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Original Research Paper

The Stability and Convergence of The Numerical Computation for the Temporal Fractional Black-Scholes Equation

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Abstract. In this paper, the temporal fractional Black-Scholes model (TFBSM) is discussed in the limited specific domain which the time derivative of this template is the Caputo fractional function. The value variance of the associated fractal transmission method was applied to forecast TFBSM. For solving, at first the semi-discrete scheme is obtained by using linear interpolation with a temporally $\tau^{2-\alpha}$ order accuracy. Then, the full scheme is collected by approximating the spatial derivative terms with the help of the Chebyshev collocation system focused on the fourth form. Finally, the unconditional stability and convergence order are evaluated by performing the energy process. As an implementation of this method, two examples of the TFBSM were reported to demonstrate the accuracy of the developed scheme. Simulation and comparison show that the suggested strategy is very accurate and effective.

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1 Introduction

The importance in estimating financial derivatives, containing pricing options, stems from the act that it is possible to use financial derivatives to mitigate losses incurred by underlying finance price fluctuations. This safeguarding mechanism is called hedging. There are a range of financial instruments, such as swaps, forwards, futures and options, on the market.

An option is a financial convention that admited its possessor the right, on or before a specified date, to buy or sell a determined amount of a specific asset at a fixed price, known as the exercise price, called the maturity date. Options that can be employid before maturity at any moment are referred to as American, while options that can just be employid on maturity data are European. Options which offer the right to purchase the underlying asset are defined as calls, while options that give the right to sell the underlying asset are known as puts.

As pricing options exist, Black and Scholes [2] and Merton [23] in 1973 developed a formula for explaining the estimated behavior of the underlying finance called the Black-Scholes model (BSM). This instance has been commonly applied by merchants of options and ultimately contributes to a significant rise because of the precision in option trading and efficacy of the model in forecasting options prices. Fractional calculus and fractional partial differential equations were introduced with financial theory and the exploration of the stochastic system's fractal assembly and the financial region by replacing the fractional Brownian movement for the normal Brownian motion involved in classical design. Based on the non-locality of fractional integrals and derivatives, numerical methods represent a strong tool to solve them [25, 28, 9]. For example Hermite wavelets methods [18], homotopy perturbation Sumudu transform method [33], Legendre scaling functions as a basis [29], the compact finite difference scheme [27], the fourth kind of Shifted Chebyshev in collocation method [26], the Fibonacci collocation method [8] and spectral methods [10, 11] were presented for the resolution of fractional differential equations. Moreover, many models can be modeled through fractional derivatives or equations, such as the dynamical model of fractional host-parasitoid population [16], fractional order SEIR epidemic of measles [15], diffusion model arising in transport phenomena [32, 19], Lotka Volterra population model [14], the model of tumor and effector cells [17].

During the last years, further researchers have extended the BSM. For example, the moving least-squares approach is utilized for pricing double barrier options [12]. In [24], the Chebyshev collocation method is used to solve the time-fractional Black–Scholes. The pricing of the European call option was firstly carried out using a TFBSM [34]. Liang et al. In [22] proposed a specific state of the bi-fractional BSM of the TFBSM. Cartea in 2013 conducted another investigation into this model, presenting that a partial-integral-differential equation could describe the worth of European-style derivatives that includes a non-local time-to-maturity technician named the fractional derivative of the Caputo notion [3]. In addition, Leonenko et al. provided powerful explicit solutions, implementing spectral methods in fractional Pearson diffusions founded on the correlating time-fractional of diffusion model which was actually applied to develop BSM [21]. The authors also have made use of a non-Markovian stable inverse time variance to give stochastic solutions. In the current paper, we investigate TFBSM as

$$\begin{aligned}&{}_0D_t^{\alpha}u(x,t)-\frac{1}{2}\sigma^2\frac{\partial^2 u(x,t)}{\partial x^2}-(r-\frac{1}{2}\sigma^2)\frac{\partial u(x,t)}{\partial x}+ru(x,t)=f(x,t),\\&0<x<1,\quad 0<t\leq T,\quad 0<\alpha\leq 1,\end{aligned}\tag{1}$$

with the initial condition $u(x,0)=\phi(x)$ and the following boundary conditions

$$u(0,t)=\eta_0(t),\quad u(1,t)=\eta_1(t),\tag{2}$$

in which $r>0$ and $f(x,t)$ are the known constant and the source term, respectively.

