begin{aligned}
k(t) &= -\mathcal{I}_q^{\sigma}[z](\tau) - \int_{\tau}^{t}\frac{(t-s)_q^{(\sigma-1)}}{\Gamma_q(\sigma)}z(s)\,\mathrm{d}_q s + \frac{t}{\lambda}\mathcal{I}_q^{\sigma}[z](\tau) \\
&\quad + \int_{\tau}^{t}\frac{t(1-s)_q^{(\sigma-1)}}{\lambda\Gamma_q(\sigma)}z(s)\,\mathrm{d}_q s \\
&\quad + \int_{t}^{1}\frac{t(1-s)_q^{(\sigma-1)}}{\lambda\Gamma_q(\sigma)}z(s)\,\mathrm{d}_q s - \frac{t}{\lambda}\mathcal{I}_q^{\sigma-\eta}[z](\tau) \\
&= \int_0^{\tau}\left[\frac{t(1-s)_q^{(\sigma-1)}}{\lambda\Gamma_q(\sigma)} - \frac{(t-s)_q^{(\sigma-1)}}{\Gamma_q(\sigma)} - \frac{t(\tau-s)_q^{(\sigma-\eta-1)}}{\lambda\Gamma_q(\sigma-\eta)}\right]z(s)\,\mathrm{d}_q s
\end{aligned}$$

$$+\int_\tau^t \left[\frac{t(1-s)_q^{(\sigma-1)}}{\lambda\Gamma_q(\sigma)} - \frac{\lambda(t-s)_q^{(\sigma-1)}}{\lambda\Gamma_q(\sigma)}\right] z(s)\,\mathrm{d}_q s$$
$$+\int_t^1 \frac{t(1-s)_q^{(\sigma-1)}}{\lambda\Gamma_q(\sigma)} z(s)\,\mathrm{d}_q s.$$

This implies that, $k(t) = \int_0^1 G_q(t,s)z(s)\,\mathrm{d}_q s = k_0(t)$ for all t. □

Remark 3.2. If $k \in \mathcal{B}$, then

$${}^C\mathcal{D}_q^\beta[k](t) = \frac{1}{\Gamma_q(1-\beta)} \int_0^t (t-s)_q^{-\beta} k'(s)\mathrm{d}_q s$$

and so

$$\left|{}^C\mathcal{D}_q^\beta[k](t)\right| \le \frac{\|k'\|}{\Gamma(1-\beta)} \int_0^t (t-qs)^{-\beta}\,\mathrm{d}_q s = \frac{\|k'\|}{\Gamma(2-\beta)} t^{1-\beta}.$$

Thus, ${}^C\mathcal{D}_q^\beta[k] \in \mathcal{A}$ and

$$\left|{}^C\mathcal{D}_q^\beta[k]\right| \le \frac{\|k'\|}{\Gamma_q(2-\beta)}.$$

Since $\int_0^1 f(r)\,\mathrm{d}r = m \in (0,\infty)$,

$$\left|\int_0^t f(r)k(r)\,\mathrm{d}r\right| \le \|k\| \int_0^t f(r)\,\mathrm{d}r \le m\|k\|.$$

Now, we give our main result.

Theorem 3.3. *The singular problem* (1) *has a solution whenever the following assumptions hold.*

1) *There exist the maps $f_i : J \to \mathbb{R}$ with $\int_0^1 f_i(r)\,\mathrm{d}r < \infty$ for all $i = 1,2,3,4$ such that*

$$\left|\Omega(t,k_1,k_2,k_3,k_4) - \Omega(t,l_1,l_2,l_3,l_4)\right| \le \sum_{i=1}^4 f_i(t)\left|k_i(t) - l_i(t)\right|,$$

for all (k_1,k_2,k_3,k_4), $(l_1,l_2,l_3,l_4) \in \mathbb{R}^4$ and $t \in J$.

2) There exist $g \in \mathcal{L}$ and $\Theta \in \mathcal{A}^4$ such that

$$|\Omega(t, k_1, k_2, k_3, k_4)| \leq g(t)\Theta(k_1, k_2, k_3, k_4),$$

for each $(k_1, k_2, k_3, k_4) \in \mathbb{R}^4$, almost all $t \in J$. Also

$$\|\Theta\|_{\mathcal{A}} = \sup\Big\{ |\Theta(k_1, k_2, k_3, k_4)| \, : (k_1, k_2, k_3, k_4) \in \mathbb{R}^4 \Big\} < \infty. \tag{18}$$

Proof. We first define a map $T : \mathcal{B} \to \mathcal{B}$ by

$$\begin{aligned} T_k(t) = \int_0^1 G_q(t, s)\tilde{\Omega}(k, s) \, \mathrm{d}_q s &= -\mathcal{I}_q^{\sigma}[\tilde{\Omega}](k, t) \\ &+ \frac{t}{\lambda}\big[\mathcal{I}_q^{\sigma}[\tilde{\Omega}](k, 1) - \mathcal{I}_q^{\sigma-\eta}[\tilde{\Omega}](k, \tau)\big], \end{aligned} \tag{19}$$

for each $k \in \mathcal{B}$ and $t \in J$ where

$$\tilde{\Omega}(z, \bar{t}) = \Omega\bigg(\bar{t}, z(\bar{t}), z'(\bar{t}), {}^{C}\mathcal{D}_q^{\zeta}[z](\bar{t}), \int_0^{\bar{t}} f(r)z(r) \, \mathrm{d}r\bigg).$$

Suppose that $k_1, k_2 \in \mathcal{B}$. Then we have

$$\begin{aligned} |T_{k_1}(t) - T_{k_2}(t)| \leq{}& \mathcal{I}_q^{\sigma}\left[\left|\tilde{\Omega}(k_1, s) - \tilde{\Omega}(k_2, s)\right|\right] \\ &+ \frac{t}{\lambda}\mathcal{I}_q^{\sigma}\left[\left|\tilde{\Omega}(k_1, 1) - \tilde{\Omega}(k_2, 1)\right|\right] \\ &+ \frac{t}{\lambda}\mathcal{I}_q^{\sigma-\eta}\left[\left|\tilde{\Omega}(k_1, \tau) - \tilde{\Omega}(k_2, \tau)\right|\right] \\ \leq{}& \mathcal{I}_q^{\sigma}\bigg[f_1(t)\big|k_1(t) - k_2(t)\big| + f_2(t)\big|k_1'(t) - k_2'(t)\big| \\ &+ f_3(t)\left|{}^{C}\mathcal{D}_q^{\zeta}[k_1](t) - {}^{C}\mathcal{D}_q^{\zeta}[k_2](t)\right| \\ &+ f_4(t)\left|\int_0^t f(r)(k_1(r) - k_2(r)) \, \mathrm{d}r\right|\bigg] \\ &+ \frac{t}{\lambda}\bigg[\mathcal{I}_q^{\sigma}\bigg[f_1(1) \, |k_1(1) - k_2(1)| + f_2(1) \, \big|k_1'(1) - k_2'(1)\big| \\ &+ f_3(1)\left|{}^{C}\mathcal{D}_q^{\zeta}[k_1](1) - {}^{C}\mathcal{D}_q^{\zeta}[k_2](1)\right| \end{aligned}$$

$$
\begin{aligned}
&\quad + f_4(1)\bigg| \int_0^1 f(r)(k_1(r) - k_2(r))\, \mathrm{d}r \bigg|\bigg] \\
&\quad + \mathcal{I}_q^{\sigma-\eta}\bigg[f_1(\tau)\,|k_1(\tau) - k_2(\tau)| + f_2(\tau)\,|k_1'(\tau) - k_2'(\tau)| \\
&\quad + f_3(\tau)\bigg|{}^{C}\mathcal{D}_q^{\zeta}[k_1](\tau) - {}^{C}\mathcal{D}_q^{\beta} k_2(\tau)\bigg| \\
&\quad + f_4(\tau)\bigg| \int_0^1 f(r)(k_1(r) - k_2(r))\, \mathrm{d}r \bigg|\bigg]\bigg] \\
&\leq \|k_1 - k_2\| \mathcal{I}_q^{\alpha}(f_1(t) + m f_4(t)) \\
&\quad + \|k_1' - k_2'\| \mathcal{I}_q^{\sigma}\left[f_2(t) + \frac{f_3(t)}{\Gamma_q(2-\zeta)} \right] \\
&\quad + \|k_1 - k_2\| \frac{t}{\lambda} \mathcal{I}_q^{\sigma}[f_1(1) + m f_4(1)] \\
&\quad + \|k_1' - k_2'\| \frac{t}{\lambda} \mathcal{I}_q^{\sigma}\left[f_2(1) + \frac{f_3(1)}{\Gamma_q(2-\zeta)} \right] \\
&\quad + \|k_1 - k_2\| \frac{t}{\lambda} \mathcal{I}_q^{\sigma-\eta}[f_1 + m f_4](\tau) \\
&\quad + \|k_1' - k_2'\| \mathcal{I}_q^{\sigma-\eta}\left(f_2(\tau) + \frac{f_3(\tau)}{\Gamma_q(2-\zeta)} \right) \\
&\leq \|k_1 - k_2\| \int_0^1 (1-s)_q^{(\sigma-\eta-1)} \\
&\quad \times \left[\frac{2f_1(s) + 2m f_4(s)}{\Gamma_q(\sigma)} + \frac{f_1(s) + m f_4(s)}{\lambda \Gamma_q(\sigma-\eta)} \right] \mathrm{d}_q s \\
&\quad + \|k_1' - k_2'\| \int_0^1 (1-s)_q^{(\sigma-\eta-1)} \left[\frac{2f_2(s)}{\Gamma_q(\sigma)} + \frac{f_2(s)}{\lambda \Gamma_q(\sigma-\eta)} \right. \\
&\quad \left. + \frac{2f_3(s)}{\Gamma_q(\sigma)\Gamma_q(2-\zeta)} + \frac{f_3(s)}{\lambda \Gamma_q(\sigma-\eta)\Gamma_q(2-\zeta)} \right] \mathrm{d}_q s \\
&\leq \Lambda_1 \left(\|k_1 - k_2\| + \|k_1' - k_2'\| \right) = \Lambda_1 \|k_1 - k_2\|_*,
\end{aligned}
$$

where

$$
\begin{aligned}
\Lambda_1 = \max\bigg\{ &\int_0^1 (1-s)_q^{\sigma-\eta-1} \bigg(\frac{2f_1(s) + 2m f_4(s)}{\Gamma_q(\sigma)} \\
&+ \frac{f_1(s) + m f_4(s)}{\lambda \Gamma_q(\sigma-\eta)} \bigg) \mathrm{d}_q s, \int_0^1 (1-s)_q^{(\sigma-\eta-1)} \bigg(\frac{2f_2(s)}{\Gamma_q(\sigma)}
\end{aligned}
$$

$$+ \frac{f_2(s)}{\lambda\Gamma_q(\sigma-\eta)} + \frac{2f_3(s)}{\Gamma_q(\sigma)\Gamma_q(2-\zeta)}$$
$$+ \frac{f_3(s)}{\lambda\Gamma_q(\sigma-\eta)\Gamma_q(2-\zeta)}\Bigg)\mathrm{d}_q s\Bigg\} < \infty.$$

