A novel inverse DEA-R model for inputs/output estimation

**Abstract**

In this paper, we propose inverse data envelopment analysis (DEA) models in the presence of ratio data. We present the inputs/output estimation process based on ratio based DEA (DEA-R) models. We first present a multiple objective linear programming (MOLP) model to determine the level of inputs based on the perturbed outputs, assuming that the relative efficiency of the under evaluation decision making unit (DMU) preserve. We also present the relationship between the Pareto solutions of the proposed MOLP model and the optimal level of inputs and outputs of the new DMU. We presented criterion models to determine the efficiency of the new DMU in the inputs/output estimation process based on inverse DEA-R models in the presence of ratio data. We showed that in the presence of ratio data the selection of criterion model can be important, in order to we provide a new criterion model in the inputs/output estimation process in the presence of ratio data, and so on the amount of calculations is reduced. We have shown that the results for the new criterion model are the same as the existing criterion model presented in the paper. In order to show the validity of the proposed approach in the inputs/output estimation process based on the inverse DEA-R models, we provide an application of our models in a real life for a set of data regarding to medical centers in Taiwan and finally we present the research results.

**AMS Subject Classification:** 90C05; 90C08.  
**Keywords and Phrases:** Data envelopment analysis, Multiple objective linear programming, Input/output estimation, Rati data; Criterion model, Inverse DEA.

**1. Introduction**

Traditional DEA models determine the efficiency of DMUs based on their inputs and outputs. However, in inverse DEA, the efficiency of the DMU is predetermined by the decision-maker (DM), and based on this score of efficiency, the optimal level of inputs or outputs are determined. This amount of efficiency that is predetermined by the DM is called the target efficiency. The concept of inverse DEA was first present by Wei et al. [39] and then by Yan et al. [38] developed on the issue of resource allocation. Hadi-Vencheh and Foroughi [24] proposed a generalized inverse DEA mode based on the model of Wei et al. [39]. They showed that some special cases of the inverse DEA model proposed by Wei et al. [39] may fail in some situations and then they revised these failures. Lertworasirikul et al. [28] considered the issue of inverse DEA by considering two different strategies. In the first strategy, by determining the specific level of efficiency for each unit under evaluation DMU, they determined the best possible level of inputs corresponding to a given level of outputs. In the second strategy, again considering a specific level for the efficiency for the under evaluation DMU, they determined the best possible level of outputs corresponding to a level of given inputs and presented their models as resource allocation models. But the early models that they presented were nonlinear models. Due to the problems in solving nonlinear models, they presented new their inverse DEA model in the form of MOLP model. In the following, Ghiyasi [20] points out the drawbacks of Lertworasirikul et al. [28] and then revised the use of MOLP in the proposed inverse DEA model considering the variable return to scale technology (VRS) (Banker, Charnes and Cooper [9]).

Gattouf et al. [18] presented a new model of inverse DEA on mergers and acquisitions to estimate the optimal level of inputs and outputs for the merged entity for a given target efficiency value. Amin et al. [4] presented a general model on mergers and acquisitions. They presented a generalized firm restructuring in two scenarios in the form of consolidation or a split. They considering a set of DMUs that called pre-restructuring DMUs, they produced a set of new DMUs that called post-restructuring DMUs, and the level of inputs and outputs from post-restructuring DMUs are determined based on the level of inputs and outputs of pre-restructuring DMUs also the efficiency scores of post-restructuring DMUs are predetermined by the DM as target efficiency scores. Emrouznejad et al. [15] proposed a new application of inverse DEA in environmental efficiency to determine the optimal allocation of CO2 emissions reduction by Chinese manufacturing industries. Wegener and Amin [35] suggested an inverse DEA model for minimizing greenhouse gas emissions with an application in oil and gas. Other applications of inverse DEA including an application in resource allocation (Ghiyasi [21, 23]), new product target setting given expected changes of production frontier (Lim [29]), inter-temporal dependence (Jahanshahloo et al. [26]), revenue setting problems of chain stores, inverse DEA models based on cost and revenue efciency (Ghiyasi [22]), application of the inverse DEA to sensitivity analysis of DMUs (Eyni et al. [16]). Amin and Al-Muharrami [2] addresses the model of inverse DEA in the mergers and acquisitions of firms with negative data. Amin et al. [3] suggested a combined inverse DEA and goal programming approach for target setting of mergers as allows DM to incorporate their preferences. Emrouznejad and Yang [14] presented a literature review of DEA and inverse DEA.

Amin and Ibn Boamah [6] proposed a new model of inverse DEA for estimating potential merger gains based on cost efficiency and used the proposed approach in the Canadian banking industry. Amin and Ibn Boamah [7] presented an inverse DEA approach for the two-stage network in the US banking sector.

In the real world, there are many cases in which data are ratio and the ratio of input data to output data or vice versa is important to the DM or input/output data is presented in the form of ratio or percentage data. Traditional DEA models can no longer be used to evaluate the efficiency of DMUs if the ratio of input to output or vice versa is important to the DM, or if the input and output data are ratio data. We need to develop DEA models and in this situation we use DEA-R models. In general, we divide ratio data into three categories as follows.

The first category includes ratio data in which the input and output data of ratio numbers are in the form of a fraction and the numerator and denominator corresponding to these fractions are known, but the DM can use this ratio data in the form of decimal numbers in the model. In this case, the data are used in both absolute and ratio forms in the efficiency evaluation model. In the presence of ratio data, the principle of convexity in underlying assumptions of the production possibility set (PPS) is not established in DEA. Among the articles that have been presented in this category to deal with ratio data in DEA models, the following articles can be mentioned.

