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A Look at $P(X > Y)$ in the Binomial Case

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Abstract: In this article we consider $P(X > Y)$ for two independent random variables $X \sim B(n+m, p_1)$ and $Y \sim B(n, p_2)$. This is a useful measure in biomedical studies and engineering reliability. The calculation of this probability is discussed by using a combinatorial identity and the approximate value of that is given when n is large. Finally some special cases are discussed.

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1. Introduction

There are some interesting problems in probability and statistics regarding two independent random variables X and Y . One of them is about the exact value, or parametric estimate, or non-parametric estimate of $P(X > Y)$. This is a useful measure, for example, in biomedical studies where X represents the result of an old treatment and Y the result of a new treatment. This probability is also useful for measuring the reliability of engineering systems ([6; pp 27-30]).

Suppose that f and F are the density and distribution of X, respectively and g and G of Y. Since X and Y are independent, conditioning on Y, we have easily

$$P(X > Y) = 1 - P(X \leqslant Y) = \begin{cases} 1 - \int_{-\infty}^{\infty} F(z)g(z)dz & (\textit{continuous case}) \\ 1 - \sum\limits_{z} F(z)g(z) & (\textit{discrete case}) \end{cases}$$

For example, if $X \sim N(\mu_1, \sigma^2)$ and $Y \sim N(\mu_2, \sigma^2)$ are independent, then

$$P(X > Y) = 1 - \Phi\left(\frac{\mu_2 - \mu_1}{\sigma\sqrt{2}}\right),$$

where Φ is the standard normal distribution. As another example, if $X \sim Exp(\theta_1)$ and $Y \sim Exp(\theta_2)$ are independent, then

$$P(X > Y) = \frac{\theta_2}{\theta_1 + \theta_2}.$$

However, when X and Y are discrete, the calculation of $P(X > Y)$ is not always straightforward or the result is not so simple. The purpose of this article is to study $P(X > Y)$ when $X \sim B(n + m, p_1)$ and $Y \sim B(n, p_2)$ are independent.

In Section 2, a simple example is given to show the method of calculation. In Section 3, a general case is considered and the complexity of the problem is discussed. In Section 4, an approximate value for $P(X > Y)$ is suggested when n is large by using the Bernoulli representation of X and Y and the Central Limit Theorem.

2. A Simple Example

We first look at the following simple example before we discuss about the general case.

Example 1. *Let* $X \sim B\left(4, \frac{1}{2}\right)$ *and* $Y \sim B\left(3, \frac{1}{2}\right)$ *be two independent Bernoulli variables. We can easily find the joint probability table of* X *and* Y *as follows:*

$y \backslash x$	0	1	2	3	4	$P(Y = y)$
0	$\frac{1}{128}$	$\frac{4^*}{128}$	$\frac{6^*}{128}$	$\frac{4^*}{128}$	$\frac{1^*}{128}$	$\frac{1}{8}$
1	$\frac{3}{128}$	$\frac{12}{128}$	$\frac{18^*}{128}$	$\frac{12^*}{128}$	$\frac{3^*}{128}$	$\frac{3}{8}$
2	$\frac{3}{128}$	$\frac{12}{128}$	$\frac{18}{128}$	$\frac{12^*}{128}$	$\frac{13^*}{128}$	$\frac{3}{8}$
3	$\frac{1}{128}$	$\frac{4}{128}$	$\frac{6}{128}$	$\frac{4}{128}$	$\frac{1^*}{128}$	$\frac{1}{8}$
$P(X = x)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$	1

Using this table, we find all the events for which $X > Y$ and their probabilities (marked by *). For example, $P(X = 3, Y = 1) = 12/128$. Therefore, we obtain

$$P(X > Y) = \frac{4^*}{128} + \frac{6^*}{128} + ... + \frac{1^*}{128} = \frac{1}{2}.$$

Actually, we have

$$\begin{aligned} P(X > Y) &= \sum_{y=0}^{3} \sum_{x=1}^{4-y} P(Y = y, X = y + x) \\ &= \sum_{y=0}^{3} \sum_{x=1}^{4-y} \binom{3}{y} \binom{4}{y+x} \left(\frac{1}{2}\right)^7 = \frac{1}{2} \end{aligned}$$

In Section 3 we find a general formula for $P(X > Y)$.