On the other hand, we get

$$\begin{aligned}
\left|T'_{k_1}(t) - T'_{k_2}(t)\right| &\leq \int_0^1 \frac{\partial G_q(t,s)}{\partial t} |\tilde{\Omega}(k_1,s) - \tilde{\Omega}(k_2,s)|\,\mathrm{d}_q s \\
&\leq \|k_1 - k_2\| \int_0^1 (1-s)_q^{(\sigma-2)} \Bigg(\frac{f_1(s)}{\Gamma_q(\sigma-1)} + \frac{f_1(s)}{\Gamma_q(\sigma)} \\
&\quad + \frac{f_1(s)}{\lambda\Gamma(\sigma-\eta)} + \frac{mf_4(s)}{\Gamma_q(\sigma-1)} + \frac{mf_4(s)}{\Gamma_q(\sigma)} + \frac{mf_4(s)}{\lambda\Gamma_q(\sigma-\eta)} \Bigg) \mathrm{d}_q s \\
&\quad + \|k'_1 - k'_2\| \int_0^1 (1-s)_q^{(\sigma-2)} \Bigg(\frac{f_2(s)}{\Gamma_q(\sigma-1)} + \frac{f_2(s)}{\Gamma_q(\sigma)} \\
&\quad + \frac{f_2(s)}{\lambda\Gamma_q(\sigma-\eta)} + \frac{f_3(s)}{\Gamma_q(\sigma-1)\Gamma_q(2-\zeta)} \\
&\quad + \frac{f_3(s)}{\Gamma_q(\sigma)\Gamma_q(2-\zeta)} + \frac{f_3(s)}{\lambda\Gamma_q(\sigma-\eta)\Gamma_q(2-\zeta)} \Bigg) \mathrm{d}_q s \\
&\leq \Lambda_2(\|k_1 - k_2\| + \|k'_1 - k'_2\|) = \Lambda_2 \|k_1 - k_2\|_*,
\end{aligned}$$

where

$$\begin{aligned}
\Lambda_2 = \max\Bigg\{ &\int_0^1 (1-s)_q^{(\sigma-2)} \Bigg(\frac{f_1(s)}{\Gamma_q(\sigma-1)} + \frac{f_1(s)}{\Gamma_q(\sigma)} \\
&+ \frac{f_1(s)}{\lambda\Gamma_q(\sigma-\eta)} + \frac{mf_4(s)}{\Gamma_q(\sigma-1)} + \frac{mf_4(s)}{\Gamma_q(\sigma)} + \frac{mf_4(s)}{\lambda\Gamma_q(\sigma-\eta)} \Bigg) \mathrm{d}_q s, \\
&\int_0^1 (1-s)_q^{(\sigma-2)} \Bigg(\frac{f_2(s)}{\Gamma_q(\sigma-1)} + \frac{f_2(s)}{\Gamma_q(\sigma)} + \frac{f_2(s)}{\lambda\Gamma_q(\sigma-\eta)} \\
&+ \frac{f_3(s)}{\Gamma_q(\sigma-1)\Gamma_q(2-\zeta)} + \frac{f_3(s)}{\Gamma_q(\sigma)\Gamma_q(2-\beta)} \\
&+ \frac{f_3(s)}{\lambda\Gamma_q(\sigma-\eta)\Gamma_q(2-\zeta)} \Bigg) \mathrm{d}_q s \Bigg\} < \infty.
\end{aligned}$$

Put

$$M_1 = \frac{1}{\Gamma_q(\sigma)} + \frac{1}{\lambda\Gamma_q(\sigma)} + \frac{1}{\lambda\Gamma_q(\sigma-\eta)},$$
$$M_2 = \frac{1}{\Gamma_q(\sigma-1)} + \frac{1}{\lambda\Gamma_q(\sigma)} + \frac{1}{\lambda\Gamma_q(\sigma-\eta)}, \tag{20}$$

$$\begin{aligned}
m_0 &= \int_0^1 (1-s)_q^{(\sigma-\eta-1)} g(s)\,\mathrm{d}_q s \\
&= (1-q)\sum_{k=0}^{\infty} q^k (1-q^k)_q^{(\sigma-\eta-1)} g(q^k) \\
&= (1-q)\sum_{k=0}^{\infty} \left(q^k g(q^k) \left[\prod_{i=0}^{\infty} \frac{1-(q^k)q^i}{1-(q^k)q^{\sigma-\eta-1+i}} \right] \right) \\
&= (1-q)\sum_{k=0}^{\infty} \left(q^k g(q^k) \left[\prod_{i=0}^{\infty} \frac{1-q^{k+i}}{1-q^{k+\sigma-\eta-1+i}} \right] \right),
\end{aligned} \tag{21}$$

and

$$r_0 = m_0 \|\Theta\|_{\mathcal{A}} \max\{M_1, M_2\}, \tag{22}$$

$\Lambda_0 = \max\{\Lambda_1, \Lambda_2\}$. Algorithm 9 shows the method of calculation m_0. Since $g \in \mathcal{L}$, $m_0 < \infty$. Then we have

$$\|T_{k_1}(t) - T_{k_2}(t)\|_* \leq \Lambda_0 \|k_1 - k_2\|_*$$

and so $\|T_{k_1}(t) - T_{k_2}(t)\|_* \to 0$ as $\|k_1 - k_2\|_* \to 0$. Consider $k \in \mathcal{B}$ and

$$B_{r_0} = \Big\{ k \in \mathcal{B} : \|k\|_* \leq r_0 \Big\}.$$

Then, we have

$$\begin{aligned}|T_k(t)| &\le \mathcal{I}_q^{\sigma}\left[g(t)\Theta\big(t,k(t),k'(t),{}^{C}\mathcal{D}_q^{\beta}[k](t),\int_0^t f(r)k(r)\,\mathrm{d}r\big)\right]\\&\quad+\frac{1}{\lambda}\left[\mathcal{I}_q^{\sigma}\left[g(1)\Theta\big(1,k(1),k'(1),{}^{C}\mathcal{D}_q^{\zeta}[k](1),\int_0^1 f(r)k(r)\,\mathrm{d}r\big)\right]\right.\\&\quad\left.+\mathcal{I}_q^{\sigma-\eta}\left[g(\tau)\Theta\big(\tau,k(\tau),k'(\tau),{}^{C}\mathcal{D}_q^{\zeta}[u](\tau),\int_0^{\tau} f(r)k(r)\,\mathrm{d}r\big)\right]\right]\\&\le\|\Theta\|_{\mathcal{A}}\left(\frac{1}{\Gamma_q(\sigma)}+\frac{1}{\lambda\Gamma_q(\sigma)}+\frac{1}{\lambda\Gamma_q(\sigma-\eta)}\right)\\&\quad\times\int_0^1(1-s)_q^{(\sigma-\eta-1)}g(s)\,\mathrm{d}_qs\\&=m_0\|\Theta\|_{\mathcal{A}}M_1,\end{aligned}$$

for each $t\in J$. Note that,

$$\int_0^1(1-qs)^{(\sigma-1)}g(s)\,\mathrm{d}_qs\le m_0.$$

Also, we can conclude that

$$\begin{aligned}T_k'(t)&=\int_0^1\frac{\partial G_q(t,s)}{\partial t}\Omega\left(s,k(s),k'(s),{}^{C}\mathcal{D}_q^{\zeta}[k](s),\int_0^s f(r)k(r)\,\mathrm{d}r\right)\mathrm{d}_qs\\&=-\mathcal{I}_q^{\sigma-1}\Omega\left(t,k(t),k'(t),{}^{C}\mathcal{D}_q^{\zeta}k(t),\int_0^t f(r)k(r)\,\mathrm{d}r\right)\\&\quad+\frac{1}{\lambda}\left[\mathcal{I}_q^{\sigma}\Omega\left(1,k(1),k'(1),{}^{C}\mathcal{D}_q^{\zeta}k(1),\int_0^1 f(r)k(r)\,\mathrm{d}r\right)\right.\\&\quad\left.+\mathcal{I}_q^{\sigma-\eta}\Omega\left(\tau,k(\tau),k'(\tau),{}^{C}\mathcal{D}_q^{\zeta}k(\tau),\int_0^s f(r)k(\tau)\,\mathrm{d}r\right)\right],\end{aligned}$$

and so

$$\begin{aligned}|T_k'(t)|&\le\|\Theta\|_{\mathcal{A}}\left(\frac{1}{\Gamma_q(\sigma-1)}+\frac{1}{\lambda\Gamma_q(\sigma)}+\frac{1}{\lambda\Gamma_q(\sigma-\eta)}\right)\\&\quad\times\int_0^1(1-s)_q^{(\sigma-\eta-1)}g(s)\,\mathrm{d}_qs=m_0\|\Theta\|_{\mathcal{A}}M_2.\end{aligned}$$

Hence, $\|T_k\|_* = \max\{\|T_k\|, \|T'_k\|\} \le r_0$. Therefore, T maps B_{r_0} into B_{r_0}. Similarly one can check that T maps bounded sets into bounded sets. Let $t_1, t_2 \in J$ with $t_1 \le t_2$. Then, we have

$$\begin{aligned}
&\|T_k(t_1) - T_k(t_2)\| \\
&\qquad \le \frac{1}{\Gamma_q(\sigma)} \int_0^{t_1} \Big[(t_2 - s)_q^{(\sigma-1)} - (t_1 - s)_q^{(\sigma-1)}\Big] |\tilde{\Omega}(k,s)| \,\mathrm{d}_q s \\
&\qquad\quad + \frac{1}{\Gamma_q(\sigma)} \int_{t_1}^{t_2} (t_1 - s)_q^{(\sigma-1)} |\tilde{\Omega}(k,s)| \,\mathrm{d}_q s \\
&\qquad\quad + \frac{|t_2 - t_1|}{\Delta \Gamma_q(\sigma)} \int_0^1 (1 - s)_q^{(\sigma-1)} |\tilde{\Omega}(k,s)| \,\mathrm{d}_q s \\
&\qquad\quad + \frac{|t_2 - t_1|}{\lambda \Gamma_q(\sigma - \eta)} \int_0^\mu (\mu - s)_q^{(\sigma-\eta-1)} |\tilde{\Omega}(k,s)| \,\mathrm{d}_q s \\
&\qquad \le \|\Theta\|_{\mathcal{A}} \Bigg[\frac{1}{\Gamma_q(\sigma)} \int_0^{t_1} \Big[(t_2 - s)_q^{(\sigma-1)} - (t_1 - s)_q^{(\sigma-1)}\Big] g(s) \,\mathrm{d}_q s \\
&\qquad\quad + \frac{1}{\Gamma_q(\sigma)} \int_{t_1}^{t_2} (t_1 - s)_q^{\sigma-1} g(s) \,\mathrm{d}_q s \\
&\qquad\quad + |t_2 - t_1| \left(\frac{1}{\lambda_q \Gamma_q(\sigma)} + \frac{1}{\lambda \Gamma_q(\sigma - \eta)}\right) \\
&\qquad\quad \times \int_0^1 (1 - s)_q^{(\sigma-\eta-1)} g(s) \,\mathrm{d}_q s\Bigg].
\end{aligned}$$