Emrouznejad and Amin [13], Hatami-Marbini and Toloo [25], Khoshnevis and Teirlinck [27].

The above articles modified DEA models to evaluate efficiency in the presence of ratio data. In this category, the numerator and denominator corresponding to these fractions corresponding to the ratio data are known, but the nature of the data is ratio.

The second category includes ratio data in which the ratio data are in the form of a fraction and the numerator and denominator corresponding to these fractional numbers may not be available and we have ratio numbers only available as decimal numbers or percentages. It is known and we must use these decimal numbers as ratio data in the model. From a series of articles that modified DEA models and change the underlying assumptions of the PPS in constant and variable return to scale technologies in the presence of ratio data. These papers provide new DEA models to calculate efficiency in the presence of absolute and ratio data. Among the articles that have been presented in this category to deal with ratio data in DEA models, the following articles can be mentioned. Olesen, Petersen and Podinovski [33, 34].

The third category includes ratio data in which the ratio data are in the form of a fraction and the numerator and denominator corresponding to these fractional numbers are important for the DM and the DM cannot use these fractions as decimal numbers in the model. These ratio data are in the form of the ratio of components input to components output or vice versa. These models were initially presented as ratio analysis models (Fernandez-Castro and Smith [17]). In these models, we must use the ratio of inputs to outputs and vice versa in the model. Among the articles that have been presented in this category to deal with ratio data in DEA models, the following articles can be mentioned.

Fernandez-Castro and Smith [17], Despic, Despic, and Paradi [12], Wei et al. [36, 37, 38], Mozaffari et al. [32], Mozaffari, Gerami and Jablonsky [31], Gerami et al. [19, 20], Mozaffari et al. [21].

Despic, Despic and Paradi [12] presented DEA-R models by combining ratio analysis and DEA models. They proposed DEA-R models in the output orientation to calculate the efficiency of DMUs in the presence of ratio data as the ratio of components output to components input.

Wei et al. [36, 37, 38] examined DEA-R models in the input orientation. They showed that by using DEA-R models in the input orientation, we can avoid the available problems in of traditional DEA models such as efficiency underestimation and pseudo-inefficiency. They showed that DEA-R models in the input orientation have higher efficiency scores than their corresponding scores from CCR models in the input orientation. Mozaffari et al. [32] used DEA-R models to evaluate cost and revenue efficiency. Gerami et al. [20] used DEA-R models to evaluate the efficiency of the hospital supply chain in the presence of ratio data. Gerami, Mozaffari, and Wanke [21] proposed DEA-R models to evaluate the efficiency of two-stage network structures in the presence of ratio data.

In the first category of ratio data, we can refer to the ratio of the number of research projects presented by some professors in a course to the total number of professors in a university, and these ratios are important for the DM. DM used of the numerator and denominator corresponding ratio data in the model. In the second category of ratio data can be referred to the percentage of successful operations performed to the total number of operations performed in a hospital during a treatment period. But the DM only uses the decimal form of this data and this data is only available in the form of decimal numbers. In the third category of data, we can refer to some concepts in economics such as immediate and current profit, or the number of patients treated to the total number of patients admitted to a hospital during a treatment period. It should be noted that in the third category of data, we use the input and output data of each of the DMUs directly in the model and put this data as the form of the ratio of components input to components output or vice versa in the model, but what is important is that we assume that the input and output data are definite and their ratio is important for the DM.

In this paper, we use the ratio data in the third category and assume that the input and output data are definite numbers and their ratio is important for the DM, and we put this data as the form of the ratio of components input to components output in the models and do not use the fractional or decimal form of these numbers.

It can be said that the main contribution of the article is as follows. In this paper, we examine one of the most important issues in DAE, namely inverse DEA, and estimate inputs and outputs if some of the input and output components change and the DM wants create a new DMU with a relative efficiency score that predetermined and is equal to relative efficiency score of the initial unit. In the process of estimating the level of inputs and outputs, we can choose two different strategies in inverse DEA models in the presence of ratio data. In the first strategy, by determining the specific level of efficiency for each unit under evaluation DMU, they determined the best possible level of inputs corresponding to a given level of outputs. In the second strategy, again considering a specific level for the efficiency for the under evaluation DMU, they determined the best possible level of outputs corresponding to a level of given inputs. That is, if we want the efficiency of the DMU to remain unchanged, we determine the optimal level of input or output based on DEA-R models. We obtain the necessary and sufficient conditions for inverse DEA-R in the input orientation models. As we know, one of the important issues in inverse DEA is the selection of a criterion model for comparing the efficiency scores ​​of DMU before and after the process of estimating inputs and outputs. In this paper, we first develop inverse DEA models in the presence of ratio data, and by providing a suitable criterion model in the presence of ratio data, we show that we can significantly reduce the computations and thus show that the new criterion model presented have the same results as the previous criterion models. Finally, we provide a case study to examine the validity of the proposed models.

The remainder of the paper unfolds as follows. In the Section 2, we examine DEA-R models in the input and output orientations and present the relationship between these models and traditional DEA models. Section 3 proposes the inverse DEA-R models and the inputs/output estimation process based on inverse DEA-R models in the presence of ratio data, in following, we present the criterion models for evaluating the efficiency of the new units created. Section 4 provides a numerical example, in this way, we illustrate the inputs/output estimation process based on inverse DEA-R models in the presence of ratio data. Section 4 provides a real  
world data empirical investigation and shows the applicability and potential use of the proposed models, we present an application of the proposed approach related to medical centers in Taiwan and at the end, we present the results of the research.