3. A General Case

Let $X \sim B(n+m, p_1)$ and $Y \sim B(n, p_2)$ be two independent binomial random variables. Following the pattern of the above simple example, we obtain:

$$\begin{aligned} P(X > Y) &= \sum_{y=0}^{n} \sum_{x=1}^{m+n-y} P(Y = y, X = y + x) \\ &= \sum_{y=0}^{n} \sum_{x=1}^{m+n-y} P(Y = y)P(X = y + x) \\ &= \sum_{y=0}^{n} \sum_{x=1}^{m+n-y} \binom{n}{y} p_2^y q_2^{n-y} \binom{m+n}{y+x} p_1^{y+x} q_1^{m+n-y-x} \\ &= \sum_{y=0}^{n} \sum_{x=1}^{m+n-y} \binom{n}{y} \binom{m+n}{y+x} p_2^y q_2^{n-y} p_1^{y+x} q_1^{m+n-y-x} \end{aligned}$$

This double sum is too complicated and it cannot be simplified. We consider some special cases.

(I) For $p_1 = q_1 = p_2 = q_2 = \frac{1}{2}$, we have:

$$P(X > Y) = \left(\frac{1}{2}\right)^{2n+m} \sum_{y=0}^{n} \sum_{x=1}^{m+n-y} \binom{n}{y} \binom{m+n}{y+x}$$

Here, we can reduce the double sum to a single sum. For this purpose, we use the fact that

$$\binom{N}{k} = 0 \qquad ; \qquad k > N$$

and we write

$$\sum_{y=0}^{n}\sum_{x=1}^{m+n-y}\binom{n}{y}\binom{m+n}{y+x}=\sum_{y=0}^{n}\sum_{x=1}^{m+n}\binom{n}{y}\binom{m+n}{y+x}.$$

Now, we are able to interchange the summation signs and to have

$$\sum_{x=1}^{m+n}\sum_{y=0}^{n}\binom{n}{y}\binom{m+n}{y+x}.$$

Next,we use the following combinatorial identity Number (10), given in

[5], page 217:

$$\sum_{k=0}^{M}\binom{M}{K}\binom{N}{R+K}=\binom{M+N}{M+R}.$$

This identity can be proved easily by the usual box-and-balls argument

if we replace $\binom{M}{K}$ by $\binom{M}{M-K}$. Thus, we have:

$$P(X>Y)=\left(\frac{1}{2}\right)^{2n+m}\sum_{x=1}^{m+n}\binom{2n+m}{n+x}.$$

(II) It is interesting to observe that for $m = 1$ and any integer $n \geqslant 1$,

we have $P(X > Y) = \frac{1}{2}$. This follows from the two identities

$$\binom{N}{K}=\binom{N}{N-K}\quad,\quad\sum_{K=0}^{N}\binom{N}{K}=2^N$$

and the fact that

$$\begin{aligned}\sum_{x=1}^{n+1}\binom{2n+1}{n+x} &= \binom{2n+1}{n+1}+\binom{2n+1}{n+2}+...+\binom{2n+1}{2n+1}\\ &= \binom{2n+1}{n}+\binom{2n+1}{n-1}+...+\binom{2n+1}{0}\\ &= \frac{1}{2}\left(2^{2n+1}\right)=2^{2n}.\end{aligned}$$

You could obtain this result by looking at the $(2n+1)$ th row of a Pascal Triangle. For $m=2$ and $m=3$ some rather simple results are obtained by an argument similar to the case $m=1$.

4. Approximation of $P(X>Y)$

As we discussed in Section 3, we cannot simplify $P(X>Y)$ in a general case. However, we can find an approximate value for this probability when n is large.

For this purpose we first consider the Bernoulli representation of X and Y. Then we apply conditional probability and the Central Limit Theorem.

It is well known that the independent random variables $X \sim B(n+m, p_1)$ and $Y \sim B(n, p_2)$ can be expressed in the following way:

$$\begin{aligned} X &= X_1 + X_2 + ... + X_n + X_{n+1} + ... + X_{n+m} \\ Y &= Y_1 + Y_2 + ... + Y_n, \end{aligned}$$

where $X_1, ..., X_{n+m}$ are independent Bernoulli variables with success probability p_1 and $Y_1, ..., Y_n$ are independent Bernoulli variables with success probability p_2 ; X_i 's are independent from Y_j 's.