Since $g \in \mathcal{L}$, $\int_0^1 (1 - s)_q^{(\sigma-\eta-1)} g(s) \,\mathrm{d}_q s < \infty$. Also, we have

$$\begin{aligned}
\sup_{t_1, t_2 \in J} &\left(\int_0^{t_1} \Big[(t_2 - s)_q^{(\sigma-1)} - (t_1 - s)_q^{(\sigma-1)}\Big] g(s) \,\mathrm{d}_q s\right) \\
&\le \int_0^1 (1 - s)_q^{(\sigma-1)} g(s) \,\mathrm{d}_q s < \infty.
\end{aligned}$$

Since $(t_2 - s)_q^{(\sigma-1)} - (t_1 - s)_q^{(\sigma-1)} \to 0$, as $t_2 \to t_1$, for each $\varepsilon > 0$ there exists $\delta > 0$ such that $|t_2 - t_1| < \delta$ implies

$$(t_2 - s)_q^{(\sigma-1)} - (t_1 - s)_q^{(\sigma-1)} < \varepsilon.$$

If $0 < \delta < \varepsilon$ and $|t_2 - t_1| < \delta$, then

$$\int_0^{t_1} \Big[(t_2 - s)_q^{(\sigma-1)} - (t_1 - s)_q^{(\sigma-1)}\Big] g(s) \,\mathrm{d}_q s \le \varepsilon \int_0^1 g(s) \,\mathrm{d}_q s,$$

and so

$$\int_0^{t_1} \left[(t_2 - s)_q^{(\sigma-1)} - (t_1 - s)_q^{(\sigma-1)}\right] g(s)\, \mathrm{d}_q s \to 0,$$

as $t_2 \to t_1$. Similarly we conclude that

$$\int_{t_1}^{t_2} (t_1 - s)_q^{(\sigma-1)} g(s)\, \mathrm{d}_q s$$

and

$$\int_0^1 (1 - s)_q^{(\sigma-\eta-1)} g(s)\, \mathrm{d}_q s,$$

tend to 0 as $t_2 \to t_1$. Thus, $|T_k(t_2) - T_k(t_1)| \to 0$ as $t_2 \to t_1$. Note that,

$$\begin{aligned} &|T'_k(t_2) - T'_k(t_1)| \\ &\qquad \leq \frac{\|\Theta\|_{\mathcal{A}}}{\Gamma_q(\sigma - 1)} \left[\int_0^{t_1} \left[(t_2 - s)_q^{(\sigma-2)} - (t_1 - s)_q^{(\sigma-2)}\right] g(s)\, \mathrm{d}_q s \right. \\ &\qquad\quad \left. + \int_{t_1}^{t_2} (t_1 - s)_q^{(\sigma-2)} g(s)\, \mathrm{d}_q s \right]. \end{aligned}$$

By using a similar way, we conclude that $|T'_k(t_2) - T'_k(t_1)| \to 0$ as $t_2 \to t_1$. Hence,

$$\|T_k(t_2) - T_k(t_1)\|_* \to 0,$$

as $t_2 \to t_1$ and so T is equi-continuous on B_{r_0}. Hence, $T : B_{r_0} \to B_{r_0}$ is completely continuous. At present, Lemma 2.5 implies that T has a fixed point on B_{r_0} which is the solution of the problem (1). The proof is completed. $\square$

Note that in Theorem 3.3, the map $\Omega(t, ., ., ., .)$ could be discontinuous at points of a subset of J of measure zero. One can obtain solutions of the problem (1) under some different conditions. For example in next result, the map $\Omega(t, ., ., ., .)$ could be discontinuous at $t = 0$.

Theorem 3.4. *Let $\Omega : J \times \mathcal{B}^4 \to \mathbb{R}$ is a map. Then the problem (1) has a solution, whenever the following assumptions are hold for all $(k_1, k_2, k_3, k_4) \in \mathcal{B}^4$ and almost all $t \in J$.*

1) $\Omega(t, ., ., ., .) : J \to \mathbb{R}$ is continuous and $\Omega(t, k_1, k_2, k_3, k_4) \geq 0$.

2) There exist $g \in \mathcal{L}$ and $\Theta_1, \Theta_2 : \mathbb{R}^4 \to [0,\infty)$ such that Θ_1 and Θ_2 are nondecreasing in all components,

$$\lim_{k\to\infty} \Theta_1(k,k,k,k)/k = 0, \qquad \lim_{l\to\infty} \Theta_2(l,l,l,l) = \ell < \infty$$

and

$$\Omega(t,k_1,k_2,k_3,k_4) \le g(t)\Theta_1(k_1,k_2,k_3,k_4) + \Theta_2(k_1,k_2,k_3,k_4).$$

Proof. For each $k \in \mathcal{B}$ and $i \ge 1$ define

$$(k)_i(t) = \begin{cases} \min\left\{\frac{-1}{i}, k(t)\right\}, & k(t) < 0 \\ \max\left\{\frac{1}{i}, k(t)\right\}, & k(t) \ge 0. \end{cases}$$

Put

$$\Omega_i(t,k_1,k_2,k_3,k_4) = \Omega\big(t,(k_1)_i,(k_2)_i,(k_3)_i,(k_4)_i\big),$$

for all i, t and k_1,k_2,k_3,k_4. By simple method, we conclude that $(k)_i(t) \to k(t)$ and each Ω_i is a regular function on J. A regular function at a point a is a function that is regular in some neighborhood of a. For each i, consider the regular fractional $q-$integro–differential equation

$$ {}^C\mathcal{D}_q^{\sigma}[k](t) + \Omega_i\left(t, k(t), k'(t), {}^C\mathcal{D}_q^{\zeta}[k](t), \int_0^t f(r)k(r)\,\mathrm{d}r\right) = 0, \tag{23}$$

under the boundary condition of the problem (1). Suppose that $\|g\|_1 = m > 0$ and $\varepsilon_0 > 0$ be given. Choose $r_1 > 0$ and $r_2 > 0$ such that $\left|\frac{\ell}{k}\right| < \frac{1}{2}\varepsilon_0$ for each $|k| > r_1$ and

$$\frac{\Theta_1(k,k,k,k)}{k} < \frac{1}{2\|g\|_1}\varepsilon_0,$$

for each $|k| > r_2$, respectively. Take $r_0 := \max\{r_1, r_2\}$, then for all $|k| > r_0$, we obtain

$$\frac{\ell + \|g\|_1\Theta_1(k,k,k,k)}{k} < \varepsilon_0.$$

Put $\Lambda_0 := \max\{M_1, M_2\}$, here M_1 and M_2 are defined in Eq. (20), and $\varepsilon_0 = \frac{1}{\Lambda_0}$. If

$$r > r_0 \max\left\{1, \frac{1}{\Gamma_q(2-\zeta)}, m\right\}, \tag{24}$$

then
$$\frac{1}{r}\left[\ell + \|g\|_1 \Theta\left(r, r, \frac{r}{\Gamma_q(2-\zeta)}, mr\right)\right] < \frac{1}{\Lambda_0}. \tag{25}$$
At present, consider the set
$$B_r = \Big\{k \in \mathcal{B} : \|k\|_* < r\Big\}.$$
For each $i \geq 1$, define $T_i : \bar{B}_r \to \mathcal{B}$ as (19) in which we replaced Ω by Ω_i. If $\{k_i\}$ is a convergent sequence in $\bar{B}_r$, then $k_i \to k$ and $k_i' \to k'$ uniformly on J. Since
$$\|{}^{C}\mathcal{D}_q^{\zeta}[k_i](t) - {}^{C}\mathcal{D}_q^{\zeta}[k](t)\| \leq \frac{\|k_i - k'\|}{\Gamma_q(2-\zeta)}$$
and ${}^{C}\mathcal{D}_q^{\zeta}[k_i](t) \to {}^{C}\mathcal{D}_q^{\zeta}[k](t)$. Also, we have
$$\begin{aligned}\left|\int_0^t f(r)k_i(r)\,\mathrm{d}r - \int_0^t f(r)k(r)\,\mathrm{d}r\right| &\leq \int_0^t f(r)|k_i(r) - k(r)|\,\mathrm{d}r \\ &\leq m\|k_i - k\|\end{aligned}$$
and so
$$\lim_{i\to\infty}\int_0^t f(r)k_i(r)\,\mathrm{d}r = \int_0^t f(r)k(r)\,\mathrm{d}r.$$
Thus, $\lim_{i\to\infty}\tilde{\Omega}_i(k_i,t) = \tilde{\Omega}_i(k,t)$. Note that,
$$\begin{aligned}|T_n[k_i](t) - T_n[k](t)| \leq \int_0^1 \Bigg[&\frac{(t-s)_q^{(\sigma-1)}}{\Gamma_q(\sigma)} + \frac{t(t-s)_q^{(\sigma-1)}}{\lambda\Gamma_q(\sigma)} \\ &+ \frac{t(\tau-s)_q^{(\sigma-\eta-1)}}{\lambda\Gamma_q(\sigma-\eta)}\Bigg]\left|\tilde{\Omega}_n(k_i,s) - \tilde{\Omega}_n(k,s)\right|\,\mathrm{d}_q s \\ \leq M_1 &\int_0^1 |\tilde{\Omega}_n(k_i,s) - \tilde{\Omega}_n(k,s)|\,\mathrm{d}_q s.\end{aligned}$$
By using a similar method, we have
$$\left|T_n'[k_i](t) - T_n'[k](t)\right| \leq M_2 \int_0^1 \left|\tilde{\Omega}_n(k_i,s) - \tilde{\Omega}_n(k,s)\right|\,\mathrm{d}_q s.$$

Thus,

$$\|T_n[k_i](t) - T_n[k](t)\|_* \to 0,$$

as $k_i \to k$. Hence, $\{T_n[k_i]\}_{i=1}^{\infty}$ is relatively compact in $\bar{B}_r$ and so T_i is a completely continuous operator on $\bar{B}_r$ for all i. Suppose that $i \geq 1$ be given and there exist $z \in \partial B_r$ and $0 < c < 1$ such that $z = cT_i[z]$. Since $\|z\|_* = r$, $\|z\| \leq r$, $\|z'\| \leq r$,

$$\|{}^{C}\mathcal{D}_q^{\zeta}[z]\| \leq \frac{\|z'\|}{\Gamma_q(2-\zeta)} \leq \frac{r}{\Gamma_q(2-\zeta)}$$

and $\| \int f(r) z(r)\, \mathrm{d}r\| \leq mr$. By using the assumption, we have

$$\begin{aligned}
|z(t)| = |cT_n[z](t)| &= \left| c\int_0^1 G_q(t,qs)\tilde{\Omega}_i(z,s)\,\mathrm{d}_q s\right| \\
&< M_1 \int_0^1 \Theta_2\Big(r, r, \frac{r}{\Gamma_q(2-\zeta)}, mr\Big)\,\mathrm{d}_q s \\
&\quad + \int_0^1 f(s)\Theta_1\Big(z(s), z'(s), {}^{C}\mathcal{D}_q^{\sigma}[z](s), \int_0^s f(r)z(r)\,\mathrm{d}r\Big)\,\mathrm{d}_q s \\
&\leq M_1\left(\ell + \|g\|_1 \Theta_1\left(r, r, \frac{r}{\Gamma_q(2-\zeta)}, mr\right)\right)
\end{aligned}$$

and

$$\begin{aligned}
|z'(t)| = |cT_n'[z](t)| &= \left| c\int_0^1 \frac{\partial G_q(t,s)}{\partial t}\tilde{\Omega}_i(z,s)\,\mathrm{d}_q s\right| \\
&< M_2\Big(\ell + \|g\|_1 \Theta_1\Big(r, r, \frac{r}{\Gamma_q(2-\zeta)}, mr\Big)\Big).
\end{aligned}$$

