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Original Research Paper

On a K-Dimensional System of Hybrid Fractional Differential Equations with Multi-Point Boundary Conditions

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Abstract. The fractional Sturm-Liouville equations have considerable role applications in some different phenomena such as mechanical and electrical engineering, medicine and physics. Thus, it is good we review different versions of this equation. We study a k-dimensional system of Sturm-Liouville hybrid equations by using the α-admissible method. We investigate the existence of solutions for the k-dimensional system of hybrid equations with some multi-point boundary value conditions. We provide an example to illustrate our main result.

AMS Subject Classification: 34A08; 34A12.
Keywords and Phrases: α-ψ-contraction, Fractional hybrid version, Multi-point boundary condition, The system of Sturm-Liouville equations.

1 Introduction

Human life at this time has become inextricably linked to mathematics and has improved people's living standards. Mathematics has also found its applications in various sciences such as laboratory sciences, chemistry, physics, and engineering. During last years, researchers have studied the complex fractional differential equations which increase their

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ability to model most real-world phenomena. Among the fractional differential equations that are widely used in engineering, physics and wave and quantum theory is the Strom-Liouville fractional differential equation. ([20, 33]). Over the past twenty years, researchers have paid close attention to examining the existence of solutions for fractional differential equations with different boundary conditions (see for examples, [3, 4, 5, 12, 15, 17, 18, 21, 22, 23, 24, 25, 26, 27, 32, 35, 36, 37]).

New and advanced models of different events are being studied and developed by researchers in mathematics by using fractional differential equations with specific or general boundary conditions (see for examples, [2, 6, 7, 10, 11, 29]). In recent years, systems of hybrid differential equations and non-hybrid systems with different hybrid and non-hybrid boundary conditions have been considered by researchers ([1, 8, 9, 13, 14, 19, 34, 38]).

As we know, the fractional Caputo derivative of order $b-1 \leq \varrho < b$ for the function v is defined by

$$D^{\alpha} v(t) = I^{b-\varrho} \frac{d^b}{dt^b} v(r) = \int_0^r \frac{(r-s)^{b-\varrho-1}}{\Gamma(b-\varrho)} \frac{d^b v(s)}{dt^b} ds$$

and the Riemann-Liouville fractional integral of order $\varrho > 0$ for a function $v \in L^1[0, K]$ is given by $I^{\varrho} v(r) = \int_0^r \frac{(r-s)^{\varrho-1}}{\Gamma(\varrho)} v(s) ds$ (see [28, 31]).

In 2011, Zhao et al. studied the fractional problem ${}^c D^{\varrho}\left(\frac{v(r)}{l(r, v(r))}\right) = h(r, v(r))$ with boundary initial condition $v(0) = 0$, where $0 < \varrho < 1$, ${}^c D^{\varrho}$ denotes the Caputo fractional derivative, $l \in C(I \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and $h \in C(I \times \mathbb{R}, \mathbb{R})$ ([38]). In 2019, the Sturm-Liouville problem ${}^c D^{\varrho}(m(r) v'(r)) + p(r) v(r) = f(r) h(v(r))$ via the multi-point boundary conditions $v'(r) = 0$, $\sum_{i=1}^{u} \zeta_i v(a_i) = \tau \sum_{j=1}^{n} \phi_j v(z_j)$ investigated, where $\varrho \in (0, 1]$, ${}^c D^{\varrho}$ denotes the fractional Caputo derivative, $m \in C^1(I, \mathbb{R})$, $p(r)$ and $f(r)$ are absolutely continuous functions on $I = [0, K]$ with $K > 0$, $m(r) \neq 0$ for all $r \in I$, $f : \mathbb{R} \to \mathbb{R}$ is differentiable on the interval I, $0 \leq a_1 < a_2 < \cdots < a_u < c$, $d \leq z_1 < z_2 < \cdots < z_n < K$, $c < d$ and $\zeta_1, \ldots, \zeta_u$, $\tau_1, \ldots, \tau_n$ and ρ are real constants ([16]).

Let $\varrho \in (0,1)$, ${}^cD^\varrho$ is the Caputo fractional derivative of order ϱ, $I = [0,K]$ with $K < \infty$, $m, \tilde{m} \in C^1(I,\mathbb{R})$, $\tilde{m}(r)$, $p(r)$ and $f(r)$ are absolutely continuous functions on I with $m(r) \neq 0$ for all $r \in I$, $h, \tilde{h} : \mathbb{R} \to \mathbb{R}$ are defined and differentiable on the interval I and $0 \le a_1 < a_2 < \cdots < a_u < c$, $d \le z_1 < z_2 < \cdots < z_n < K$, $c < d$ and $\zeta_1, \ldots, \zeta_u$, $\tau_1, \ldots, \tau_n$ and ρ are real constants with $\sum_{i=1}^{u} \zeta_i - \rho \sum_{j=1}^{n} \tau_j \neq 0$. Now by mixing the ideas in these works and main idea of [13], we review the k-dimensional hybrid differential system

$$\begin{cases} {}^cD_1^\varrho\Big(m_1(r)\Big(\dfrac{v_1(r)}{l_1(r,v_1(r))}\Big)' - \tilde{m}_1(r)\tilde{h}_1(v_1(r))\Big) \\ +q_1(r)v_1(r) = f_1(r)h_1(v_1(r)), \\ {}^cD_2^\varrho\Big(m_2(r)\Big(\dfrac{v_2(r)}{l_2(r,v_2(r))}\Big)' - \tilde{m}_2(r)\tilde{h}_2(v_2(r))\Big) \\ +q_2(r)v_2(r) = f_2(r)h_2(v_2(r)), \\ . \\ {}^cD_k^\varrho\Big(m_k(r)\Big(\dfrac{v_k(r)}{l_k(r,v_k(r))}\Big)' - \tilde{m}_k(r)\tilde{h}_k(v_2(r))\Big) \\ +q_k(r)v_k(r) = f_k(r)h_k(v_k(r)),\ (r \in I), \end{cases} \tag{1}$$