Now, let $U = X_1 + X_2 + ... + X_n$ and $W = X_{n+1} + X_{n+2} + ... + X_{n+m}$.

It is clear that $U \sim B(n, p_1)$, $W \sim B(m, p_1)$, and $Y \sim B(n, p_2)$ are independent with $X = U + W$. We observe that

$$\begin{aligned} P(X > Y) &= P(U + W > Y) = P(Y - U < W) \\ &= \sum_{k=0}^{m} P(Y - U < W | W = k) P(W = k) \\ &= \sum_{k=0}^{m} P(Y - U < k) P(W = k) \\ &= \sum_{k=0}^{m} P(Y - U < k) \left(\begin{array}{c} m \\ k \end{array} \right) p_1^k q_1^{m-k}. \end{aligned}$$

Using the above Bernoulli representations, we can write

$$Y - U = (Y_1 - X_1) + (Y_2 - X_2) + ... + (Y_n - X_n) = \sum_{i=1}^{n} V_i,$$

where $V_1, V_2, ..., V_n$ are independent and identically distributed as

$V = v$	-1	0	1
$P(V = v)$	$p_1 q_2$	$p_1 p_2 + q_1 q_2$	$p_2 q_1$

with $E(V) = p_2 q_1 - p_1 q_2 = a$ and $Var(V) = p_1 q_1 + p_2 q_2 = b$. Now, by the Central Limit Theorem an approximate value for $P(Y - U < k)$ can be computed as follows:

$$\begin{aligned} P(Y - U < k) &\approx P(Y - U \leqslant k - 0.5) \\ &= P\left(\frac{Y - U - na}{\sqrt{nb}} \leqslant \frac{k - 0.5 - na}{\sqrt{nb}} \right) \\ &\approx P\left(Z \leqslant \frac{k - 0.5 - na}{\sqrt{nb}} \right) \end{aligned}$$

$$= \quad \Phi\left(\frac{k-0.5-na}{\sqrt{nb}}\right) = h(k;a,b),$$

where $Z \sim N(0,1)$ has distribution Φ . Thus, we have:

$$P(X>Y) \approx \sum_{k=0}^{m} h(k,a,b) \begin{pmatrix} m \\ k \end{pmatrix} p_1^k q_1^{m-k}$$

The exact value of the probability, for $m = 1$ and $n \geqslant 1$, and independent $X \sim B(n+1,p)$ and $Y = B(n,p)$ is

$$\begin{aligned} P(X>Y) \qquad &= qP(Y-U<0) + pP(Y-U<1) \\ &= qP(Y-U<0) + p[1-P(Y-U<0)] \\ &= p + (q-p)P(Y-U<0) < \frac{1}{2} \end{aligned}$$

This follows from the fact that $Y-U$, i.e., the difference of two independent random variables Y and U with common distribution $B(n,p)$, is symmetric about zero with positive probabilities at $0, \pm 1, \pm 2, ..., \pm n$. For $p = q = \frac{1}{2}$ we have $P(X>Y) = \frac{1}{2}$. This is the same answer we obtained in Section 3 by a combinatorial analysis.

It may be useful to observe that for two independent binomial variables $Y \sim B(n_1,p_1)$ and $U \sim B(n_2,p_2)$, the probability function of $Y-U$ with $p_1 = p_2 = \frac{1}{2}$ is

$$P(Y-U=k) = \left(\frac{1}{2}\right)^{n_1+n_2} \begin{pmatrix} n_1+n_2 \\ n_2+k \end{pmatrix}, \quad k = 0, \pm 1, \pm 2, ..., \pm n.$$

For obtaining this probability function, it is easy to show that $Y - U + n_2$ has binomial distribution $B(n_1 + n_2, \frac{1}{2})$. This can be proved by using the moment generating function of $Y - U + n_2$ or the fact that $Y + n_2 - U$ is the sum of two independent binomial variables with distributions $B(n_1, \frac{1}{2})$ and $B(n_2, \frac{1}{2})$. Now, $P(Y - U = k) = P(Y - U + n_2 = k + n_2)$ gives the result. Of course, for a general case, we cannot find a simple expression ([1;p 55]).

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