Hence,

$$\|z\|_* < \max\{M_1, M_2\}\Big(\ell + \|g\|_1\Theta_1\Big(r, r, \frac{r}{\Gamma_q(2-\beta)}, mr\Big)\Big),$$

and so

$$r < \Lambda_0\Big(\ell + \|g\|_1\Theta_1\Big(r, r, \frac{r}{\Gamma_q(2-\zeta)}, mr\Big)\Big).$$

Thus,

$$\ell + \|g\|_1\Theta_1\Big(r, r, \frac{r}{\Gamma_q(2-\zeta)}, mr\Big) > \frac{r}{\Lambda_0},$$

which is a contradiction to (25). This implies that $z \notin \partial B_r$. By using Lemma 2.6, T_i has a fixed point $k_i \in \bar{B}_r$ for each i, that is the problem (23) has a solution. Let $(k)_i$ be the solution of the problem (23). As we proved, $\{(k)_i\}$ is relatively compact and $(k)_i \to k$ for some $k \in \mathcal{B}$. Thus, $k \in \bar{B}_r$. Similar to last result, we can show that

$$\lim_{i\to\infty} {}^{C}\mathcal{D}_q^{\beta}[k_i](t) = {}^{C}\mathcal{D}_q^{\zeta}[k](t), \qquad \lim_{i\to\infty} k_i'(t) = k(t),$$

and

$$\lim_{i\to\infty} \int_0^t f(r)k_i(r)\,\mathrm{d}r = \int_0^t f(r)k(r)\,\mathrm{d}r,$$

for each $t \in J$. Consequently, we get $\lim_{i\to\infty} \tilde{\Omega}_i(k,t) = \tilde{\Omega}(k,t)$ and

$$|G_q(t,s)\tilde{\Omega}_i(k,t) - \tilde{\Omega}(k,t)| \leq M_1\left[g(s)\Theta_2\left(r, r, \frac{r}{\Gamma_q(2-\zeta)}, mr\right)\right] < \infty.$$

By applying the Lebesgue dominated theorem, we obtain

$$k(t) = \int_0^1 G_q(t,s)\tilde{\Omega}(k,s)\mathrm{d}_q s,$$

for all $t \in J$. This completes the proof. □

4 Illustrative Examples With Numerical Results

Herein, we give some examples to show the validity of the main results. In this way, we give a computational technique for checking the problem (1). For problems for which the analytical solution is not known, we will use, as reference solution, the numerical approximation obtained with a tiny step h by the implicit trapezoidal PI rule, which, as we will see, usually shows an excellent accuracy ([15]).

All the experiments are carried out in MATLAB Ver. 8.5.0.197613 (R2015a) on a computer equipped with a CPU AMD Athlon(tm) II X2 245 at 2.90 GHz running under the operating system Windows 7.

Example 4.1. Consider the fractional $q-$integro–differential problem

$$
\begin{aligned}
{}^{C}\mathcal{D}_q^{\frac{25}{14}}[k](t) + g(t)\Bigg[& \frac{|k(t)|}{3+|k(t)|} + \frac{|k'(t)|}{3+|k'(t)|} \\
& + \frac{|{}^{C}\mathcal{D}_q^{\frac{9}{14}}[k](t)|}{3+|{}^{C}\mathcal{D}_q^{\frac{9}{14}}[k](t)|} + \frac{|z_k(t)|}{3+|z_k(t)|}\Bigg] = 0, \qquad (26)
\end{aligned}
$$

for $t \in J$, $k \in C^1(J)$ and for each $q \in J$, under conditions

$$
k(0) = 0, \qquad k(1) = {}^{C}\mathcal{D}_q^{\frac{5}{7}}[k]\left(\frac{8}{9}\right),
$$

where $z_k(t) = \int_0^t f(r)k(r)\,\mathrm{d}r$, $f(t) = \frac{3}{\sqrt{t+1}} \in \mathcal{L}$ and $m = \|f\|_1 = 3$. Clearly in the problem $\sigma = \frac{25}{14} \in [1,2)$, $\zeta = \frac{9}{14} \in J$, $\tau = \frac{8}{9} \in J$, $\eta = \frac{5}{7} \in J$. We define $g(t)$ by

$$
g(t) = \begin{cases}
\dfrac{1}{t^{p_1}}, & t \in (0, \gamma_1], \\
\dfrac{1}{(t-\gamma_1)^{p_2}}, & t \in (\gamma_1, \gamma_2], \\
\vdots & \vdots \\
\dfrac{1}{(t-\gamma_k)^{p_{N_0+1}}}, & t \in (\gamma_k, 1).
\end{cases}
$$

where $k \geq 1$, $p_1, \ldots, p_{N_0+1} \in J$ and $\gamma_1, \gamma_2, \ldots, \gamma_{N_0}$, be real numbers such that

$$
0 < \gamma_1 < \gamma_2 < \cdots < \gamma_{N_0} < 1.
$$

For $N_0 = 4$, we take

$$
g(t) = \begin{cases}
\frac{1}{t^{\frac{1}{8}}}, & t \in (0, \frac{1}{4}], \\
\frac{1}{(t-\frac{1}{4})^{\frac{3}{7}}}, & t \in (\frac{1}{4}, \frac{1}{2}], \\
\frac{1}{(t-\frac{1}{2})^{\frac{5}{7}}}, & t \in (\frac{1}{2}, \frac{3}{4}], \\
\frac{1}{(t-\frac{3}{4})^{\frac{9}{10}}}, & t \in (\frac{3}{4}, 1).
\end{cases} \qquad (27)
$$

Algorithm 10 shows the MATLAB codes for calculation of $g(t)$. Now, define

$$\Theta(k_1, k_2, k_3, k_4) = \sum_{i=1}^{4} \frac{|k_i|}{3 + |k_i|},$$

for $(k_1, k_2, k_3, k_4) \in \mathbb{R}^4$. One can see that Θ satisfies in Eq. (18). Then we have

$$\begin{aligned}
&|\Omega(t, k_1, k_2, k_3, k_4) - \Omega(t, l_1, l_2, l_3, l_4)| \\
&\qquad = \Bigg| g(t) \Bigg[\frac{|k_1(t)|}{3 + |k_1(t)|} + \frac{|k_2'(t)|}{3 + |k_2'(t)|} \\
&\qquad\quad + \frac{|{}^C\mathcal{D}_q^{\frac{9}{14}}[k_3](t)|}{3 + |{}^C\mathcal{D}_q^{\frac{9}{14}}[k_3](t)|} + \frac{|z_{k_4}(t)|}{3 + |z_{k_4}(t)|} \Bigg] \\
&\qquad\quad - g(t) \Bigg[\frac{|l_1(t)|}{3 + |l_1(t)|} + \frac{|l_2'(t)|}{3 + |l_2'(t)|} \\
&\qquad\quad + \frac{|{}^C\mathcal{D}_q^{\frac{9}{14}}[l_3](t)|}{3 + |{}^C\mathcal{D}_q^{\frac{9}{14}}[l_3](t)|} + \frac{|z_{l_4}(t)|}{3 + |z_{l_4}(t)|} \Bigg] \Bigg| \\
&\qquad \leq |g(t)| \Bigg[\frac{1}{3} |k_1(t) - l_1(t)| + \frac{1}{3} |k_2(t) - l_2(t)| \\
&\qquad\quad + \frac{1}{3} |k_3(t) - l_3(t)| + \frac{1}{3} |k_4(t) - l_4(t)| \Bigg] \\
&\qquad \leq |g(t)| \sum_{i=1}^{4} |k_i(t) - l_i(t)|.
\end{aligned}$$

Thus, by applying the definition of $g(t)$, we get

$$\begin{aligned}
&|\Omega(t, k_1, k_2, k_3, k_4) - \Omega(t, l_1, l_2, l_3, l_4)| \\
&\qquad \leq \begin{cases}
\frac{1}{t^{\frac{1}{8}}} \sum_{i=1}^{4} |k_i(t) - l_i(t)|, & t \in (0, \frac{1}{4}], \\
\frac{1}{(t - \frac{1}{4})^{\frac{3}{7}}} \sum_{i=1}^{4} |k_i(t) - l_i(t)|, & t \in (\frac{1}{4}, \frac{1}{2}], \\
\frac{1}{(t - \frac{1}{2})^{\frac{5}{7}}} \sum_{i=1}^{4} |k_i(t) - l_i(t)|, & t \in (\frac{1}{2}, \frac{3}{4}], \\
\frac{1}{(t - \frac{3}{4})^{\frac{9}{10}}} \sum_{i=1}^{4} |k_i(t) - l_i(t)|, & t \in (\frac{3}{4}, 1).
\end{cases}
\end{aligned}$$

Therefore,

$$f_i(t) = \frac{1}{\sqrt[8]{t}}, \quad \frac{1}{\sqrt[7]{(t-0.25)^3}}, \quad \frac{1}{\sqrt[7]{(t-0.5)^5}}, \quad \frac{1}{\sqrt[10]{(t-0.75)^9}},$$

for $t \in (0, \frac{1}{4}]$, $(\frac{1}{4}, \frac{1}{2}]$, $(\frac{1}{2}, \frac{3}{4}]$, $(\frac{3}{4}, 1)$ and $i = 1, 2, 3, 4$ respectively.

Table 1: Some numerical results of λ, M_1, M_2, m_0, r_0 in Example 4.1 for $t \in \overline{J}$ and $q = 0.1$.