In principle, the system (1) is a model of time-fractional advection-diffusion for $r-\frac{1}{2}\sigma^2<0$ and reaction-diffusion for $r=\frac{1}{2}\sigma^2$, $r\neq 0$. In the Eq. (1), we let $u(x,t)$ be the value of an option as a function of time and stock price and r, σ and t be the risk-free interest rate, the volatility of the standard deviation of the stock return and the time in the year, respectively.

The right Caputo fractional derivative ${}_0D_t^{\alpha}$ is defined as

$$ {}_0D_t^{\alpha}u(x,t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1}\frac{\partial^n u(x,\tau)}{\partial \tau^n}d\tau, \quad n-1<\alpha<n. $$

Seeing as of the retention feature of fractional derivatives, it is partly hard to get a precise solution to this subject. So several researchers have therefore desirable methods to estimate these tight problems. One of this methods is modified Legendre multiwavelet for solving the pricing discrete double barrier option [30]. The more method used to solve the analytical form [4, 7]. Generally, the solutions obtained through the mentioned schemes take in the form of an infinite series, which makes them hard to solve. For this purpose, greater attention is being paid to developing efficient numerical computation for the solution of fractional BSM. Some of those strategies will be described below. Option pricing through temporal fractional BSM is overcome by a θ finite-difference structure of 2nd order precision and an implicit scheme of finite difference with accurate first-order precision in [36] and [31], respectively. In 2014, Bhowmik applied an explicit–implicit numerical method to solve a partial integro-differential system which correlates to an assumption of option pricing [1]. Chen in 2015 used a predictor-corrector for approaching American options pricing in [5]. In addition, the authors in [35] present a discreet implicit numerical solution to this option.

This article contains the following sections. The time discretization based on a quadratic interpolation and full scheme of discretization founded on the Chebyshev polynomials of the fourth kind for solving the equation are created in section 2. In Sections 3,the stability and convergence study of temporal discretization are demonstrated. Finally, in the late part, to illustrate the expertise of the new approach, we provide two numerical examples.

2 The Temporal and Full Scheme

Designating $u^j = u(x, t_j), \mathfrak{p} = \frac{\Gamma(2-\alpha)}{2}\sigma^2, \mathfrak{q} = \Gamma(2-\alpha)(r - \frac{1}{2}\sigma^2), \mathfrak{r} = r\Gamma(2-\alpha), F^M = \Gamma(2-\alpha)f(x, t_M)$ and $\overline{\mathcal{S}}_{M,j} = -\mathcal{S}_{M,j}$ and using the linear scheme for approximating ${}_0D_t^{\alpha}u(x,t)$ that is determined in [20] that nodes of time with the step size $\tau = \frac{T}{M}$ be $t_j = j\tau,\ j = 0, 1, \ldots, M$, we can get the following semi-discrete scheme of Eq. (1) for $0 < \alpha \leq 1$ as

$$(\mathcal{S}_{M,M} + \mathfrak{r}\tau^{\alpha})u^M - \mathfrak{p}\tau^{\alpha}\frac{\partial^2 u^M}{\partial x^2} - \mathfrak{q}\tau^{\alpha}\frac{\partial u^M}{\partial x} = \sum_{j=0}^{M-1}\overline{\mathcal{S}}_{M,j}u^j + \tau^{\alpha}F^M + \tau^{\alpha}\mathcal{R}^M, \tag{3}$$

where a nonnegative C exists such that $\mathcal{R}^M \leq C\mathcal{O}(\tau^{2-\alpha})$ and

$$\mathcal{S}_{M,j} = \begin{cases} 1, & j = M, \\ (M-j-1)^{1-\beta} - 2(M-j)^{1-\beta} + (M-j+1)^{1-\beta}, & 1 \leq j < M, \\ (M-1)^{1-\beta} - (M)^{1-\beta}, & j = 0. \end{cases} \tag{4}$$

The subsequent semi-discrete design can be obtained by eliminating $\mathcal{R}^M$ in Eq. (3), as

$$\begin{cases} (\mathcal{S}_{M,M} + \mathfrak{r}\tau^{\alpha})U^M - \mathfrak{p}\tau^{\alpha}\frac{\partial^2 U^M}{\partial x^2} - \mathfrak{q}\tau^{\alpha}\frac{\partial U^M}{\partial x} = \sum\limits_{j=0}^{M-1}\overline{\mathcal{S}}_{M,j}U^j + \tau^{\alpha}F^M, \\ U^0(x) = \phi(x), \quad 0 < x < 1, \\ U^j(0) = \eta_0(t_j), \quad U^j(1) = \eta_1(t_j), \quad j = 0, 1, \ldots, M, \end{cases} \tag{5}$$