with the sigma boundary value conditions

$$\begin{cases} \Big(\dfrac{v_i(t)}{l_i(r,v_i(r))}\Big)'_{r=0} = \Big(\dfrac{\tilde{m}_i(r)}{m_i(t)}\tilde{h}_i(v_i(r))\Big)_{r=0}, \qquad (1 \le i \le k) \\ \displaystyle\sum_{i=1}^{u} \zeta_i\Big(\frac{v_i(a_i)}{l_i(a_i,u_i(a_i))}\Big) = \rho_i \sum_{j=1}^{n} \tau_j \Big(\frac{v_i(z_j)}{l_i(z_j,v_i(z_j))}\Big). \end{cases} \tag{2}$$

Let I be an interval in $\mathbb{R}$.Consider the space $W = C(I,\mathbb{R})$ via the norm $\|w\| = \sup_{r \in I} |w(t)|$ and the norm $\|w\| = \int_0^K |w(s)|ds$ on $L_1[0,K]$, where $|w(t)|$ is the usual norm on $\mathbb{R}^n$. Consider the Banach product space $W^k = \big(W \times W \times ... \times W, \|.\|_*\big)$ with the norm $\|w_1, w_2, ..., w_k\| = \max\{\|w_1\|, \|w_2\|, ..., \|w_k\|\}$. The Riemann-Liouville fractional integral of order ϱ for a function h is defined by $I^\varrho h(r) = \frac{1}{\Gamma(\varrho)}\int_0^r (r-s)^{\varrho-1}h(s)ds$ ($\varrho > 0$) and the Caputo derivative of order ϱ for a function h is defined by ${}^cD^\varrho h(r) = I^{n-\varrho}\frac{d^n}{dr^n}h(r) = \frac{1}{\Gamma(n-\varrho)}\int_0^r \frac{h^{(n)}(s)}{(r-s)^{\varrho-n+1}}ds$, where $n = [\varrho]+1$ ([28], [31]). Assume that Ψ is a family of non-descending functions $\psi : [0,+\infty) \to [0,+\infty)$ such that $\sum_{n=1}^{\infty}\psi^n(t) < +\infty$ for all $r > 0$, where ψ^n is the n-th iterate of ψ. Let $K : W \to W$ be a selfmap and $\alpha : W \times W \to [0,+\infty)$ a function. We say that K is α-admissible whenever $\alpha(w,x) \geq 1$ implies $\alpha(Kw,Kx) \geq 1$ ([30]). Let $\psi \in \Psi$ and $\alpha : X \times X \to [0,+\infty)$ be a map. A self-map $K : W \to W$ is called an α-ψ-contraction whenever $\alpha(w,x)d(Kw,Kx) \leq \psi(d(w,x))$ for all $w, x \in W$ ([30]). We need next result.

Lemma 1.1. *[30] Suppose that (W,d) is a complete metric space, $\psi \in \Psi$, $\alpha : X \times X \to [0,+\infty)$ is a map and $K : W \to W$ is an α-admissible α-ψ-contraction. Assume that there exists $w_0 \in W$ such that $\alpha(w_0, Kw_0) \geq 1$ and $\alpha(w_n, w) \geq 1$ for all n whenever $\{w_n\}$ is a sequence in W such that $\alpha(w_{n-1}, w_n) \geq 1$ for all $n \geq 1$ and $w_n \to w$. Then K has a fixed point.*

2 Main Results

To study the problem (1)-(2), we consider the following assumptions.

(A_1) The maps $h_1, \ldots, h_k, \tilde{h}_i, \ldots \tilde{h}_k, : \mathbb{R} \to \mathbb{R}$ are bounded are differentiable on $[0, K]$ and the functions $\frac{\partial h_1}{\partial t}, \ldots, \frac{\partial h_k}{\partial t}$ and $\frac{\partial \tilde{h}_1}{\partial t}, \ldots, \frac{\partial \tilde{h}_k}{\partial t}$ are bounded on $[0, K]$ with $|\frac{\partial h_i}{\partial v_i}| \leq \mathcal{S}$ and $|\frac{\partial \tilde{h}_i}{\partial v_i}| \leq \tilde{\mathcal{S}}$ for all $i = 1, \ldots, k$ and two constants $\mathcal{S}$ and $\tilde{\tilde{\mathcal{S}}}$.

(A_2) The map $m_1, \ldots .m_k \in C^1(I, \mathbb{R})$ have this property that $m_i(r) \neq 0$ for all r and $\inf_{r\in I}|m_i(r)| = m_i$ for all $i = 1, \ldots, k$. Also, $\tilde{m}_i(r)$, $p_i(r)$ and $h_i(r)$ are absolutely continuous functions on I for all $i = 1, \ldots, k$.

(A_3) The functions $l_1, \dots, l_k : I \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$ are continuous in the two variables and there are mappings $\xi_1, \dots, \xi_k \geq 0$ such that $|l_i(r,w) - l_i(r,x)| \leq \xi_i(r)|w-x|$ for all (r,w,x) in $I \times \mathbb{R} \times \mathbb{R}$ and $i = 1, \dots, k$.