	$q = 0.1$				
n	λ	M_1	M_2	m_0	r_0
1	0.518684	2.786200	1.324100	0.117800	0.328400
2	0.520697	2.774900	1.322800	<u>0.133800</u>	0.371300
3	0.520898	2.773800	<u>1.322600</u>	0.135900	0.377100
4	0.520918	<u>2.773700</u>	1.322600	0.136200	0.377800
5	<u>0.520920</u>	2.773700	1.322600	0.136300	0.377900
6	0.520920	2.773700	1.322600	0.136300	<u>0.378000</u>
7	0.520920	2.773700	1.322600	0.136300	0.378000
8	0.520920	2.773700	1.322600	0.136300	0.378000
⋮	⋮	⋮	⋮	⋮	⋮
115	0.520920	2.773700	1.322600	0.136300	0.378000
116	0.520920	2.773700	1.322600	0.136300	0.378000
117	0.520920	2.773700	1.322600	0.136300	0.378000
118	0.520920	2.773700	1.322600	0.136300	0.378000
119	0.520920	2.773700	1.322600	0.136300	0.378000
120	0.520920	2.773700	1.322600	0.136300	0.378000

In addition by using Eqs. (17), (20) and (21), we obtain

$$\lambda = 1 - \frac{\tau^{1-\eta}}{\Gamma_q(2-\eta)} = 1 - \frac{\left(\frac{8}{9}\right)^{\left(\frac{2}{7}\right)}}{\Gamma_q\left(\frac{9}{7}\right)} \cong \begin{cases} 0.520920, & q = \frac{1}{10}, \\ 0.812929, & q = \frac{1}{2}, \\ 0.954051, & q = \frac{6}{7}. \end{cases}$$

On the other hand,

$$\begin{aligned} M_1 &= \frac{1}{\Gamma_q(\sigma)} + \frac{1}{\lambda\Gamma_q(\sigma)} + \frac{1}{\lambda\Gamma_q(\sigma-\eta)} \\ &= \frac{1}{\Gamma_q\left(\frac{25}{14}\right)} + \frac{1}{\lambda\Gamma_q\left(\frac{25}{14}\right)} + \frac{1}{\lambda\Gamma_q\left(\frac{15}{14}\right)}, \end{aligned}$$

$$\begin{aligned} M_2 &= \frac{1}{\Gamma_q(\sigma-1)} + \frac{1}{\lambda\Gamma_q(\sigma)} + \frac{1}{\lambda\Gamma_q(\sigma-\eta)} \\ &= \frac{1}{\Gamma_q\left(\frac{11}{14}\right)} + \frac{1}{\lambda\Gamma_q\left(\frac{25}{14}\right)} + \frac{1}{\lambda\Gamma_q\left(\frac{15}{14}\right)} \end{aligned}$$

Table 2: Some numerical results of λ, M_1, M_2, m_0, r_0 in Example 4.1 for $t \in \overline{J}$ and $q = 0.5$.

n	$q = 0.5$				
	λ	M_1	M_2	m_0	r_0
1	0.775851	1.138408	0.469201	0.425149	0.484000
2	0.796366	1.086752	0.467331	0.567061	0.616300
3	0.805064	1.064907	0.465153	0.645533	0.687400
4	0.809092	1.054781	0.463887	0.688662	0.726400
5	0.811034	1.049897	0.463219	0.712290	0.747800
6	0.811987	1.047499	0.462878	0.725212	0.759700
7	0.812460	1.046309	0.462706	0.732272	0.766200
8	0.812695	1.045718	0.462619	0.736128	0.769800
⋮	⋮	⋮	⋮	⋮	⋮
15	0.812927	1.045132	0.462534	0.740695	0.774100
16	0.812928	1.045130	0.462533	0.740726	0.774200
17	0.812929	1.045129	0.462533	0.740742	0.774200
18	0.812929	1.045128	0.462533	0.740751	0.774200
19	0.812929	1.045128	0.462533	0.740756	0.774200
20	0.812929	1.045128	0.462533	0.740759	0.774200
21	0.812929	1.045128	0.462533	0.740760	0.774200
22	0.812929	1.045128	0.462533	0.740761	0.774200
23	0.812929	1.045128	0.462533	0.740762	0.774200
24	0.812929	1.045128	0.462533	0.740762	0.774200
⋮	⋮	⋮	⋮	⋮	⋮
115	0.812929	1.045128	0.462533	0.740762	0.774200
116	0.812929	1.045128	0.462533	0.740762	0.774200
117	0.812929	1.045128	0.462533	0.740762	0.774200
118	0.812929	1.045128	0.462533	0.740762	0.774200
119	0.812929	1.045128	0.462533	0.740762	0.774200
120	0.812929	1.045128	0.462533	0.740762	0.774200

and from Eq. (21),

$$
\begin{aligned}
m_0 &= \int_0^1 (1-s)_q^{(\sigma-\eta-1)} g(s)\,\mathrm{d}_q s \\
&= \int_0^1 (1-s)_q^{\left(\frac{25}{14}-\frac{5}{7}-1\right)} g(s)\,\mathrm{d}_q s \\
&= \int_0^1 (1-s)_q^{\left(\frac{1}{14}\right)} g(s)\,\mathrm{d}_q s \\
&= (1-q)\sum_{k=0}^{\infty}\left(q^k g(q^k)\left[\prod_{i=0}^{\infty}\frac{1-q^{k+i}}{1-q^{k+\sigma-\eta-1+i}}\right]\right) \\
&= (1-q)\sum_{k=0}^{\infty}\left(q^k g(q^k)\left[\prod_{i=0}^{\infty}\frac{1-q^{k+i}}{1-q^{k+i+\frac{1}{14}}}\right]\right).
\end{aligned}
$$

Table 3: Some numerical results of λ, M_1, M_2, m_0, r_0 in Example 4.1 for $t \in \overline{J}$ and $q = \frac{6}{7}$.

	$q = \frac{6}{7}$				
n	λ	M_1	M_2	m_0	r_0
1	0.886658	0.377221	0.066857	0.832489	0.314000
2	0.909096	0.341934	0.094234	1.095316	0.374500
3	0.921506	0.321630	0.103825	1.451197	0.466700
4	0.929373	0.308273	0.107981	2.181391	0.672500
⋮	⋮	⋮	⋮	⋮	⋮
61	0.954048	0.262157	0.111235	2.985013	0.782500
62	0.954049	0.262156	0.111234	2.985045	0.782500
63	0.954049	0.262155	0.111234	2.985073	0.782600
64	0.954049	0.262155	0.111234	2.985097	0.782600
65	0.954049	0.262155	0.111234	2.985119	0.782600
66	0.954050	0.262154	0.111234	2.985137	0.782600
67	0.954050	0.262154	0.111234	2.985154	0.782600
68	0.954050	0.262153	0.111234	2.985168	0.782600
69	0.954050	0.262153	0.111234	2.985181	0.782600
⋮	⋮	⋮	⋮	⋮	⋮
75	0.954050	0.262153	0.111234	2.985229	0.782600
76	0.954050	0.262152	0.111234	2.985234	0.782600
77	0.954050	0.262152	0.111234	2.985238	0.782600
78	0.954051	0.262152	0.111234	2.985242	0.782600
79	0.954051	0.262152	0.111234	2.985245	0.782600
80	0.954051	0.262152	0.111234	2.985248	0.782600
⋮	⋮	⋮	⋮	⋮	⋮
112	0.954051	0.262152	0.111234	2.985267	0.782600
113	0.954051	0.262152	0.111234	2.985268	0.782600
114	0.954051	0.262152	0.111234	2.985268	0.782600
115	0.954051	0.262152	0.111234	2.985268	0.782600
116	0.954051	0.262152	0.111234	2.985268	0.782600
117	0.954051	0.262152	0.111234	2.985268	0.782600
118	0.954051	0.262152	0.111234	2.985268	0.782600
119	0.954051	0.262152	0.111234	2.985268	0.782600
120	0.954051	0.262152	0.111234	2.985268	0.782600

Thus, we have

$$\begin{aligned} M_1 &\cong 2.773800, \quad 1.045128, \quad 0.262152, \\ M_2 &\cong 1.322600, \quad 0.462533, \quad 0.111234, \\ m_0 &\cong 0.133800, \quad 0.740762, \quad 2.985268, \end{aligned}$$

and finally by using Eq.(22), we get

$$\begin{aligned} r_0 &= m_0 \|\Theta\|_{\mathcal{A}} \max\{M_1, M_2\} \\ &= 0.371134\|\Theta\|_{\mathcal{A}}, \quad 0.774191\|\Theta\|_{\mathcal{A}}, \quad 0.782593\|\Theta\|_{\mathcal{A}}, \end{aligned}$$

for $q = \frac{1}{10}, \frac{1}{2}, \frac{6}{7}$ respectively, which are shown in Tables 1, 2 and 3. These results are obtaned by Algorithms 9, 10 and 11. These results are marked with underline in Tables 1, 2 and 3. Now, for showing the numerical results, we consider the problem (26) as follows:

$$\begin{aligned} {}^{C}\mathcal{D}_q^{\frac{25}{14}}[k](t) + g(t)\Big[\frac{|k(t)|}{3+|k(t)|} + \frac{|k'(t)|}{3+|k'(t)|} \\ + \frac{|{}^{C}\mathcal{D}_q^{\frac{9}{14}}[k](t)|}{3+|{}^{C}\mathcal{D}_q^{\frac{9}{14}}[k](t)|} + \frac{|z_k(t)|}{3+|z_k(t)|}\Big] \\ \leq {}^{C}\mathcal{D}_q^{\frac{25}{14}}[k](t) + g(t)\Big[|k(t)| + |k'(t)| \\ + |{}^{C}\mathcal{D}_q^{\frac{9}{14}}[k](t)| + |z_k(t)|\Big]. \end{aligned} \tag{28}$$

Let $t_1 = \frac{1}{8}$, $t_2 = \frac{4}{11}$, $t_3 = \frac{5}{9}$ and $t_4 = \frac{16}{19}$. Then from definition of $g(t)$ in Eq. (27), we have

$$g(t_1) = 1.2668, \quad g(t_2) = 2.5396, \quad g(t_3) = 7.8817, \quad g(t_4) = 8.5535,$$

which, upon substitution in Eq. (28), leads to

$$\begin{aligned} {}^{C}\mathcal{D}_q^{\frac{25}{14}}[k](t) + \frac{1}{\sqrt[8]{8}}\Big[|k(t)| + |k'(t)| + |{}^{C}\mathcal{D}_q^{\frac{9}{14}}[k](t)|\Big] \\ = -\frac{1}{\sqrt[8]{8}}\Big|\int_0^t f_1(r)k(r)\,\mathrm{d}r\Big|, \end{aligned}$$

$$\begin{aligned} {}^{C}\mathcal{D}_q^{\frac{25}{14}}[k](t) + \frac{1}{\sqrt[7]{\left(\frac{5}{44}\right)^3}}\Big[|k(t)| + |k'(t)| + |{}^{C}\mathcal{D}_q^{\frac{9}{14}}[k](t)|\Big] \\ = -\frac{1}{\sqrt[7]{\left(\frac{5}{44}\right)^3}}\Big|\int_0^t f_2(r)k(r)\,\mathrm{d}r\Big|, \end{aligned}$$

$$\begin{aligned} {}^{C}\mathcal{D}_q^{\frac{25}{14}}[k](t) + \frac{1}{\sqrt[7]{\left(\frac{1}{18}\right)^5}}\Big[|k(t)| + |k'(t)| + |{}^{C}\mathcal{D}_q^{\frac{9}{14}}[k](t)|\Big] \\ = -\frac{1}{\sqrt[7]{\left(\frac{1}{18}\right)^5}}\Big|\int_0^t f_3(r)k(r)\,\mathrm{d}r\Big|, \end{aligned}$$

$$
\begin{aligned}
&{}^{C}\mathcal{D}_q^{\frac{25}{14}}[k](t) + \frac{1}{\sqrt[10]{\left(\frac{7}{76}\right)^9}}\left[|k(t)| + |k'(t)| + |{}^{C}\mathcal{D}_q^{\frac{9}{14}}[k](t)|\right] \\
&\qquad = -\frac{1}{\sqrt[10]{\left(\frac{7}{76}\right)^9}}\left|\int_0^t f_4(r)k(r)\,\mathrm{d}r\right|. \qquad (29)
\end{aligned}
$$

Table 4 shows numerical values of $k(t)$ for each Equations in (29).

Table 4: Some numerical results of $T_k(t) = \int_0^1 G_q(t,s)\tilde{\Omega}(k,s)\,\mathrm{d}_q s$ in Example 4.1 for $t \in \overline{J}$.