where the approximate solution of Eq. (3) is $U^j, \quad j = 0, 1, \ldots, M$.
Next, to obtain the space-discrete scheme of Eq. (5), we apply the shifted Chebyshev polynomials of the fourth kind (SCPFK) $\mathcal{W}_i(x),\ i = 0, 1, \ldots, N$ as the following scheme

$$\mathcal{W}_i^*(x) = \mathbf{\Lambda}_i \sum_{k=0}^{i-1}\sum_{\xi=0}^{k}\Upsilon_{i,k,\xi} \times x^{k-\xi}, \quad x \in [0,1], \quad i = 1, 2, \ldots.$$

$$\mathbf{\Lambda}_i = \frac{(2^{2i-2})\Gamma(i+0.5)(i-1)!}{(2i-2)!}, \quad \Upsilon_{i,k,\xi} = \frac{(-1)^{\xi}\Gamma(i+k)}{(k-\xi)!\xi!(i-k-1)\Gamma(k+1.5)}.$$

Now only by using the first $N+1$-terms of SCPFK at duration $[0,1]$ is the following expansion of $u(x,t_j)$ from around space variable defined as

$$u(x,t_j)=\sum_{i=0}^{N}\upsilon_i(t_j)\mathcal{W}_i^*(x), \tag{6}$$

where $\upsilon_i(t_j)$ is the unknown coefficients that are defined as

$$\upsilon_i(t_j)=\frac{2}{\pi}\int_0^1\sqrt{\frac{1-x}{x}}u(x,t_j)\mathcal{W}_i^*(x)dx,\quad j=0,1,\dots,M. \tag{7}$$

To get a full-discrete scheme Eq. (5), we approximate the first and second order space derivative, $\frac{\partial^l u^M}{\partial x^l}, l=1,2$, based on SCPFK. By using Eq. (6), we have

$$\frac{\partial^\xi(u(x,t_j))}{\partial x^\xi}=\sum_{i=\xi}^{N}\sum_{k=0}^{i-\xi}\sum_{l=0}^{k}\upsilon_i(t_j)N_{i,k,l}^{\xi}x^{k-l},\quad \xi\in\mathbb{N}, \tag{8}$$

where $N_{i,k,l}^{\xi}$ is given by

$$N_{i,k,l}^{\xi}=\frac{(-1)^l\ 2^{2i}\ (i)!\ \Gamma(i+0.5)\ \Gamma(i+k+\xi+1)\ \Gamma(k-l+\xi+1)}{(2i)!\ (i-k-\xi)!\ (k+\xi-l)!\ \Gamma(k+\xi+1.5)\ \Gamma(k-l+1)\Gamma(l+1)}.$$

With taking the collocation points $\{x_s=\frac{-\cos(\pi\times\frac{s+i+0.5}{s+0.5})+1}{2}\}_{s=1}^{N+1-\xi}$ that are the roots of SCPFK $\mathcal{W}_{N+1-\xi}^*(x)$ and substituing Eq. (8) in Eq. (5) we arrive in a point (x_s,t_j) at

$$\begin{aligned}&(\mathcal{S}_{j,j}+\mathfrak{r}\tau^\alpha)\sum_{i=0}^{N}\upsilon_i^j\mathcal{W}_i^*(x_s)-\mathfrak{p}\tau^\alpha\sum_{i=2}^{N}\sum_{k=0}^{i-2}\sum_{l=0}^{k}\upsilon_i^jN_{i,k,l}^2x_s^{k-l}\\&-\mathfrak{q}\tau^\alpha\sum_{i=1}^{N}\sum_{k=0}^{i-1}\sum_{l=0}^{k}\upsilon_i^jN_{i,k,l}^1x_s^{k-l}=\sum_{m=0}^{j-1}\sum_{i=0}^{N}\overline{\mathcal{S}}_{j,m}\upsilon_i^m\mathcal{W}_i^*(x_s)+\tau^\alpha F(x_s,t_j),\\&j=1,2,\dots,M,\qquad s=0,1,\dots,N,\end{aligned} \tag{9}$$

where $\upsilon_i^j=\upsilon_i(t_j)$ are the unknown coefficients. The above relation with the following boundary conditions gives $N+1$ linear algebraic equations which one can be determined the unknown coefficients $\upsilon_i^i, i=$

$0, 1, 2, \ldots, N$ in each step of time j. Notice that we replace Eq. (6) in (2) to specify the boundary conditions as

$$\sum_{i=0}^{N}(-1)^i \upsilon_i^j = \eta_0(t_j), \quad \sum_{i=0}^{N}(2i+1)\upsilon_i^j = \eta_1(t_j), \quad j = 1, 2, \ldots, M. \quad (10)$$

In addition, the initial solution υ_i^0 is obtained by combining relationship $u(x, 0) = \phi(x)$ with Eq. (7).