(A_4) There exists a real number $t > 0$ such that

$$(\|\xi_i\|t + l_{i,0})(\otimes_{i,1}t + \otimes_{i,2}) \leq t \text{ and } (2\otimes_{i,1}t + \otimes_{i,2})\|\xi_i\| + l_0\otimes_{i,1} < 1,$$

where

$$\otimes_{i,1} = \frac{K}{m_i}\left(\tilde{\mathcal{S}}\|\tilde{m_i}\| + \frac{K^{\varrho_i}(\|p_i\| + \mathcal{S}\|h_i\|)}{\Gamma(\varrho_i+2)}\right)\left(|B|\big(\sum_{i=1}^{u}|\zeta_i| + |\rho_i|\sum_{j=1}^{n}|\tau_j|\big) + 1\right),$$

$$\otimes_{i,2} = \frac{K}{m_i}\left(\tilde{h_0}\|\tilde{m_i}\| + \frac{K^{\varrho_i}\|h_i\|h_0}{\Gamma(\varrho_i+2)}\right)\left(|B|\left(\sum_{i=1}^{u}|\zeta_i| + |\rho_i|\sum_{j=1}^{n}|\tau_j|\right) + 1\right),$$

$h_{i,0} = |h_i(0)|$, $\tilde{h}_{i,0} = |\tilde{h}_i(0)|$ and $l_{i,0} = \sup_{r\in I} l_i(r,0)$ for $1 \leq i \leq k$.

Now, we provide our main result.

Theorem 2.1. *Assume that the assumptions (A_1)-(A_2) hold. Then, the hybrid system (1) with boundary conditions (2) has a solution $v = (v_1, \dots, v_n)$, where*

$$\begin{aligned}
v_i(r) = l_i(r, v_i(r))\Bigg[&B\rho_i\sum_{j=1}^{n}\tau_j\int_0^{z_j}\frac{\tilde{m}_i(s)}{m_i(s)}\tilde{h}_i(v_i(s))ds \\
&- B\sum_{i=1}^{u}\zeta_i\int_0^{a_i}\frac{\tilde{m}_i(s)}{m_i(s)}\tilde{h}_i(v_i(s))ds \\
&+ B\sum_{i=1}^{u}\zeta_i\int_0^{a_i}\frac{1}{m_i(s)}I_i^{\varrho}(p_i(s)v_i(s))ds - B\rho\sum_{j=1}^{n}\tau_j\int_0^{z_j}\frac{1}{m_i(s)}I_i^{\varrho}(p_i(s)v_i(s))ds \\
&+ B\rho_i\sum_{j=1}^{n}\tau_j\int_0^{z_j}\frac{1}{m_i(s)}I_i^{\varrho}(h_i(s)h_i(v_i(s)))ds \\
&- B\sum_{i=1}^{u}\zeta_i\int_0^{a_i}\frac{1}{m_i(s)}I_i^{\varrho}(h_i(s)h_i(v_i(s)))ds \\
&+ \int_0^{r}\frac{\tilde{m}_i(s)}{m_i(s)}\tilde{h}_i(v_i(s))ds - \int_0^{r}\frac{1}{m_i(s)}I_i^{\varrho}(p_i(s)v_i(s))ds \\
&+ \int_0^{r}\frac{1}{m_i(s)}I_i^{\varrho}(h_i(s)h_i(v_i(s)))ds\Bigg],
\end{aligned} \tag{3}$$

for all $i = 1, ..., k$ *and* $B = \dfrac{1}{\sum_{i=1}^{u} \zeta_i - \rho_i \sum_{j=1}^{n} \tau_j}$. *Also,*

$$\frac{v_i}{l_i(r, v_i(r))} \in C^1(I, \mathbb{R})$$

and $\left(\dfrac{v_i(r)}{l_i(r, v_i(r))}\right)'' \in L_1(I, \mathbb{R})$ *for all* i. *If* $(l_i(r, v_i(r)))' \in C(I, \mathbb{R})$, *then* $v_i \in C^1(I, \mathbb{R})$ $(i = 1, \ldots, n)$.

Proof. Define the map $\Delta_k : W^k \to W^k$ by

$$\Delta_k v_k(r) = \Big(l_1(r, v_1(r))H_1 v_1(r), \ldots, l_k(r, v_k(r))H_k v_k(r)\Big),$$

where

$$\begin{aligned}
H_i v_i(r) &= B\rho_i \sum_{j=1}^{n} \tau_j \int_0^{z_j} \frac{\tilde{m}_i(s)}{m_i(s)} \tilde{h}_i(v_i(s))ds - B \sum_{i=1}^{u} \zeta_i \int_0^{a_i} \frac{\tilde{m}_i(s)}{m_i(s)} \tilde{h}_i(v_i(s))ds \\
&+ B \sum_{i=1}^{u} \zeta_i \int_0^{a_i} \frac{1}{m_i(s)} I_i^{\varrho}(p_i(s)v_i(s))ds - B\rho_i \sum_{j=1}^{n} \tau_j \int_0^{z_j} \frac{1}{m_i(s)} I_i^{\varrho}(p_i(s)v_i(s))ds \\
&+ B\rho_i \sum_{j=1}^{n} \tau_j \int_0^{z_j} \frac{1}{m_i(s)} I_i^{\varrho}(f_i(s)h_i(v_i(s)))ds \\
&- B \sum_{i=1}^{u} \zeta_i \int_0^{a_i} \frac{1}{m_i(s)} I_i^{\varrho}(f_i(s)h_i(v_i(s)))ds \\
&+ \int_0^{r} \frac{\tilde{m}_i(s)}{m_i(s)} \tilde{h}_i(v_i(s))ds - \int_0^{r} \frac{1}{m_i(s)} I_i^{\varrho}(p_i(s)v_i(s))ds \\
&+ \int_0^{t} \frac{1}{m_i(s)} I_i^{\varrho}(f_i(s)h_i(v_i(s)))ds.
\end{aligned}$$

for all $i = 1, ..., k$. In accordance with (A_4), there exists $t > 0$ such that

$$(\|\xi\|t + l_{i,0})(\mathcal{A}_{i,1}t + \otimes_{i,2}) \leq t \text{ and } (2\otimes_{i,1}t + \otimes_{i,2})\|\xi\| + l_{i,0}\otimes_{i,1} < 1.$$