$\left(0, \frac{1}{4}\right]$		$\left(\frac{1}{4}, \frac{1}{2}\right]$		$\left(\frac{1}{2}, \frac{3}{4}\right]$		$\left(\frac{3}{4}, 1\right)$	
t	$k(t)$	t	$k(t)$	t	$k(t)$	t	$k(t)$
0.00000	0.00000	0.25000	0.00000	0.50000	0.00000	0.75000	0.00000
0.03125	0.00135	0.28125	0.00086	0.53125	0.00069	0.78125	0.00018
0.06250	0.00445	0.31250	0.00296	0.56250	0.00252	0.81250	0.00061
0.09375	0.00877	0.34375	0.00612	0.59375	0.00569	0.84375	0.00125
0.12500	0.01402	0.37500	0.01030	0.62500	0.01148	0.87500	0.00207
0.15625	0.02004	0.40625	0.01553	0.65625	0.01584	0.90625	0.00307
0.18750	0.02669	0.43750	0.02198	0.68750	0.01835	0.93750	0.00423
0.21875	0.03387	0.46875	0.03007	0.71875	0.02090	0.96875	0.00554
0.25000	0.04150	0.50000	NaN	0.75000	0.02372	1.00000	0.00701

Also, one can see that the curve of $k(t)$ with respect to t in Fig. 1 for

$$
t \in \left\{ \left(0, \frac{1}{4}\right], \left(\frac{1}{4}, \frac{1}{2}\right], \left(\frac{1}{2}, \frac{3}{4}\right], \left(\frac{3}{4}, 1\right) \right\},
$$

respectively (Algorithm 12). By using Theorem 3.4, one can see that the singular $q-$integro-differential problem (26) has a solution.

In the next example we consider the discontinuous map $\Omega(t,.,.,.,.)$ at points of a subset of J of measure zero. Then, we obtain solutions of the problem (1) under some different conditions in Theorem 3.4 when the map $\Omega(t,.,.,.,.)$ is discontinuous at $t = 0$.

Example 4.2. Consider the singular fractional $q-$integro-differential

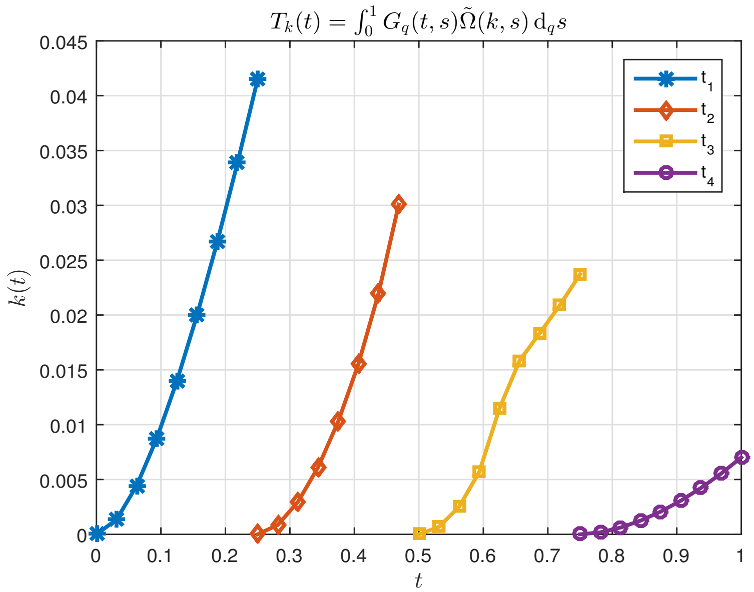

Figure 1: $k(t)$ with respect to t for Equations in (29) in Example 4.1 for $t \in \left(0, \frac{1}{4}\right]$, $\left(\frac{1}{4}, \frac{1}{2}\right]$, $\left(\frac{1}{2}, \frac{3}{4}\right]$, $\left(\frac{3}{4}, 1\right)$ respectively according to Table 4.

problem

$$
\begin{aligned}
&{}^{C}\mathcal{D}_q^{\frac{10}{7}}[k](t) + \frac{1}{\sqrt[5]{t}}\left[\frac{1}{2}|k|^{\frac{1}{3}} + \frac{8}{5}|k'|^{\frac{2}{5}} + \frac{1}{10}|{}^{C}\mathcal{D}_q^{\frac{4}{15}}[k](t)|^{\frac{3}{4}} + \frac{15}{6}|z_k(t)|^{\frac{7}{9}}\right] \\
&\quad + \frac{3}{2}\left(t^2 + \Gamma_q\left(\frac{4}{3}\right)\right)\left[\frac{1}{1+k^2(t)} + \frac{1}{2+[k'(t)]^2}\right. \\
&\quad \left. + \frac{1}{1+({}^{C}\mathcal{D}_q^{\frac{4}{15}}[k](t))^2} + \frac{1}{2+[z_k(t)]^2}\right] = 0, \qquad (30)
\end{aligned}
$$

for $t, q \in J$, with boundary conditions $k(0) = 0$ and

$$
k(1) = {}^{C}\mathcal{D}_q^{\frac{6}{11}}[k]\left(\frac{5}{8}\right).
$$

It is clear that $\sigma = \frac{10}{7} \in [1,2)$, $\zeta = \frac{4}{15} \in J$, $\eta = \frac{6}{11} \in J$, $\tau = \frac{5}{8} \in J$,

$z_k(t) = \int_0^t f(r)k(r)\,\mathrm{d}r$. Put $g(t) = \frac{1}{\sqrt[5]{t}}$ and

$$\Theta_1(k_1, k_2, k_3, k_4) = \sum_{i=1}^{4} \beta_i |k_i|^{p_i},$$
$$\Theta_2(k_1, k_2, k_3, k_4) = \sum_{i=1}^{4} \frac{3(t^2+5)}{2+k_i^2},$$

for $t \in J$. Hence, we get $m = \|g(t)\|_1 = \frac{5}{4}$,

$$\lim_{k\to\infty} \frac{\Theta_1(k,k,k,k)}{k} = 0,$$

$$\ell = \lim_{k\to\infty} \Theta_2(k,k,k,k) = 6\left(1 + \Gamma_q\left(\frac{4}{3}\right)\right) < \infty$$

and

$$\Omega(t, k_1, k_2, k_3, k_4) \leq \frac{1}{\sqrt[5]{t}} \Theta_1(k_1, k_2, k_3, k_4) + 6\left(1 + \Gamma_q\left(\frac{4}{3}\right)\right).$$

Table 5: Some numerical results of λ, M_1, M_2, ε_0, $\frac{1}{\Gamma_q(2-\zeta)}$ in Example 4.2 for $t \in \overline{J}$ and $q = \frac{1}{8}$.

n	$q=\frac{1}{8}$				
	λ	M_1	M_2	ε_0	$\frac{1}{\Gamma_q(2-\zeta)}$
1	0.486680	1.313820	−0.985848	0.761140	0.801186
2	0.488798	1.312489	−0.901121	0.761911	0.800024
3	0.489061	1.312306	−0.891080	0.762017	0.799879
4	0.489094	1.312283	−0.889834	0.762031	0.799861
5	0.489098	1.312280	−0.889678	0.762032	<u>0.799859</u>
6	<u>0.489099</u>	<u>1.312279</u>	−0.889659	<u>0.762033</u>	0.799859
7	0.489099	1.312279	<u>−0.889656</u>	0.762033	0.799859
8	0.489099	1.312279	−0.889656	0.762033	0.799859
9	0.489099	1.312279	−0.889656	0.762033	0.799859
⋮	⋮	⋮	⋮	⋮	⋮
116	0.489099	1.312279	−0.889656	0.762033	0.799859
117	0.489099	1.312279	−0.889656	0.762033	0.799859
118	0.489099	1.312279	−0.889656	0.762033	0.799859
119	0.489099	1.312279	−0.889656	0.762033	0.799859
120	0.489099	1.312279	−0.889656	0.762033	0.799859

One can see that in Problem (30) $\gamma = \frac{1}{5} \in J$, $\beta_1 = \frac{1}{2}$, $\beta_2 = \frac{8}{5}$, $\beta_3 = \frac{1}{10}$, $\beta_4 = \frac{15}{6} \in [0, \infty)$, $p_1 = \frac{1}{3}$, $p_2 = \frac{2}{5}$, $p_3 = \frac{3}{4}$, $p_4 = \frac{7}{9} \in [0, 1)$. At first by

Table 6: Some numerical results of λ, M_1, M_2, ε_0, $\frac{1}{\Gamma_q(2-\zeta)}$ in Example 4.2 for $t \in \overline{J}$ and $q = \frac{1}{2}$.

	$q = \frac{1}{2}$				
n	λ	M_1	M_2	ε_0	$\frac{1}{\Gamma_q(2-\zeta)}$
1	0.730226	0.605317	−0.209663	0.541370	1.652027
2	0.747934	0.571575	−0.081753	0.531167	1.749551
3	0.755656	0.555956	−0.040986	0.525888	1.798705
4	0.759277	0.548472	−0.024310	0.523228	1.823247
⋮	⋮	⋮	⋮	⋮	⋮
14	0.762750	0.541210	−0.009626	0.520565	1.847713
15	0.762752	0.541206	−0.009619	0.520564	1.847725
16	<u>0.762753</u>	0.541204	−0.009616	0.520563	1.847731
17	0.762753	0.541204	−0.009614	0.520563	1.847734
18	0.762753	<u>0.541203</u>	−0.009613	0.520563	1.847735
19	0.762753	0.541203	<u>−0.009612</u>	<u>0.520562</u>	<u>1.847736</u>
20	0.762753	0.541203	−0.009612	0.520562	1.847736
21	0.762753	0.541203	−0.009612	0.520562	1.847736
22	0.762753	0.541203	−0.009612	0.520562	1.847736
23	0.762753	0.541203	−0.009612	0.520562	1.847736
⋮	⋮	⋮	⋮	⋮	⋮
116	0.762753	0.541203	−0.009612	0.520562	1.847736
117	0.762753	0.541203	−0.009612	0.520562	1.847736
118	0.762753	0.541203	−0.009612	0.520562	1.847736
119	0.762753	0.541203	−0.009612	0.520562	1.847736
120	0.762753	0.541203	−0.009612	0.520562	1.847736

using Eqs, (17), (20) and (21), we obtain

$$\lambda = 1 - \frac{\tau^{1-\eta}}{\Gamma_q(2-\eta)} = 1 - \frac{\left(\frac{5}{8}\right)^{\left(\frac{5}{11}\right)}}{\Gamma_q\left(\frac{5}{11}\right)} \cong \begin{cases} 0.489099, & q = \frac{1}{8}, \\ 0.762753, & q = \frac{1}{2}, \\ 0.862603, & q = \frac{9}{13}, \end{cases}$$

$$\begin{aligned} M_1 &\cong 1.312279, \quad 0.541203, \quad 0.300304, \\ M_2 &\cong -0.889656, \quad -0.009612, \quad 0.016764, \\ \varepsilon_0 &\cong 0.762033, \quad 0.520562, \quad 3.329960, \end{aligned}$$

$$\frac{1}{\Gamma_q(2-\zeta)} \approxeq 0.790859, \quad 1.847736, \quad 0.253916,$$

and finally

$$\Lambda_0 = \max\{M_1, M_2\} = 1.312279, \quad 0.541203, \quad 0.300304,$$

for $q = \frac{1}{8}, \frac{1}{2}, \frac{9}{13}$ respectively, which are shown in Tables 5, 6 and 7. These results are obtaned by Algorithm 13. Note that, the value of r must be

Table 7: Some numerical results of λ, M_1, M_2, ε_0, $\frac{1}{\Gamma_q(2-\zeta)}$ in Example 4.2 for $t \in \overline{J}$ and $q = \frac{9}{13}$.