**2. Ratio-based DEA models.**

Suppose we have decision units as . The input and output vectors corresponding to as and. We suppose that . Suppose the ratios , in the input orientation and the ratios in the output orientation are defined. Suppose, we consider the multiples corresponding to the ratios , as . Fernandez-Castro and Smith [17] proposed ratio analysis model in the input orientation as follows.

, (1)

.

We consider the variable corresponding to first constraint in model (1) as . The dual model (1) is as follows.

, (2)

.

By considering and placing from the optimization point of the model (2) is converted as follows.

, (3)

.

Model (3) is called the DEA-R model in the input orientation in the envelopment form. Model (3) by Wei et al. [36, 37, 38] and Mozaffari et al. [31] were also studied. We now examine the relationship between the above model (1) to traditional DEA model.

**Theorem 1.** Model (1) is equivalent to the CCR multiplier model in the input orientation.

Proof: If we define Then we have DMUs with one output and input. Considering the multiples corresponding to the input components of the new units as , and the multiples corresponding to the output component of the new units as . Then model (1) is converted as follows.

, (4)

.

That model (4) is CCR multiplier model in the input orientation (Charnes, Cooper and Rhodes [10]) in evaluation If the set of DMUs be as , which completes the proof.

Similarly, suppose, we consider the multiples corresponding to the ratios , as . Fernandez-Castro and Smith [17] proposed ratio analysis model in the output orientation as follows.

, (5)

.

We consider the variable corresponding to first constraint in model (5) as . The dual model (5) is as follows.

, (6)

.

By considering , and placing , from the optimization point, the model (5) is converted as follows.

, (7)

.

Model (7) is called the DEA-R model in the output orientation in the envelopment form. Model (7) by Despic, Despic and Paradi [12] were also studied. We now examine the relationship between the above model (5) to traditional DEA model.

**Theorem 2.** Model (5) is equivalent to the CCR multiplier model in the output orientation.

Proof: If we define . Then we have DMUs with one input and output. Considering the multiples corresponding to the output components of the new units as , and the multiples corresponding to the input component of the new units as . Then model (5) is converted as follows.

, (8)

.

That model (8) is CCR multiplier model in the output orientation (Charnes, Cooper and Rhodes [10]) in evaluation ,

if we consider the set of DMUs as , which completes the proof.

**3. The inputs/output estimation process based on inverse DEA-R.**

In this section, we present inverse DEA-R models in the presence of ratio data in the input orientation. In this regard, we present inputs/output estimation process based on inverse DEA-R models. We provide criterion models to evaluate the efficiency of new units. In other words, we need to find the new input level of under evaluation DMU that guarantees unchanged relative efficiency for this DMU.

Suppose we have DMUs as that each DMU consume input vector to product output vector. We suppose that . Suppose the ratios , are the ratio ith input component to rth output component of

. We show under evaluation DMU as , also, assume perturbs its output level into , , . Now we want to know how much we need to change the input level of , so that the relative efficiency of this unit remains unchanged. In other words, we first perturbs the output level of to a certain extent, and then, we must determine the input level of the new DMU namely , , , in such a way that the relative efficiency of the new unit is equal to the relative efficiency of .

We proposed the following MOLP model in the inverse DEA-R and in the presence of ratio data to determine , , , as follows.

(9)

.

**Definition 1.**  is called a weak efficient solution in evaluation with model (3) if the optimal value of model (3) is equal to one.

**Definition 2.** Suppose that and are a feasible solution of model (9). If there does not exist a feasible solution of model (9) such that then will be a weakly efficient solution of model (9).

A weak efficient solution of model (9) are as new input values from for a disturbed output level , , , to preserve relative efficiency of

after the output changes. At first, to check the relative efficiency of the new unit namely , we present the following criterion model.

, (10)

, .

**Theorem 3.** Suppose that perturbs its output from to , , . Then is a weak efficient solution of MOLP model (9) if and only if .

Proof: First assume that is a weak efficient solution of MOLP model (9). We show that the efficiency score of and are equal, i.e.. Put , it is easily seen that is a feasible solution for model (10), so we will have . Now suppose , let that is an optimal solution of model (10), so according to the constraints of model (10), we will have

, (11)

.

Given that and is a weak efficient solution of MOLP model (9), so we will have

(12)

.

By comparing relations (11) and (12) we will have

(13)

Now, we put

.

And so according to relation (13), we will have

, (14)

Given that and . Then .

Therefore, we have

(15)

.

Therefore that  will be a feasible solution for model (9). According to relation (15) there exists a such that

(16)

.

Therefore, according to relation (16), is a feasible solution of model (9), which and but this is impossible because is a weak efficient solution of model (9). Therefore, the contradiction assumption is invalid and we will have .

Conversely, let , we show that a feasible solution of model (9). By contradiction assume is not a weakly efficient solution of model (9). Therefore, a feasible solution of model (9) will exist as such that . Given that is a feasible solution of model (9), so we will have

(17)

.

Then there exists a such that

(18)

.

Let , according to relation (18), we have

.

Then, is a feasible solution of model (10), so we will have. But this is against the assumption that is the optimal value of model (10). Therefore, the contradiction assumption is invalid and we will have is not a weakly efficient solution of model (9)

and the proof is complete. ∎

Suppose we have decision units as . Each DM uses the input vector to product the output vector . Then we define the set as follows.

. (19)

We define a division data set, which are m × s dimension vectors as follows.

with,. (20)

The technology set has the properties inclusion of observations, a free-disposal and convexity.