Consider the closed ball E_t, where $E_t = \{v_i \in W^k : \|v_i\| \leq t\}$. it is clear that E_t is bounded and closed subset of W^k. Define the function $\varrho : W^k \times W^k \to [0, \infty)$ by $\varrho(v_i, q_i) = 1$ whenever $v_i, q_i \in E_t$ and $\varrho(v_i, q_i) = 0$ otherwise. Then, we have

$$\begin{aligned}
|\tilde{h}_i(v_i(s))| &= |\tilde{h}_i(v_i(s)) - \tilde{h}_i(0) + \tilde{h}_i(0)| \leq |\tilde{h}_i(v_i(s)) - \tilde{h}_i(0)| + |\tilde{h}_i(0)| \\
&\leq \widetilde{\mathcal{S}}|v_i(s))| + |\tilde{h}_i(0)| \leq \widetilde{\mathcal{S}}\|v_i\| + \tilde{h_{i,0}},
\end{aligned}$$

$|l_i(s,v_i(s))| \leq \|\xi\|\|v_i\| + l_0$ and $|h_i(v_i(s))| \leq \mathcal{S}\|v_i\| + h_{i,0}$. We show that the Δ_k operator satisfy the conditions of Lemma 1.1. We show that $\|\Delta_k v_k\| \leq t$ whenever $v_i \in E_t$. Let $v_i \in E_t$. Then,

$$|B||\rho_i|\sum_{j=1}^{n}|\tau_j|\int_0^{z_j}\frac{|\tilde{m}_i(s)|}{|m_i(s)|}|\tilde{h}_i(v_i(s))|ds \leq \frac{|B||\rho_i|\|\tilde{m}_i\|(\widetilde{\mathcal{S}}\|v_i\|+\tilde{h_0}_i)\sum_{j=1}^{n}|\tau_j|z_j}{m_i}$$
$$\leq \frac{|B||\rho_i|\|\tilde{m}_i\|(\widetilde{\mathcal{S}}t+\tilde{h_0}_i)\sum_{j=1}^{n}|\tau_j|K}{m_i} \tag{4}$$
$$= \frac{K|B||\rho_i|\|\tilde{m}_i\|\widetilde{\mathcal{S}}\sum_{j=1}^{n}|\tau_j|}{m_i}t + \frac{K|B||\rho|\|\tilde{m}_i\|\tilde{h_0}\sum_{j=1}^{n}|\tau_j|}{m_i}$$

and

$$|B|\sum_{i=1}^{u}|\zeta_i|\int_0^{a_i}\frac{|\tilde{m}_i(s)|}{|m_i(s)|}|\tilde{h}_i(v_i(s))|ds$$
$$\leq \frac{K|B|\|\tilde{m}_i\|\widetilde{\mathcal{S}}\sum_{i=1}^{u}|\zeta_i|}{m_i}t + \frac{K|B|\|\tilde{m}_i\|\tilde{h_0}\sum_{i=1}^{u}|\zeta_i|}{m_i}.$$

Since $I_i^{\varrho}(1) = \int_0^s \frac{(s-\phi)^{\varrho_i-1}}{\Gamma(\varrho_i)}d\phi = \frac{s_i^{\varrho}}{\Gamma(\varrho_i+1)}$, we obtain

$$|B|\sum_{i=1}^{u}|\zeta_i|\int_0^{a_i}\frac{1}{|m_i(s)|}I_i^{\varrho}(|p_i(s)||v_i(s)|)ds$$
$$\leq \frac{|B|\|p_i\|\|v_i\|}{m_i}\sum_{i=1}^{u}|\zeta_i|\int_0^{a_i}(\int_0^s\frac{(s-\phi)^{\varrho_i-1}}{\Gamma(\varrho_i)}d\phi)ds$$
$$\leq \frac{K^{\varrho_i+1}|B|\|p_i\|\sum_{i=1}^{u}|\zeta_i|}{m_i\Gamma(\varrho_i+2)}t$$

and

$$|B||\rho_i|\sum_{j=1}^{n}|\tau_j|\int_0^{z_j}\frac{1}{|m_i(s)|}I_i^{\varrho}(|p_i(s)||v_i(s)|)ds$$
$$\leq \frac{K^{\varrho_i+1}|B||\rho_i|\|p_i\|\sum_{j=1}^{n}|\tau_j|}{m_i\Gamma(\varrho_i+2)}t.$$

Also,

$$|B||\rho_i|\sum_{j=1}^{n}|\tau_j|\int_0^{z_j}\frac{1}{|m_i(s)|}I_i^{\varrho}(|f_i(s)||h_i(v_i(s))|)ds \leq \frac{K^{\varrho_i+1}\mathcal{S}|B||\rho_i|\|f_i\|\sum_{j=1}^{n}|\tau_j|}{m_i\Gamma(\varrho_i+2)}t$$
$$+\frac{K^{\varrho_i+1}|B||\rho_i|\|f_i\|h_{i,0}\sum_{j=1}^{n}|\tau_j|}{m_i\Gamma(\varrho_i+2)}, |B|\sum_{i=1}^{u}|\zeta_i|\int_0^{a_i}\frac{1}{|m_i(s)|}I_i^{\varrho}(|f_i(s)||h_i(v_i(s))|)ds$$
$$\leq \frac{K^{\varrho_i+1}\mathcal{S}|B|\|f_i\|\sum_{i=1}^{u}|\zeta_i|}{m_i\Gamma(\varrho_i+2)}t + \frac{K^{\varrho_i+1}|B|\|f_i\|h_{i,0}\sum_{i=1}^{u}|\zeta_i|}{m_i\Gamma(\varrho_i+2)},$$