	$q = \frac{9}{13}$				
n	λ	M_1	M_2	ε_0	$\frac{1}{\Gamma_q(2-\zeta)}$
1	0.811891	0.393339	−0.248387	2.542333	0.292963
2	0.832639	0.358201	−0.095360	2.791727	0.277974
3	0.843725	0.337586	−0.042607	2.962205	0.269433
4	0.850289	0.324902	−0.018185	3.077850	0.264184
⋮	⋮	⋮	⋮	⋮	⋮
29	0.862602	0.300306	0.016762	3.329935	0.253917
30	<u>0.862603</u>	0.300305	0.016762	3.329944	<u>0.253916</u>
31	0.862603	0.300305	0.016763	3.329949	0.253916
32	0.862603	0.300305	0.016763	3.329952	0.253916
33	0.862603	<u>0.300304</u>	<u>0.016764</u>	3.329954	0.253916
34	0.862603	0.300304	0.016764	3.329956	0.253916
35	0.862603	0.300304	0.016764	3.329957	0.253916
36	0.862603	0.300304	0.016764	3.329958	0.253916
37	0.862603	0.300304	0.016764	3.329959	0.253916
38	0.862603	0.300304	0.016764	3.329959	0.253916
39	0.862603	0.300304	0.016764	3.329959	0.253916
40	0.862603	0.300304	0.016764	3.329959	0.253916
41	0.862603	0.300304	0.016764	3.329959	0.253916
42	0.862603	0.300304	0.016764	<u>3.329960</u>	0.253916
43	0.862603	0.300304	0.016764	3.329960	0.253916
⋮	⋮	⋮	⋮	⋮	⋮
116	0.862603	0.300304	0.016764	3.329960	0.253916
117	0.862603	0.300304	0.016764	3.329960	0.253916
118	0.862603	0.300304	0.016764	3.329960	0.253916
119	0.862603	0.300304	0.016764	3.329960	0.253916
120	0.862603	0.300304	0.016764	3.329960	0.253916

more than

$$r > r_0 \max\left\{1, \frac{1}{\Gamma_q(2-\zeta)}, m\right\} = 1.847736 r_0,$$

for $q \in J$ according to Tables 5, 6 and 7. Now, for showing the numerical results, we consider the problem (30) as follows:

$$\begin{aligned}
&{}^{C}\mathcal{D}_q^{\frac{8}{7}}[k](t) + \frac{1}{\sqrt[5]{t}}\left[\frac{1}{2}|k|^{\frac{1}{3}} + \frac{8}{5}|k'|^{\frac{2}{5}} + \frac{1}{10}\left|{}^{C}\mathcal{D}_q^{\frac{4}{15}}[k](t)\right|^{\frac{3}{4}} + \frac{15}{6}|z_k(t)|^{\frac{7}{9}}\right] \\
&\qquad + \frac{3}{2}\left(t^2 + \Gamma_q\left(\frac{4}{3}\right)\right)\left[\frac{1}{1+k^2(t)} + \frac{1}{2+[k'(t)]^2}\right. \\
&\qquad \left. + \frac{1}{1+\left({}^{C}\mathcal{D}_q^{\frac{4}{15}}[k](t)\right)^2} + \frac{1}{2+[z_k(t)]^2}\right]
\end{aligned}$$

$$\leq {}^{C}\mathcal{D}_q^{\frac{8}{7}}[k](t) + \frac{1}{2}|k|^{\frac{1}{3}} + \frac{8}{5}|k'|^{\frac{2}{5}} + \frac{1}{10}\left|{}^{C}\mathcal{D}_q^{\frac{4}{15}}[k](t)\right|^{\frac{3}{4}}$$

$$+ \frac{15}{6}|z_k(t)|^{\frac{7}{9}} + 6\left(t^2 + \Gamma_q\left(\frac{4}{3}\right)\right).$$

Thus,

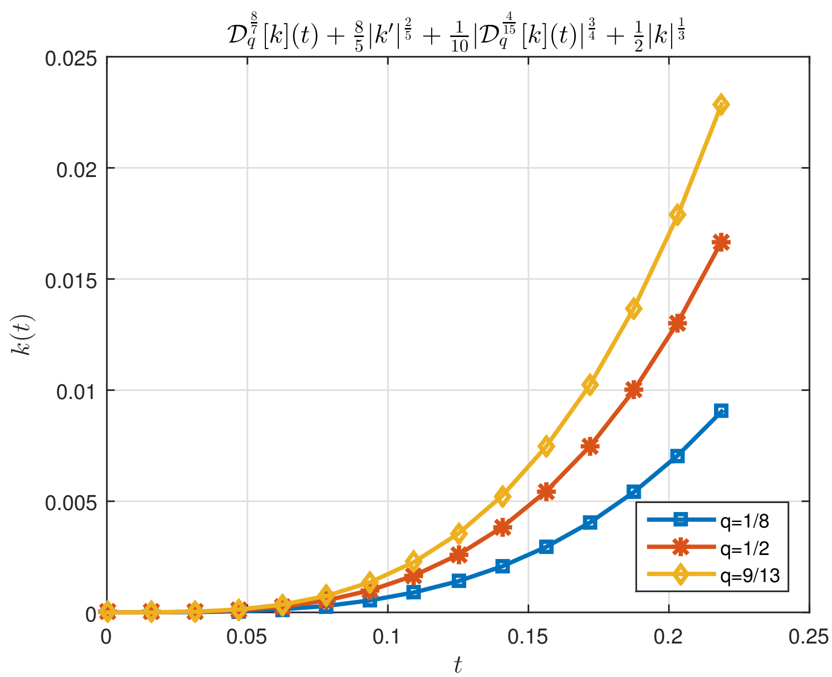

Figure 2: $k(t)$ with respect to t for Equations in (30) in Example 4.2 for $q \in \left\{\frac{1}{8}, \frac{1}{2}, \frac{9}{13}\right\}$, respectively according to Table 8.

$$ {}^{C}\mathcal{D}_q^{\frac{8}{7}}[k](t) + \frac{8}{5}|k'|^{\frac{2}{5}} + \frac{1}{10}\left|{}^{C}\mathcal{D}_q^{\frac{4}{15}}[k](t)\right|^{\frac{3}{4}} + \frac{1}{2}|k|^{\frac{1}{3}}$$

$$= -\frac{15}{6}|z_k(t)|^{\frac{7}{9}} - 6\left(t^2 + \Gamma_q\left(\frac{4}{3}\right)\right). \tag{31}$$

Table 8 shows numerical values of $k(t)$ in Eq. (30). Further, one can see that the curve of $k(t)$ with respect to t in Fig. 8 (Algorithm 14)). We

Table 8: Some numerical results of $k(t)$ in Example 4.2 for $t \in \overline{J}$, $q \in \{\frac{1}{8}, \frac{1}{2}, \frac{9}{13}\}$ and $n = 1, 2, \cdots, 10$.

	$q=\frac{1}{8}$		$q=\frac{1}{2}$		$q=\frac{9}{13}$	
$(n=1)$	t	$k(t)$	t	$k(t)$	t	$k(t)$
1	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
1	0.01563	0.00000	0.01563	0.00000	0.01563	0.00000
1	0.03125	0.00001	0.03125	0.00003	0.03125	0.00003
1	0.04688	0.00005	0.04688	0.00010	0.04688	0.00013
1	0.06250	0.00014	0.06250	0.00026	0.06250	0.00035
⋮	⋮	⋮	⋮	⋮	⋮	⋮
1	0.92188	1.09153	0.92188	1.98635	0.92188	2.70714
1	0.93750	1.15415	0.93750	2.09920	0.93750	2.86045
1	0.95313	1.21922	0.95313	2.21636	0.95313	3.01956
1	0.96875	1.28679	0.96875	2.33790	0.96875	3.18460
1	0.98438	1.35690	0.98438	2.46392	0.98438	3.35565
1	1.00000	1.42961	1.00000	2.59450	1.00000	3.53284
$(n=2)$						
2	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
2	0.01563	0.00000	0.01563	0.00000	0.01563	0.00000
2	0.03125	0.00001	0.03125	0.00003	0.03125	0.00004
2	0.04688	0.00005	0.04688	0.00011	0.04688	0.00016
2	0.06250	0.00014	0.06250	0.00028	0.06250	0.00040
⋮	⋮	⋮	⋮	⋮	⋮	⋮
2	0.93750	1.15987	0.93750	2.27442	0.93750	3.28841
2	0.95313	1.22525	0.95313	2.40123	0.95313	3.47111
2	0.96875	1.29314	0.96875	2.53279	0.96875	3.66059
2	0.98438	1.36359	0.98438	2.66918	0.98438	3.85696
2	1.00000	1.43665	1.00000	2.81048	1.00000	4.06035
⋮						
$(n=10)$						
10	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
10	0.01563	0.00000	0.01563	0.00000	0.01563	0.00001
10	0.03125	0.00001	0.03125	0.00003	0.03125	0.00005
10	0.04688	0.00005	0.04688	0.00011	0.04688	0.00019
10	0.06250	0.00014	0.06250	0.00030	0.06250	0.00051
⋮	⋮	⋮	⋮	⋮	⋮	⋮
10	0.93750	1.16069	0.93750	2.44245	0.93750	4.11806
10	0.95313	1.22611	0.95313	2.57853	0.95313	4.34649
10	0.96875	1.29405	0.96875	2.71968	0.96875	4.58336
10	0.98438	1.36455	0.98438	2.86601	0.98438	4.82882
10	1.00000	1.43766	1.00000	3.01760	1.00000	5.08300

can see that Θ_1, Θ_2 and g satisfy the conditions of Theorem 3.4. Thus, the problem (30) has a solution.

5 Conclusion

The $q-$integro-differential boundary equations and their applications represent a matter of high interest in the area of fractional $q-$calculus due to their various applications in areas of science and technology. Indeed, the $q-$integro-differential boundary value problems often occur in mathematical modeling of a variety of physical operations. In this context, we prove the existence of a solution for a new class of the singular $q-$integro-differential Eqsuations (19) and (30) on a time scale. In addition to, we prove the main results in context of completely continuous functions and with the help of the Lebesgue dominated theorem. Examples are presented and MATLAB algorithms ([15]) are implemented to demonstrate the validity of the proposed results. The results are verified by constructing two examples along with their numerical simulations that demonstrated perfect consistency with the theoretical findings. To this end, the authors investigated a complicated case by utilizing an appropriate basic theory which facilitates a particular interest in this paper.

Acknowledgements
The authors express their gratitude to the editor of the journal for their helpful comments and the referees for their helpful suggestions which improved final version of this paper.