Now to obtain the efficiency score based on the concept of radial efficiency, we obtain the value in such a way that the unit under evaluation i.e. in the form be on the efficiency frontier of the set . Therefore, we solve model (21) as follows.

. (21)

By considering multiplier corresponding to the ratio input to output of as , model (21) is equivalent to the following model.

, , (22)

, .

The model (22) is the identical to the DEA-R in input orientation namely model (3) which was introduced in the second section.

We now present a new criterion model compared to model (10). If the created new unit means

belong to the set , that is, the created new unit is an internal point or a point on the efficient frontier of the set .

In this case, we can present the criterion model (23) to check whether the relative efficiency of the unit under evaluation changes after perturbation of its inputs and outputs or not.

, (23)

.

In model (23), is the relative efficiency of . Model (23) compared to model (10) has one variable less.

**Theorem 4.** Suppose that perturbs its output from to , , . Then is a weak efficient solution of MOLP model (9) if and only if .

Proof: First assume that , we show that is a weak efficient solution of MOLP model (9). Assume that is not a weak efficient solution of MOLP model (9). Thus there is a feasible solution of model (9) such that . So there exists a such that Given that is a feasible solution of model (9), so we will have

(24)

.

Assuming that , therefore

(25)

.

Therefore is a feasible solution for model (23) and we will have .

This contradicts with the optimality ofin model (23) since . Therefore, the contradiction assumption is invalid and then is a weak efficient solution of MOLP model (9).

Conversely, assume that is a weak efficient solution of MOLP model (9). We show that . Given that is a weak efficient solution of MOLP model (9), so we have

(26)

.

This set of constraints in (26) will be the same as the set of constraints in model (23). Therefore is a feasible solution of the model (23) and according to the optimality , we will have . Now suppose that , thus there exists a such that . Given that is the optimal value of the model (23), assume that the optimal solution corresponding to this optimal value is . Therefore, the set of constraints from model (23) will be as following.

, (27)

.

Given that therefore

, (28)

.

Therefore is a feasible solution of model (9) and, because otherwise if there exist a , such that , then

The relation (29) concludes that , which is inconsistent with that

. Therefore . Therefore, given that , which is a contradiction with the fact that is a weak efficient solution of MOLP model (9). Therefore, the contradiction assumption is invalid and we will have and the proof is complete. ∎

As you know, in proposed inverse DEA-R models in this paper, we determine the level of inputs based on the perturbed outputs, assuming that the relative efficiency of the under evaluation DMU i.e. preserve. We replace the under evaluation DMU i.e. with a new unit as . So that the efficiency of these units are equal. We are now looking to introduce a new criterion model that has less number of variables than previous criterion models, and with fewer calculations we can compare the efficiency of the created new DMU with the efficiency of the original DMU i.e. . In the new model, we remove the unit under evaluation, i.e. from between all DMUs, and the we put new unit, instead of the primitive DMU i.e. in a set of DMUs. So the difference between models (10) and (23) with the new created model is that in models (10) and (23) we evaluate the new unit in the presence of all DMUs, but in the proposed new criterion model, we evaluate a new unit, in the presence of all units and new unit , with the exception of the under evaluation unit i.e. . In the new model does not exist among DMUs, that is, we remove from the set .

Now, we proposed the new criterion model for evaluating the efficiency of the new unit, i.e. in the absence of the unit under evaluation, i.e. as follows.

, (30)

.

**Theorem 5.** Suppose that perturbs its output from to , , . Then is a weak efficient solution of MOLP model (9) if and only if .

Proof: We now consider two cases. In the first case, suppose that is efficient in evaluation with model (3), i.e. . According to Theorem (3), we have , i.e. the optimal value of model (10) is also equal to one. Now we show that the optimal value obtained from model (30) is also equal to one, i.e. . Assuming that is an optimal solution for model (30), we know that . We show that . Suppose that

. For this purpose, we put and

Given that is an optimal solution for model (30), so it is easy to see that is a feasible solution for model (10) which results in , which cannot be true because the value of the optimal value of model (10) is equal to one. Therefore, the contradiction assumption is invalid and we have , then .

In the second case, suppose that is inefficient in evaluating with model (3). Therefore is an internal point of the set and adding a new unit does not change the set and therefore the new unit is is also an interior point of and is an inefficient unit, and removing it will not change the set . So the new unit means will be evaluated in terms of units . Therefore, the solutions of models (10) and (30) are the same, i.e. in this case, too, and the proof is complete.∎

It should be noted that they are different in criterion models (10) and (30). Model (10) has variables and constraints and model (30) has variables and constraints. Therefore, the number of calculations related to model (30) is significantly reduced compared to model (10).

The DEA-R models presented in this paper are in the input orientation based on model (3) and we have proposed the approach presented in this paper in the output orientation based on model (7) and we consider the ratio of output components to input components, which is beyond the scope of this paper and is suggested as future work.

**4. Numerical example**

In this section we use the data from the paper of Ali, Lerme, and Seiford [1] and Chen and Ali [11] to illustrate the validity of the proposed models. Suppose we have 11 DMU that use two inputs to generate two outputs. Table (1) shows the input and output data.

Table 1. Input and output data of eleven DMUs.