$$\int_0^r \frac{|\tilde{m}_i(s)|}{|m_i(s)|}|\tilde{h}_i(v_i(s))|ds \leq \frac{K\widetilde{\mathcal{S}}\|\tilde{m}_i\|}{m_i}t + \frac{K\tilde{h_{i,0}}\|\tilde{m}_i\|}{m_i},$$

$$\int_0^r \frac{1}{|m_i(s)|}I_i^{\varrho}(|p_i(s)||v_i(s)|)ds \leq \frac{K^{\varrho_i+1}\|p_i\|}{m_i\Gamma(\varrho_i+2)}t$$

and

$$\int_0^r \frac{1}{|m_i(s)|}I_i^{\varrho}(|f_i(s)||h_i(v_i(s))|)ds \leq \frac{K^{\varrho_i+1}\mathcal{S}\|f_i\|}{m_i\Gamma(\varrho_i+2)}t + \frac{K^{\varrho_i+1}\|f_i\|h_{i,0}}{m_i\Gamma(\varrho_i+2)}. \quad (5)$$

Since

$$\begin{aligned}
&|Hv_i(r)| \leq |B||\rho_i|\sum_{j=1}^{n}|\tau_j|\int_0^{z_j}\frac{|\tilde{m}_i(s)|}{|m_i(s)|}|\tilde{h}_i(v_i(s))|ds \\
&+ |B|\sum_{i=1}^{u}|\zeta_i|\int_0^{a_i}\frac{|\tilde{m}_i(s)|}{|m_i(s)|}|\tilde{h}_i(v_i(s))|ds \\
&+ |B|\sum_{i=1}^{u}|\zeta_i|\int_0^{a_i}\frac{1}{|m_i(s)|}I_i^{\varrho}(|p_i(s)||v_i(s)|)ds \\
&+ |B||\rho_i|\sum_{j=1}^{n}|\tau_j|\int_0^{z_j}\frac{1}{|m_i(s)|}I_i^{\varrho}(|p_i(s)||v_i(s)|)ds \\
&+ |B||\rho_i|\sum_{j=1}^{n}|\tau_j|\int_0^{z_j}\frac{1}{|m_i(s)|}I_i^{\varrho}(|f_i(s)||h_i(v_i(s))|)ds \\
&+ |B|\sum_{i=1}^{u}|\zeta_i|\int_0^{a_i}\frac{1}{|m_i(s)|}I_i^{\varrho}(|f_i(s)||h_i(v_i(s))|)ds \\
&+ \int_0^r\frac{|\tilde{m}_i(s)|}{|m_i(s)|}|\tilde{h}_i(v_i(s))|ds + \int_0^r\frac{1}{|m_i(s)|}I_i^{\varrho}(|p_i(s)||v_i(s)|)ds \\
&+ \int_0^r\frac{1}{|m_i(s)|}I_i^{\varrho}(|f_i(s)||h_i(v_i(s))|)ds,
\end{aligned}$$

by using (4)-(5), we get $|Hv_i(r)| \leq \otimes_{i,1}t + \otimes_{i,2}$, where

$$\begin{aligned}
\otimes_{i,1} &= \frac{K|B||\rho_i|\|\tilde{m}_i\|\widetilde{\mathcal{S}}\sum_{j=1}^{n}|\tau_j|}{m_i} + \frac{K|B|\|\tilde{m}_i\|\widetilde{\mathcal{S}}\sum_{i=1}^{u}|\zeta_i|}{m_i} \\
&+ \frac{K^{\varrho_i+1}|B|\|p_i\|\sum_{i=1}^{u}|\zeta_i|}{m_i\Gamma(\varrho_i+2)} \\
&+ \frac{K^{\varrho_i+1}|B||\rho_i|\|p_i\|\sum_{j=1}^{n}|\tau_j|}{m_i\Gamma(\varrho_i+2)} + \frac{K^{\varrho_i+1}\mathcal{S}B||\rho_i|\|f_i\|\sum_{j=1}^{n}|\tau_j|}{m_i\Gamma(\varrho_i+2)} \\
&+ \frac{K^{\varrho_i+1}\mathcal{S}B|\|f_i\|\sum_{i=1}^{u}|\zeta_i|}{m_i\Gamma(\varrho_i+2)} \\
&+ \frac{K\widetilde{\mathcal{S}}\|\tilde{m}_i\|}{m_i} + \frac{K^{\varrho_i+1}\|p_i\|}{m_i\Gamma(\varrho_i+2)} + \frac{K^{\varrho_i+1}\mathcal{S}\|f_i\|}{m_i\Gamma(\varrho_i+2)} \\
&= \frac{K\|\tilde{m}_i\|\widetilde{\mathcal{S}}}{m_i}\left(|B|(\sum_{i=1}^{u}|\zeta_i| + |\rho_i|\sum_{j=1}^{n}|\tau_j|) + 1\right) \\
&+ \frac{K^{\varrho_i+1}}{m\Gamma(\varrho_i+2)}\left[\left(|B|(\sum_{i=1}^{u}|\zeta_i| + |\rho_i|\sum_{j=1}^{n}|\tau_j|) + 1\right)(\|p_i\| + \mathcal{S}\|f_i\|)\right] \\
&= \frac{K}{m_i}\left(\widetilde{\mathcal{S}}\|\tilde{m}_i\| + \frac{K^{\varrho_i}(\|p_i\| + \mathcal{S}\|f_i\|)}{\Gamma(\varrho_i+2)}\right)\left(|B|(\sum_{i=1}^{u}|\zeta_i| + |\rho_i|\sum_{j=1}^{n}|\tau_j|) + 1\right)
\end{aligned}$$

and

$$\otimes_{i,2} = \frac{K}{m_i}\left(\tilde{h_{0,i}}\|\tilde{m}_i\| + \frac{K^{\varrho_i}\|f_i\|h_{0,i}}{\Gamma(\varrho_i+2)}\right)\left(|B|\left(\sum_{i=1}^{u}|\zeta_i| + |\rho_i|\sum_{j=1}^{n}|\tau_j|\right) + 1\right).$$