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Supplement

Algorithm 9: MATLAB lines for calculation m_0 in Eq (21).

```
function m_0 = qm0(q,sigma,eta,t, n)
S=0;
for k=0:n
    p1=1;
    for i=0:n
        p1=p1*(1-q^(k+i))/(1-q^(k+sigma-eta-1+i));
    end;
    p1=p1*q^k*gx(t*q^k);
    S=S+p1;
end;
S=(1-q)*S;
m_0=round(S, 6);
```

```
end
```

Algorithm 10: MATLAB lines for calculation $g(x)$ in Example 4.1.

```
function g_x = gx(t)
p_i=[1/8 3/7 5/7 9/10];
z=0;
if ((t >0) && (t ≤1/4))
    z= 1/t^(p_i(1));
end;
if ((t >1/4) && (t ≤1/2))
    z= 1/((t-1/4)^(p_i(2)));
end;
if ((t >1/2) && (t ≤3/4))
    z= 1/((t-1/2)^(p_i(3)));
end;
if ((t >3/4) && (t <1))
    z= 1/((t-3/4)^(p_i(4)));
end;
g_x=z;
end
```

Algorithm 11: MATLAB lines for calculation λ, M_1, M_2, m_0 in Example 4.1.

```
format long;
q=[1/10 1/2 6/7];
[xq yq]=size(q);
k=120;
sigma=25/9;
tau=8/9;
eta=5/7;
column=1;
for s=1:yq
    for n=1:k
        paramsmatrix(n, column)=n;
        C1=qGamma(q(s),2-eta,n);
        lambda =round( 1-tau^(1-eta)/C1, 6);
        paramsmatrix(n,column+1)=lambda;
        M1=1/qGamma(q(s),sigma,n) ...
        + 1/(lambda*qGamma(q(s),sigma,n)) ...
        + 1/(lambda*qGamma(q(s),sigma-eta,n));
```

```
        M2=1/qGamma(q(s),sigma-1,n) ...
        + 1/(lambda*qGamma(q(s),sigma,n)) ...
        + 1/(lambda*qGamma(q(s),sigma-eta,n));
        paramsmatrix(n, column+2)=M1;
        paramsmatrix(n, column+3)=M2;
        C2=qm0(q(s),sigma,eta,T,n);
        paramsmatrix(n, column+4)=C2;
    end;
    column=column +5;
end;
```

Algorithm 12: MATLAB lines for Example 4.1.

```
format long;
q=[1/10 1/2 6/7];
[xq yq]=size(q);
k=20;
ti=[1/8 4/11 5/9 16/19];
[xti yti]=size(ti);
sigma = [25/9 1 9/14 0];
t0 = 0; T = 1/4;
y0 = [0 , 0, 0, 0];
h = 2^(-6);
column=1;
s=1;
row=0;
for n=1:k
    A=gx(ti(s));
    B=gx(ti(s));
    C=gx(ti(s));
    D=gx(ti(s));
    lambda = [A B C D];
    f_fun = @(t,y) 1/(t^(1/8));
    J_fun = @(t,y) -1/(8*t^(9/8));
    [t, y]= mt_fde_pi1_im(sigma,lambda,f_fun,J_fun, ...
    t0,T,y0,h);
    [xt yt]=size(t);
    for j=1:yt
        paramsmatrix(row+j, column)=abs(n);
        paramsmatrix(row+j, column+1) = round(abs(A), 5);
        paramsmatrix(row+j, column+2) = round(abs(B), 5);
        paramsmatrix(row+j, column+3) = round(abs(C), 5);
        paramsmatrix(row+j, column+4) = round(abs(D), 5);
```

```
        paramsmatrix(row+j, column+5) = round(abs(t(j)), 5);
        paramsmatrix(row+j, column+6) = round(abs(y(j)), 5);
    end;
    row=row+yt;
end;
column=column+7;
t0 = 1/4; T = 1/2;
s=2;
row=0;
for n=1:k
    A=gx(ti(s));
    B=gx(ti(s));
    C=gx(ti(s));
    D=gx(ti(s));
    lambda = [A B C D];
    f_fun = @(t,y) 1/((t-1/4)^(3/7));
    J_fun = @(t,y) -3/(7*(t - 1/4)^(10/7));
    [t, y]= mt_fde_pi1_im(sigma,lambda,f_fun,J_fun, ...
    t0,T,y0,h);
    [xt yt]=size(t);
    for j=1:yt
        paramsmatrix(row+j, column)=abs(n);
        paramsmatrix(row+j, column+1) = round(abs(A), 5);
        paramsmatrix(row+j, column+2) = round(abs(B), 5);
        paramsmatrix(row+j, column+3) = round(abs(C), 5);
        paramsmatrix(row+j, column+4) = round(abs(D), 5);
        paramsmatrix(row+j, column+5) = ...
            round(abs(t(j))+t0, 5);
        paramsmatrix(row+j, column+6) = round(abs(y(j)), 5);
    end;
    row=row+yt;
end;
column=column+7;
t0 = 1/2; T = 3/4;
s=3;
row=0;
for n=1:k
    A=gx(ti(s));
    B=gx(ti(s));
    C=gx(ti(s));
    D=gx(ti(s));
    lambda = [A B C D];
    f_fun = @(t,y) 1/((t-1/7)^(5/7));
    J_fun = @(t,y) -5/(7*(t - 1/7)^(12/7));
    [t, y]= mt_fde_pi1_im(sigma,lambda,f_fun,J_fun, ...
```

```
    t0,T,y0,h);
    [xt yt]=size(t);
    for j=1:yt
        paramsmatrix(row+j, column)=abs(n);
        paramsmatrix(row+j, column+1) = round(abs(A), 5);
        paramsmatrix(row+j, column+2) = round(abs(B), 5);
        paramsmatrix(row+j, column+3) = round(abs(C), 5);
        paramsmatrix(row+j, column+4) = round(abs(D), 5);
        paramsmatrix(row+j, column+5) = ...
            round(abs(t(j))+t0, 5);
        paramsmatrix(row+j, column+6) = round(abs(y(j)), 5);
    end;
    row=row+yt;
end;
column=column+7;
t0=3/4; T=1;
s=4;
row=0;
for n=1:k
    A=gx(ti(s));
    B=gx(ti(s));
    C=gx(ti(s));
    D=gx(ti(s));
    lambda = [A B C D];
    f_fun = @(t,y) 1/((t-3/4)^(9/10));
    J_fun = @(t,y) -9/(10*(t - 3/4)^(19/10));
    [t, y]= mt_fde_pi1_im(sigma,lambda,f_fun,J_fun, ...
    t0,T,y0,h);
    [xt yt]=size(t);
    for j=1:yt
        paramsmatrix(row+j, column)=abs(n);
        paramsmatrix(row+j, column+1) = round(abs(A), 5);
        paramsmatrix(row+j, column+2) = round(abs(B), 5);
        paramsmatrix(row+j, column+3) = round(abs(C), 5);
        paramsmatrix(row+j, column+4) = round(abs(D), 5);
        paramsmatrix(row+j, column+5) = ...
            round(abs(t(j))+t0, 5);
        paramsmatrix(row+j, column+6) = round(abs(y(j)), 5);
    end;
row=row+yt;
end;
```

Algorithm 13: MATLAB lines for calculation λ, M_1, M_2, ε_0 and $\Gamma_q(2-\zeta)$ in

Example 4.2.

```
format long;
q=[1/8 1/2 9/13];
[xq yq] = size(q);
k=120;
sigma =8/3;
zeta=4/15;
tau=5/8;
eta=6/11;
column=1;
for s=1:yq
    for n=1:k
        paramsmatrix(n,column)=n;
        C1=qGamma(q(s),2-eta, n);
        lambda =round(1-tau^(1- eta)/C1, 6);
        paramsmatrix(n,column+1)=lambda;
        M1=1/qGamma(q(s),sigma,n)+ 1/(lambda * ...
            qGamma(q(s), sigma, n))+ 1/(lambda * ...
            qGamma(q(s), sigma-eta, n));
        M2=1/qGamma(q(s),sigma-1,n)+ 1/(lambda * ...
            qGamma(q(s), sigma, n))+ 1/(lambda * ...
            qGamma(q(s), sigma-eta, n));
        paramsmatrix(n, column+2)=M1;
        paramsmatrix(n, column+3)=M2;
        C2=max(M1, M2);
        paramsmatrix(n, column+4)=round(1/C2, 6);
        paramsmatrix(n, column+5)=round(1/qGamma(q(s), ...
            2-zeta, n), 6);
    end;
    column=column +6;
end;
```

Algorithm 14: MATLAB lines for Example 4.2.

```
format long;
q=[1/8 1/2 9/13];
[xq yq]=size(q);
k=10;
sigma =8/3;
zeta=4/15;
tau=5/8;
eta=6/11;
```

```
m=5/4;
t0 = 0; T = 1;
y0 = [0, 0, 0, 0];
h = 2^(-6);
column=1;
s=1;
row=0;
for n=1:k
    A=1;
    B=8/5;
    C=1/10;
    D=1/2;
    lambda = [A B C D];
    gam=qGamma(q(s), 4/3, n);
    syms x;
    f=1/(1+x);
    f_fun = @(t,y) -15*int(f, t0, T)/6 -6*(t^2 + gam);
    J_fun = @(t,y) -12*t;
    alpha=[10/3 1 4/15 0];
    t, y]= ...
        mt_fde_pi1_im(alpha,lambda,f_fun,J_fun,t0,T,y0,h);
    [xt yt]=size(t);
    for j=1:yt
        paramsmatrix(row+j, column)=abs(n);
        paramsmatrix(row+j, column+1) = round(gam, 5);
        paramsmatrix(row+j, column+2) = round(abs(t(j)), 5);
        paramsmatrix(row+j, column+3) = round(abs(y(j)), 5);
        end;
        row=row+yt;
    end;
    column=column+4;
s=2;
row=0;
for n=1:k
    A=1;
    B=8/5;
    C=1/10;
    D=1/2;
    lambda = [A B C D];
    gam=qGamma(q(s), 4/3, n);
    syms x;
    f=1/(1+x);
    f_fun = @(t,y) -15*int(f, t0, T)/6 -6*(t^2 + gam);
    J_fun = @(t,y) -12*t;
    alpha=[10/3 1 4/15 0];
```

```
    [t, y]= ...
        mt_fde_pi1_im(alpha,lambda,f_fun,J_fun,t0,T,y0,h);
    [xt yt]=size(t);
    for j=1:yt
        paramsmatrix(row+j, column)=abs(n);
        paramsmatrix(row+j, column+1) = round(gam, 5);
        paramsmatrix(row+j, column+2) = round(abs(t(j)), 5);
        paramsmatrix(row+j, column+3) = round(abs(y(j)), 5);
    end;
    row=row+yt;
end;
column=column+4;
s=3;
row=0;
for n=1:k
    A=1;
    B=8/5;
    C=1/10;
    D=1/2;
    lambda = [A B C D];
    gam=qGamma(q(s), 4/3, n);
    syms x;
    f=1/(1+x);
    f_fun = @(t,y) -15*int(f, t0, T)/6 -6*(t^2 + gam);
    J_fun = @(t,y) -12*t;
    alpha=[10/3 1 4/15 0];
    [t, y]= ...
        mt_fde_pi1_im(alpha,lambda,f_fun,J_fun,t0,T,y0,h);
    [xt yt]=size(t);
    for j=1:yt
        paramsmatrix(row+j, column)=abs(n);
        paramsmatrix(row+j, column+1) = round(gam, 5);
        paramsmatrix(row+j, column+2) = round(abs(t(j)), 5);
        paramsmatrix(row+j, column+3) = round(abs(y(j)), 5);
    end;
row=row+yt;
end;
```