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| DMU | I1 | I2 | O1 | O2 | Efficiency scores (model 3) |
| DMU1 | 40 | 30 | 160 | 100 | 1 |
| DMU2 | 30 | 60 | 180 | 70 | 1 |
| DMU3 | 93 | 40 | 170 | 60 | 0.729 |
| DMU4 | 50 | 70 | 190 | 130 | 1 |
| DMU5 | 80 | 30 | 180 | 120 | 1 |
| DMU6 | 35 | 45 | 140 | 82 | 0.94 |
| DMU7 | 105 | 75 | 120 | 90 | 0.356 |
| DMU8 | 97 | 67 | 100 | 82 | 0.361 |
| DMU9 | 100 | 50 | 140 | 40 | 0.494 |
| DMU10 | 90 | 60 | 140 | 105 | 0.512 |
| DMU11 | 98 | 65 | 140 | 50 | 0.397 |

First, we use model (3) to evaluate the efficiency of DMUs based on the ratio of input components to output components. As can be seen in the last column of Table (1), units, 1, 2, 4, and 5 are efficient units, and other units are inefficient in evaluated by model (3).

Suppose that the amount of changes in the components of the first and second outputs of the unit under evaluation, i.e. is denoted by and , respectively, and also the value of the components of the first and second outputs of the new unit i.e. are denoted by and , respectively. Assume that the amount of changes of the components of the first and second inputs corresponding to the unit under evaluation i.e. from model (9) is indicated by and , respectively, and also the value of the components of the first and second inputs of the new unit i.e. are denoted by and , respectively that theses values determine from model (9).

To illustrate the inputs/output estimation process based on inverse DEA-R models in the presence of ratio data, first consider the inefficient unit 3. As can be seen in the last column of Table (1), the efficiency score of unit 3 is equal to 0.729. Suppose this unit increases the value of its first and second outputs by 10 and 50 units, respectively. Then we have

In this case, if we want its efficiency does not change and its be equal to 0.729. According to model (9), the minimum input level of this DMU is determined as follows.

.

As can be seen, the amount of the first and second inputs decreases and increases by 31.272 and 6.296, respectively. The efficiency score of the new unit means , based on the criterion models (10), (23), and (30) are equal to, , , respectively.

As can be seen, all three criterion models obtain the efficiency score of the new unit equal to 0.729 and this shows that the solution proposed by model (9) have the relative efficiency score equal to the efficiency score of the unit under evaluation.

Now consider efficiency unit 5. As can be seen in the last column of Table (1), the efficiency value of the unit 3 is equal to one and this unit is an efficient unit. Suppose this unit increases the value of its first output by 20 units and does not change its second output. Then we have

.

In this case, if we want its efficiency does not change and its be is equal to one. According to model (9), the minimum input level of this DMU is determined as follows.

.

As can be seen, the amount of the first and second inputs decreases and increases by 30 and 7.5, respectively. The efficiency score of the new unit means , based on the criterion models (10), (23), and (30) are equal , , , respectively.

As can be seen, all three criterion models obtain the efficiency score of the new unit equal to one and this shows that the solution proposed by model (9) have the relative efficiency score equal to the efficiency score of the unit under evaluation.

In order to the sensitivity analysis of the results related to the proposed approach in this paper, we also used the inputs/output estimation process based on inverse DEA-R models in the presence of ratio data for units 6 and 9, the results are shown in Table (2).

Table 2. The results corresponding to different DMU for inputs/outputs estimation based on proposed approach.

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| DMU | DMU3 | DMU6 | DMU5 | DMU9 | DMU9 |
|  |  |  |  | 175 |  |
|  |  |  |  | 80 |  |
|  |  | 40 | 20 | 35 | 25 |
|  |  | 38 | 0 | 40 | 25 |
|  |  |  |  |  |  |
|  |  |  |  |  |  |
|  |  |  | -30 | -11 |  |
|  |  |  | 7.5 |  |  |
|  |  |  | 1 |  |  |
|  |  |  | 1 |  |  |
|  |  |  | 1 |  |  |

**5. Case study**

In this section, we apply the approach presented in this paper to the real-world data set. For this purpose, we apply the inputs/output estimation process based on inverse DEA-R models in the presence of ratio data that proposed in this paper for 21 medical centers in Taiwan. These medical centers are included private and public health centers in 2005. Also, this data has been used in the article Wei et al. [37]. Input and output data sets including two inputs and outputs are listed in Table (3).

Table 3.

The input and output variables of Taiwan medical centers in 2005. (Wei et al. [37]).

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| DMU | Sickbed | Physician | Out-patient | In-patient | Surgeries | Efficiency scores (model 3) |
| DMU1 | 2618 | 1106 | 2,029,864 | 680,136 | 38,714 | 0.814 |
| DMU2 | 1212 | 473 | 1,003,707 | 297,719 | 18,575 | 0.792 |
| DMU3 | 1721 | 531 | 1,592,960 | 408,556 | 36,658 | 0.843 |
| DMU4 | 2902 | 973 | 2,596,143 | 855,467 | 75,348 | 1 |
| DMU5 | 1389 | 447 | 1,116,161 | 337,523 | 23,803 | 0.842 |
| DMU6 | 1500 | 547 | 1,476,282 | 378,658 | 22,503 | 0.842 |
| DMU7 | 340 | 145 | 1,300,016 | 55,003 | 5,614 | 1 |
| DMU8 | 571 | 305 | 1,052,992 | 199,780 | 26,026 | 1 |
| DMU9 | 1168 | 369 | 1,849,711 | 326,109 | 30,967 | 1 |
| DMU10 | 921 | 372 | 1,089,975 | 209,323 | 23,847 | 0.746 |
| DMU11 | 920 | 316 | 334,090 | 268,723 | 15,130 | 0.981 |
| DMU12 | 3236 | 1023 | 1,954,775 | 920,215 | 56,167 | 0.98 |
| DMU13 | 495 | 130 | 332,741 | 136,351 | 23,423 | 1 |
| DMU14 | 1759 | 491 | 1,465,374 | 430,407 | 35,599 | 0.908 |
| DMU15 | 1357 | 390 | 1,277,752 | 368,174 | 36,006 | 0.986 |
| DMU16 | 2468 | 675 | 1,825,332 | 668,467 | 32,275 | 0.98 |
| DMU17 | 962 | 316 | 550,700 | 247,961 | 15,618 | 0.878 |
| DMU18 | 745 | 272 | 1,277,899 | 217,371 | 11,671 | 1 |
| DMU19 | 1662 | 590 | 1,916,888 | 418,205 | 21,551 | 0.855 |
| DMU20 | 898 | 275 | 698,945 | 209,134 | 11,748 | 0.822 |