Thus,

$$|\Delta_i v_i(r)| = |l_i(r, v_i(r))||Hv_i(r)| \leq (\|\xi\|t + l_{0,i})(\otimes_{i,1}t + \otimes_{i,2}) \leq t$$

and so $\|\Delta_k v_k\| \le t$ and so $\Delta_k E_t \subseteq E_t$. Assume that $v_i, q_i \in E_t$. Similar to above proofs, we can conclude that

$$|B||\rho_i| \sum_{j=1}^n |\tau_j| \int_0^{z_j} \frac{|\tilde{m}_i(s)|}{|m_i(s)|} |\tilde{h}_i(v_i(s)) - \tilde{h}_i(q_i(s))| ds$$
$$\le \frac{K|B||\rho_i|\|\tilde{m}_i\|\tilde{S}\sum_{j=1}^n |\tau_j|}{m_i} \|v_i - q_i\|,$$
$$|B| \sum_{i=1}^u |\zeta_i| \int_0^{a_i} \frac{|\tilde{m}_i(s)|}{|m_i(s)|} |\tilde{h}_i(v_i(s)) - \tilde{h}_i(q_i(s))| ds \le$$
$$|B| \sum_{i=1}^u |\zeta_i| \int_0^{a_i} \frac{1}{|m_i(s)|} I_i^{\varrho}(|p_i(s)||v_i(s) - q_i(s)|) ds \le \frac{K^{\varrho+1}|B|\|p_i\| \sum_{i=1}^u |\zeta_i|}{m_i \Gamma(\varrho_i + 2)} \|v_i - q_i\|,$$
$$|B||\rho_i| \sum_{j=1}^n |\tau_j| \int_0^{z_j} \frac{1}{|m_i(s)|} I_i^{\varrho}(|p_i(s)||v_i(s) - v_i(s)|) ds$$
$$\le \frac{K^{\varrho_i+1}|B||\rho_i|\|p_i\| \sum_{j=1}^n |\tau_j|}{m_i \Gamma(\varrho_i + 2)} \|v_i - q_i\|,$$
$$|B||\rho_i| \sum_{j=1}^n |\tau_j| \int_0^{z_j} \frac{1}{|m_i(s)|} I_i^{\varrho}(|f_i(s)||h_i(v_i(s)) - h_i(q_i(s))|) ds$$
$$\le \frac{K^{\varrho_i+1}\mathcal{S}|B||\rho_i|\|f_i\| \sum_{j=1}^n |\tau_j|}{m_i \Gamma(\varrho_i + 2)} \|v_i - q_i\|,$$
$$|B| \sum_{i=1}^u |\zeta_i| \int_0^{a_i} \frac{1}{|m_i(s)|} I_i^{\varrho}(|f_i(s)||h_i(v_i(s)) - h_i(q_i(s))|) ds$$
$$\le \frac{K^{\varrho_i+1}\mathcal{S}|B|\|f_i\| \sum_{i=1}^u |\zeta_i|}{m_i \Gamma(\varrho_i + 2)} \|v_i - q_i\|,$$
$$\int_0^r \frac{|\tilde{m}_i(s)|}{|m_i(s)|} |\tilde{h}_i(v_i(s)) - \tilde{h}_i(q_i(s))| ds \le \frac{K\tilde{\mathcal{S}}\|\tilde{m}_i\|}{m_i} \|v_i - q_i\|,$$
$$\int_0^r \frac{1}{|m_i(s)|} I_i^{\varrho}(|p_i(s)||v_i(s) - p_i(s)|) ds \le \frac{K^{\varrho_i+1}\|p_i\|}{m_i \Gamma(\varrho_i + 2)} \|v_i - q_i\|,$$

and

$$\int_0^r \frac{1}{|m_i(s)|} I_i^{\varrho}(|f_i(s)||h_i(v_i(s)) - h_i(q_i(s))|) ds \le \frac{K^{\varrho_i+1}\mathcal{S}\|f_i\|}{m_i \Gamma(\varrho_i + 2)} \|v_i - q_i\|.$$

Thus,

$$|Hu_i(r) - Hq_i(r)|$$
$$\le |B||\rho_i| \sum_{j=1}^n |\tau_j| \int_0^{z_j} \frac{|\tilde{m}_i(s)|}{|m_i(s)|} |\tilde{h}_i(v_i(s)) - \tilde{h}_i(q_i(s))| ds$$
$$+ |B| \sum_{i=1}^u |\zeta_i| \int_0^{a_i} \frac{|\tilde{m}_i(s)|}{|m_i(s)|} |\tilde{h}_i(v_i(s)) - \tilde{h}_i(q_i(s))| ds$$
$$+ |B| \sum_{i=1}^u |\zeta_i| \int_0^{a_i} \frac{1}{|m_i(s)|} I_i^{\varrho}(|p_i(s)||v_i(s) - q_i(s)|) ds$$
$$+ |B||\rho_i| \sum_{j=1}^n |\tau_j| \int_0^{z_j} \frac{1}{|m_i(s)|} I_i^{\varrho}(|p_i(s)||v_i(s) - q_i(s)|) ds$$
$$+ |B||\rho_i| \sum_{j=1}^n |\tau_j| \int_0^{z_j} \frac{1}{|m_i(s)|} I_i^{\varrho}(|f_i(s)||h_i(v_i(s)) - h_i(q_i(s))|) ds$$
$$+ |B| \sum_{i=1}^u |\zeta_i| \int_0^{a_i} \frac{1}{|m_i(s)|} I_i^{\varrho}(|f_i(s)||h_i(v_i(s)) - h_i(q_i(s))|) ds$$
$$+ \int_0^r \frac{|\tilde{m}_i(s)|}{|m_i(s)|} |\tilde{h}_i(v_i(s)) - \tilde{h}_i(q_i(s))| ds + \int_0^r \frac{1}{|m_i(s)|} I_i^{\varrho}(|p_i(s)||v_i(s) - v_i(s)|) ds$$
$$+ \int_0^r \frac{1}{|m_i(s)|} I_i^{\varrho}(|f_i(s)||h_i(v_i(s)) - h_i(q_i(s))|) ds \le \otimes_{i,1} \|v_i - q_i\|.$$

Therefore, $|Hv_i(r) - Hq_i(r)| \leq \otimes_{i,1}\|v_i - q_i\|$. This implies that

$$\begin{aligned}
&|\Delta_k v_i(r) - \Delta_k q_i(r)| = |l_i(r, v_i(r))Hv_i(r) - l_i(r, q_i(r))Hq_i(r)|\\
&= |l_i(r, v_i(r))Hv_i(r) - l_i(r, v_i(r))Hq_i(r)\\
&+ l_i(r, v_i(r))Hq_i(r) - l_i(r, q_i(r))Hq_i(r)|\\
&= |l_i(r, v_i(r))[Hv_i(r) - Hq_i(r)] + Hq_i(r)[l_i(r, v_i(r)) - l_i(r, q_i(r))]|\\
&\leq |l_i(r, v_i(r))||Hv_i(r) - Hq_i(r)| + |Hq_i(r)||l(r, v_i(r)) - l(r, q_i(r))|\\
&\leq (\|\xi\|t + l_{0,i})\otimes_{i,1}\|v_i - q_i\| + (\otimes_{i,1}t + \otimes_{i,2})\|\xi\|\|v_i - q_i\|\\
&= \bigg((2\otimes_{i,1}t + \otimes_{i,2})\|\xi\| + l_0\otimes_{i,1}\bigg)\|v_i - q_i\|
\end{aligned}$$

and also $\|\Delta_k v_i - \Delta_k q_i\| \leq \bigg((2\otimes_{i,1}t + \otimes_{i,2})\|\xi\| + l_0\otimes_{i,1}\bigg)\|v_i - q_i\|$ for all $v_i, q_i \in E_t$ and $i = 1, ..., k$. Consider the map $\psi_i(r) = \bigg((2\otimes_{i,1}t + \otimes_{i,2})\|\xi\| + l_0\otimes_{i,1}\bigg)r$. Then, one can easily find that $\psi \in \Psi$ and $\|\Delta_k v_i - \Delta_k q_i\| \leq \psi(\|v_i - q_i\|)$ for all $v_i, q_i \in E_t$ and $i = 1, ..., k$. Thus, we get $\varrho_i(v_i, q_i)\|\Delta_k v_i - \Delta_k q_i\| \leq \psi(\|v_i - q_i\|)$ for all $v_i, q_i \in C(I, \mathbb{R})$, that is, Δ_k is an α-ψ-contraction. Now, we show that Δ_k is α-admissible. Let $\varrho_i(v_i, q_i) \geq 1$. Then, $v_i, q_i \in E_t$ and so $\Delta_k v_i, \Delta_k q_i \in E_t$ and so $\varrho_i(\Delta_k v_i, \Delta_k q_i) \geq 1$. Let $\{v_n\}$ is a sequence in $C(I, \mathbb{R})$ such that $\varrho(v_{n-1}, v_n) \geq 1$ for all $n \geq 1$ and $v_n \to v \in C(I, \mathbb{R})$. Then, $\{v_n\}$ is a sequence in E_t. Since E_t is closed, $v \in E_t$ and so $\varrho(v_n, v) \geq 1$ for all n. Let $v_0 \in E_t$. Since $\Delta_k E_t \subset E_t$, $\Delta_k v_0 \in E_t$ and so $\varrho(v_0, \Delta_k v_0) \geq 1$. Now by using Lemma 1.1, Δ_k has a fixed point in $C(I, \mathbb{R})$ which is a solution for the system. □

3 Example

Now, we provide an example to illustrate our main result.

Example 3.1. Consider the two-dimensional hybrid system

$$\begin{cases} D^{\frac{3}{4}}\left(500\sqrt{2+r^3}\left(\dfrac{v(r)}{l(r,v(r))}\right)' - \dfrac{e^{-2r}}{200}\big(\dfrac{2}{5}\sin v(r)+3\big)\right) + e^{-\sqrt{2r}}v(r) \\ = e^{-2r}\sin r\tan^{-1}(v(r)+2) \\ \\ D^{\frac{3}{4}}\left(400\sqrt{3+r^4}\left(\dfrac{v(r)}{l(r,v(r))}\right)' - \dfrac{e^{-3r}}{300}\big(\dfrac{3}{5}\sin v(r)+3\big)\right) + e^{-\sqrt{3r}}v(r) \\ = e^{-3r}\sin r\tan^{-1}(v(r)+2) \end{cases} \tag{6}$$