In this paper we select all medical centers (21) as evaluation subjects, including seven public hospitals (33%) and private hospitals (67%). Two inputs and three outputs were selected. Note that the total inputs and outputs were less than half of all DMUs in conformity with empirical rules. The inputs include: sickbeds and physicians, outputs include: out-patients, in-patients, and surgeries. For example, consider DMU 4, this DMU serviced 2,596,143 out-patients, and 855,467 in-patients, and conducted 75,348 surgeries in 2005, with 2902 sickbeds and 973 physicians.

Due to the nature of the data, we can use DEA-R models in the input orientation, i.e. model (3) to evaluate the efficiency of these centers. In the input orientation models presented in the paper, we use the ratio of input components to output components as in Table (4) and these ratios are defined and are important for the DM and management.

Table 4.

The ratios of inputs to outputs in order to using in the input orientation.

|  |  |  |
| --- | --- | --- |
| Number of sickbeds / Number of out-patients | Number of sickbeds / Number of in-patients | Number of sickbeds / Number of surgeries |
| Number of physicians / Number of out-patients | Number of physicians / Number of in-patients | Number of physicians / Number of surgeries |

For example, the ratio of the total number of sickbeds admitted to the hospital to the number of out-patients is important for hospital management, because whatever decreases the ratio of total sickbeds or increases the number of out-patients is important for management and the treatment system. Also, the goal is to determine efficient medical centers that provide more out-patients with the least number of sickbeds. This increases hospital services, because if the numerator and denominator corresponding to these fractional numbers decreases and increases respectively, then the number of treated patients increases to the total number of patients admitted to the hospital, and this issue is important for the hospital management and consequently the cost and revenue of the hospital decreases and increases respectively. Or consider another ratio, for example consider the ratio of the number of sickbeds to the number of successful surgeries, this ratio should be a good ratio for hospital management to offer more number of successful surgeries compared to the smaller number of out-patients. Then the medical centers are introduced as successful and efficient that offer a higher number of successful surgeries with a smaller number of sickbeds and in this case, this ratio is a suitable ratio.

Or consider the ratio of the total number of physicians to the number of successful surgeries. If this ratio decreases, then the number of unsuccessful surgeries increases compared to the number of physicians, which means that the hospital performs more successful surgeries for a lower fee, including fees paid to physicians and staff and other costs. The hospital management perspective is important to reduce this ratio, because by reducing this ratio, the costs paid to the treatment staff will decrease, and in contrast, with the increase in the number of surgeries or successful operations in the hospital, the amount of services received by patients will increase and the income received from these patients will increase that is suitable from the point of view of optimization. For other ratios in Table (4) we can provide similar interpretations.

The last column in Table (3) shows the efficiency scores ​​obtained from Model (3) in the evaluation of medical centers. As can be seen, units 4, 7, 8, 9, 13, and 18 are introduced as efficient medical centers and other centers are inefficient. It should be noted that the technology used in this paper is constant returns to scales technology.

Now, in order to examine the results of the proposed approach presented in this paper, we

we apply the inputs/output estimation process based on inverse DEA-R models in the presence of ratio data that proposed in this paper. Table (5) shows these results.

At first, consider the inefficient unit 10. As can be seen in the last column of Table (3), the efficiency value of unit 10 is equal to 0.746. Suppose this unit increases the amount of its first, second, and third outputs by 120,000, 35,000, and 80, respectively. Then, we have

.

In this case, according to model (9), the minimum input level of this DMU is determined as follows, if we want its efficiency does not change and its be equal to 0.746. .

As can be seen, the first and second inputs increase to 15.073 and 130, respectively. The efficiency of the new unit means . Based on the criterion models (10), (23), and (30) are equal to , , .

As it was observed, all three criterion models obtain the efficiency score of the new unit equal to 0.746 and this shows that the solution proposed by model (9) have the relative efficiency score equal to the efficiency score of the unit under evaluation.

Now, consider efficient unit 8. As shown in the last column of Table (3), the efficiency score of unit 10 is equal to one. Assume that this unit increases the value of its first, second, and third outputs by 80,000, 43,000, and 50 units, respectively. Then we have

.

In this case, the minimum input level of a new unit corresponding to DMU 10 according to model (9) is determined as follows, if we want efficiency score of new unit does not change and its be equal to one.

.

As can be seen, the first and second inputs increase to 120 and 65.6470, respectively. The efficiency of the new unit means

.

Based on the criterion models (10), (23), and (30) are equal to , , .

As can be seen, all three criterion models obtain the efficiency score of the new unit equal to one and this shows that the solution proposed by model (9) have the relative efficiency score equal to the efficiency score of the unit under evaluation.