with boundary value conditions

$$\begin{cases} \left(\dfrac{v_i(r)}{l_i(r,v_i(r))}\right)'_{r=0} = \dfrac{1}{70000}\left(\dfrac{1}{4}v_i(0)+2\right), \\ \displaystyle\sum_{j=1}^{3}\frac{1}{3000j}\left(\frac{v_i(\frac{1}{5^j})}{l_i(\frac{1}{4^j},v_i(\frac{1}{4^j}))}\right) = \frac{2}{222}\sum_{j=1}^{4}\frac{1}{100^j}\left(\frac{v_i(\frac{2}{5^j})}{l(\frac{2}{5^j},v_i(\frac{2}{5^j}))}\right), \end{cases} \tag{7}$$

where $l_i(r,v_i(r)) = \frac{|\sin 5r|}{3\pi}\frac{|v_i(r)|}{2+|v_i(r)|} + \frac{|\sin r|}{3}e^{-3\pi r}$. Put $\varrho = \frac{3}{4}$, $K = 1$, $t = 0.1$, $\zeta_1 = \frac{1}{3000}$, $\zeta_2 = \frac{1}{3000}$, $\tau_1 = \frac{2}{20}$, $\tau_2 = \frac{2}{200}$, $\tau_3 = \frac{3}{3000}$, $m_1(r) = 500\sqrt{3+r^3}$, $m_2(r) = 400\sqrt{3+r^4}$, $\tilde{m}_1(r) = \frac{e^{-2r}}{200}$, $\tilde{m}_2(r) = \frac{e^{-3r}}{300}$, $p_i(r) = e^{-\sqrt{2r}}$, $f_i(r) = e^{-2r}\sin r$, $f_i(v_i(r)) = \tan^{-1}(v_i(r)+2)$, $\tilde{h}_{1_i}(v_i(r)) = \frac{2}{5}\sin v_i(r)+3$ and $\tilde{h}_{2_i}(v_i(r)) = \frac{3}{5}\sin v_i(r)+3$. Then, we have $|\frac{\partial h_i(v)}{\partial r}| \le 1 = \mathcal{S}$, $h_0 = \frac{\pi}{5}$, $|\frac{\partial \tilde{h}_i(v)}{\partial r}| \le \frac{3}{5} = \tilde{\mathcal{S}}$, $\tilde{h}_0 = 1$, $m = 500$, $\|\tilde{m}\| = \frac{2}{200}$, $\|p\| = 1$, $\|f\| = 1$. Also,

$$|l_i(r,v_i(r)) - l_i(r,q_i(r))| = \frac{|\sin r|}{3\pi}\frac{\big||v_i(r)|-|q_i(r)|\big|}{(1+|v_i(r)|)(1+|q_i(r)|)}$$

$$\le \frac{|\sin r|}{3\pi}|v_i(r)-q_i(r)|.$$

Note that, $\|\xi\| = \frac{1}{3\pi}$ and $l_0 = \frac{2}{3}$, $\sum_{j=1}^{2}\frac{2}{4000j} - \frac{2}{222}\sum_{j=1}^{3}\frac{2}{(20)^j} = \frac{5}{7000} - \frac{2}{2000} = -\frac{4}{8000} \neq 0$ and $B = -8000$. Then, $|B|\big(\sum_{j=1}^{2}|\zeta_j| + |\rho|\sum_{j=1}^{3}|\tau_j|\big) + 1 = 6000\big(\frac{5}{7000} + \frac{2}{222}\frac{222}{2000}\big) + 1 = 8$ and so

$$
\begin{aligned}
\otimes_1 &= \frac{K}{m}\left(\widetilde{\mathcal{S}}\|\tilde{m}\| + \frac{K^{\varrho}(\|p\| + \mathcal{S}\|f\|)}{\Gamma(\varrho+2)}\right) \\
&\times \left(|B|\big(\sum_{j=1}^{u}|\zeta_j| + |\rho|\sum_{j=1}^{n}|\tau_j|\big) + 1\right) \\
&= \frac{16}{1200}\left(\frac{1}{600} + \frac{3}{\Gamma(\frac{28}{10})}\right) \approx 0.01499506341,
\end{aligned}
$$

$$
\begin{aligned}
\otimes_2 &= \frac{K}{m}\left(\tilde{h_0}\|\tilde{m}\| + \frac{K^{\varrho}\|f\|h_0}{\Gamma(\varrho+2)}\right)\left(|B|\left(\sum_{j=1}^{u}|\zeta_j| + |\rho|\sum_{j=1}^{n}|\eta_j|\right)\right) \\
&= \frac{8}{600}\left(\frac{2}{200} + \frac{\pi}{4\Gamma(\frac{28}{10})}\right) \approx 0.005269853,
\end{aligned}
$$

$$
\begin{aligned}
(\|\xi\|r + g_0)(\mathcal{A}_1 r + \otimes_2) &\approx (\frac{0.4}{8\pi} + \frac{2}{4})(0.0159506855 \times 0.1 + 0.0063796996) \\
&\approx 0.0041143065 \leq 0.1 = t
\end{aligned}
$$

and

$$
\begin{aligned}
(2\otimes_1 r + \otimes_2)\|\phi\| + g_0\otimes_1 &\approx (2 \times 0.0.005269853 \times 0.10.005269853) \times \frac{1}{3\pi} \\
&+ \frac{1}{3} \times 0.01499506341 \approx 0.0095892469 < 1.
\end{aligned}
$$

Now by using Theorem 2.1, the problem (6)-(7) has a solution.

4 Conclusion

In today's world, most events are modeled by systems of fractional equations which increase our abilities to provide a good study of various phenomena. It is always good to focus on solving complex fractional differential equations. One of the most important types of these equations is the hybrid fractional differential equations with complex boundary conditions. In this work, we studied a k-dimensional system of hybrid fractional differential equations with hybrid boundary conditions. By using the α-ψ-technique we reviewed the system. We provided an example to illustrate our main result.

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