In order to the sensitivity analysis of the results related to the proposed approach in this paper, we also used the inputs/output estimation process based on inverse DEA-R models in the presence of ratio data for units 4 and 17, the results are shown in Table (5).

Table 5. The results corresponding to different medical centers for inputs/outputs estimation

based on proposed approach.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| DMU | DMU10 | DMU8 | DMU17 | DMU4 |
|  |  |  |  |  |
|  |  |  |  |  |
|  |  |  |  |  |
|  |  |  |  |  |
|  |  |  |  |  |
|  | 80 | 50 |  |  |
|  |  |  |  |  |
|  | 502 |  |  |  |
|  |  | 120 |  |  |
|  |  |  | 154 |  |
|  |  |  |  |  |
|  |  |  |  |  |
|  |  |  |  |  |

In this analysis, we used GAMS software to analyze the results and solve the proposed models. According to the constraints of the criterion model (30) compared the constraints of the criterion models (10) and (23), this model has a smaller number of variables and we expect that the computational rate based on the criterion model (30) compared to the criterion models (10), (23) is less and we can use this model as a criterion model in the inputs/output estimation process based on inverse DEA-R models in the presence of ratio to reduce the time and number of calculations.

**6. Conclusion**

This paper presents inverse DEA-R models in the presence of ratio data. In this paper, we used the input orientation DEA-R models. We presented the inputs/output estimation process based on ratio based DEA (DEA-R) models. We showed that by determining the specific level of efficiency for each unit under evaluation DMU, we can determine the best possible level of inputs corresponding to a given level of outputs in the inputs/output estimation process based on inverse DEA-R models in the presence of ratio input components to output components. Next, we examined the criterion models in the inputs/output estimation process based on ratio based DEA (DEA-R) models. In this way, in order to reduce calculations, we presented a new criterion model based on DEA-R model, we have shown that by using this new criterion model, we can reduce the amount of computation outputs in the inverse DEA-R in order to compare the amount of efficiency of the unit under evaluation and the new unit created. We can easily use the proposed approach given that the models presented are linear and always feasible. As future work, we can examine inverse DEA-R models in the presence of ratio output components to input components, and we can develop the above models in other technologies such as VRS technology or non-convex technology. We can also develop the proposed models in this paper based on cost and revenue efficiency concepts, or develop the proposed models in this paper for other data structures in DEA, such as fuzzy data or network data.

**References**

[1] A. I. Ali, C. S. Lerme, and L. M. Seiford, Components of efficiency evaluation in data  
envelopment analysis. *European Journal of Operational Research*, 80 (1995), 462–473.

[2] G. R. Amin, and S. Al-Muharrami, A new inverse DEA model for merger with negative data. *IMA Journal of Management Mathematics*, 29 (2) (2018), 137–149.  
[3] G. R. Amin, S. Al-Muharrami, and M. Toloo, A combined goal programming and inverse DEA method for target setting in mergers. *Expert Systems with Applications*, 115 (2019), 412–417.  
[4] G. R. Amin, A. Emrouznejad, and S. Gattouf, Modelling generalized firms’ restructuring using inverse DEA. *Journal of Productivity Analysis*, 48 (1) (2017), 51–61.  
[5] G. R. Amin, A. Emrouznejad, and S. Gattouf, Minor and major consolidations in inverse DEA: Definition and determination. *Computers & Industrial Engineering*, 103 (2017),193–200.

[6] G. R. Amin and M. Ibn Boamah, A new inverse DEA cost efficiency model for estimating  
potential merger gains: a case of Canadian banks. *Annals of Operations Research*, 295 (1) (2020), 21-36.

[7] G. R. Amin and M. Ibn Boamah, A two-stage inverse data envelopment analysis approach for estimating potential merger gains in the US banking sector. [*Managerial and Decision Economics*](https://r.search.yahoo.com/_ylt=AwrE1.E3YpZgPXkAn75XNyoA;_ylu=Y29sbwNiZjEEcG9zAzEEdnRpZAMEc2VjA3Ny/RV=2/RE=1620497080/RO=10/RU=https%3a%2f%2fonlinelibrary.wiley.com%2fjournal%2f10991468/RK=2/RS=YoeDk01u113GLHVA3nlwgC4r4AY-), (2021), DOI: [10.1002/mde.3319](http://dx.doi.org/10.1002/mde.3319).

[8] G. R. Amin and A. Oukil, Flexible target setting in mergers using inverse data envelopment analysis. *International Journal of Operational Research*, 35 (3) A. (2019), 301–317.

[9] R. D. Banker, A. Charnes and W. W. Cooper, Some models for estimating technical and scale efficiencies in data envelopment analysis. *Management Science*, 30 (9) (1984), 1078–1092.

[10] A. Charnes, A., W. W. Cooper and E. Rhodes, Measuring efficiency of decision making units. *European Journal of Operational Research*, 2 (6) (1978), 429–44.

[11] Y. Chen and A. I. Ali, Output-input ratio analysis and DEA frontier, *European Journal  
of Operational Research*, 142 (2002), 476–479.

[12] Despic, O., Despic, M. and Paradi, J. C. (2007). DEA-R: Ratio-based comparative efficiency model, its mathematical relation to DEA and its use in applications. Journal of Productivity Analysis, 28(1), 33–44.

[13] A. Emrouznejad and Gh. R. Amin, DEA models for ratio data: Convexity consideration. Applied Mathematical Modelling, 33(1) (2009), 486–498.

[14] A. Emrouznejad and G. Yang, A survey and analysis of the first 40 years of scholarly literature in DEA: 1978–2016. *Socio-Economic Planning Sciences*, (2018), 61, 4–8.

[15] A. Emrouznejad, G. Yang and G. R. Amin, A novel inverse DEA model with application to allocate the emissions quota to different regions in Chinese manufacturing industries. *Journal of the Operational Research Society*, 70 (7) (2019), 1079–1090.  
[16] M. Eyni, G. Tohidi and S. Mehrabeian, Applying inverse DEA and cone constraint to sensitivity analysis of DMUs with undesirable inputs and outputs. *Journal of the Operational Research Society*, 68 (1) (2017), 34–40.

[17] A. Fernandez-Castro and P. Smith, Towards a general non-parametric model of corporate performance. *Omega*, 22 (3) (1994), 237–49.

[18] S. Gattouf, G. R. Amin and A. Emrouznejad, A new inverse DEA method for merging banks. *IMA Journal of Management Mathematics*, 25, (2014), 73–87.

[19] J. Gerami, M. R. Mozaffari and P.F. Wanke, A multi-criteria ratio-based approach for two-stage data envelopment analysis, *Expert Systems with Applications*, 158 (2020), 113508.

[20] J. Gerami, R. Kiani Mavi, R. Farzipoor Saen and N. Kiani Mavi, A novel network DEA-R model for evaluating hospital services supply chain performance. *Annals of Operations Research*. (2020), DOI: [10.1007/s10479-020-03755-w](http://dx.doi.org/10.1007/s10479-020-03755-w).

[21] M. Ghiyasi, On inverse DEA model: the case of variable returns to scale. *Computers and  
Industrial Engineering*, 87 (2015), 407–409.

[22] M. Ghiyasi, Inverse DEA based on cost and revenue efficiency. *Computers & Industrial Engineering*, 114 (2017), 258–263.

[23] M. Ghiyasi, Efficiency improvement and resource estimation: A tradeoff analysis. *International Journal of Productivity and Quality Management*, 25 (2) (2018), 151–169.

[24] A. Hadi-Vencheh and A.A. Foroughi, A generalized DEA model for inputs/outputs   
estimation. *Mathematical and Computer Modelling*, 43 (5) (2006), 447–457.

[25] A. Hatami-Marbini and M. Toloo, Data Envelopment Analysis Models with Ratio Data: A revisit. *Computers & Industrial Engineering*, 133 (2019), 331-338.

[26] G. R. Jahanshahloo, M. Soleimani-Damaneh and S. Ghobadi, Inverse DEA under inter-temporal dependence using multiple-objective programming*. European Journal of Operational Research*, 240 (2) (2015), 447–456.

[27] P. Khoshnevis and P. Teirlinck, Performance evaluation of R&D active firms. *Socio Economic Planning Sciences*, 61 (2018), 16–28.

[28] S. Lertworasirikul, P. Charnsethikul and S. C. Fang, Inverse data envelopment analysis  
model to preserve relative efficiency values: the case of variable returns to scale. *Computers  
and Industrial Engineering*, 61 (4) (2011), 1017–1023.

[29] D. J. Lim, Inverse DEA with frontier changes for new product target setting. *European Journal of Operational Research*, 254 (2) (2016), 510–516.

[30] M. R. Mozaffari, F. Dadkhah, J. Jablonsky and P. W. Wanke, Finding efficient surfaces in DEA-R models. *Applied Mathematics and Computation*, 386 (2021),125497.

[31] M. R. Mozaffari, J. Gerami and J. Jablonsky, Relationship between DEA models without explicit inputs and DEA-R models, *Central European Journal of Operations Research*, 22 (1) (2014), 1–12.  
[32] M. R. Mozaffari, P. Kamyab, J. Jablonsky and J. Gerami, Cost and revenue efficiency in DEA-R models. *Computers & Industrial Engineering*, 78 (12) (2014), 188–194.

[33] O. B. Olesen, N. C. Petersen and V. V. Podinovski, Efﬁciency analysis with ratio measures. *European Journal of Operational Research*, 245 (2) (2015), 446–462.

[34] O. B. Olesen, N. C. Petersen and V. V. Podinovski, Efﬁciency measures and computational approaches for data envelopment analysis models with ratio inputs and outputs. *European Journal of Operational Research*, 261 (2017), 640–655.

[35] M. Wegener and G. A. Amin, Minimizing GHG emissions using inverse DEA with an application in oil and gas. *Expert Systems with Applications*, 122 (2019), 369–375.

[36] C. K. Wei, L. C. Chen, R. K. Li and C. H. Tsai, A study of developing an input  
oriented ratio-based comparative efficiency model. *Expert Systems with Applications*, 38 (3) (2011), 2473–2477.  
[37] C. K. Wei, L. C. Chen, R. K. Li and C. H. Tsai, Exploration of efficiency underestimation of CCR model: Based on medical sectors with DEA-R model. *Expert Systems with Applications*, 38 (4) (2011), 3155–3160.  
[38] C. K. Wei, L. C. Chen, R. K. Li and C. H. Tsai, Using DEA-R model in the hospital industry to study the pseudo-inefficiency problem. *Expert Systems with Applications*, 38 (3) (2011), 2172–2176.

[39] Q. Wei, J. Zhang and X. Zhang, An inverse DEA model for inputs/outputs estimate. *European Journal of Operational Research*, 121 (1) (2000), 151–163.

[40] H. Yan, Q. Wei and G. Hao, DEA models for resource reallocation and production input/output estimation. *European Journal of Operational Research*, 136 (1) (2002), 19–31.