3 The Stability of the Semi-Discrete Scheme with the Convergence Order

In the current section, we state theorems for proofing of the stability of the semi discrete of Eq. (5). Substantiation of the theorems calculate that the novel method is the unconditionally stable and convergence order is $\mathcal{O}(\tau^{2-\alpha})$. The functional space is described as following

$$\mathcal{H}_\Omega^n(\varphi) = \{\varphi \in L^2(\Omega), \quad \frac{\partial^\alpha \varphi}{\partial x^\alpha} \in L^2(\Omega), \ \forall \ |\alpha| \leq n\},$$

in which $L^2(\Omega)$ is the measurable function space which in Ω is square Lebesgue integrable. To claim the unconditional stability we need to prove $\|\varepsilon^M\| \leq C\|\varepsilon^0\|$, where C is nonnegative. For this work, with multiplying Eq. (5) in U^j and using the error function $\varepsilon^j = u^j - U^j$, $j = 0, 1, \ldots, k$, we can rewrite Eq. (5) as following

$$(1+\mathfrak{r}\tau^\alpha)\langle\varepsilon^k, \varepsilon^k\rangle - \mathfrak{p}\tau^\alpha\langle\frac{\partial^2\varepsilon^k}{\partial x^2}, \varepsilon^k\rangle - \mathfrak{q}\tau^\alpha\langle\frac{\partial\varepsilon^k}{\partial x}, \varepsilon^k\rangle = \sum_{j=0}^{k-1}\overline{\mathcal{S}}_{k,j}\langle\varepsilon^j, \varepsilon^k\rangle. \quad (11)$$

Theorem 3.1. *The obtained scheme by Eq. (5) is unconditionally stable.*

Proof. First of all, without losing to the whole issue, we would apply the condition $1 + \mathfrak{r}\tau^\alpha > 1$ and use

$$\langle\frac{\partial^2 U^1}{\partial x^2}, U^1\rangle = -\langle\frac{\partial U^1}{\partial x}, \frac{\partial U^1}{\partial x}\rangle, \qquad \langle\frac{\partial U^1}{\partial x}, U^1\rangle = 0,$$

for the second and third term in the left hand side of Eq. (11). Then we have

$$\langle \varepsilon^k, \varepsilon^k \rangle \leq \sum_{j=0}^{k-1} \overline{\mathcal{S}}_{k,j} \langle \varepsilon^j, \varepsilon^k \rangle.$$

To prove, we use the mathematical induction on k. For $k = 1$ in the the above relation and using the Cauchy–Schwarz inequality, we have

$$\|\varepsilon^1\| \leq \|\varepsilon^0\|, \tag{12}$$

because $\overline{\mathcal{S}}_{1,0} = 1$. Now we let

$$\|\varepsilon^k\| \leq \|\varepsilon^0\|, \qquad k = 1, 2, \ldots, M-1. \tag{13}$$

For $k = M$, one can get

$$\|\varepsilon^M\| \leq \sum_{j=0}^{M-1} \overline{\mathcal{S}}_{k,j} \|\varepsilon^0\|.$$

In the other hand, by using $1 < \sum_{j=0}^{M-1} \overline{\mathcal{S}}_{M,j} < 2$ that is presented in [26], we can conclude

$$\|\varepsilon^M\| \leq C\|\varepsilon^0\|,$$

where C is nonnegative. Thus the theorem is proved. □

Theorem 3.2. *$\mathcal{O}(\tau^{2-\alpha})$ is the order of convergence of the time-discrete scheme (5).*

Proof. For all $k = 1, 2, \ldots, M$, suppose U^k and u^k be the approximation and exact solutions of Eqs. (5) and (1), respectively. Then $\varepsilon^k = |u^k - U^k|, k = 1, 2, \ldots, M$ is the errors of Eq. (5) as following

$$\begin{aligned}(1 + \mathfrak{r}\tau^\alpha)\langle \varepsilon^k, \varepsilon^k \rangle - \mathfrak{p}\tau^\alpha \langle \frac{\partial^2 \varepsilon^k}{\partial x^2}, \varepsilon^k \rangle - \mathfrak{q}\tau^\alpha \langle \frac{\partial \varepsilon^k}{\partial x}, \varepsilon^k \rangle = \sum_{j=0}^{M-1} \overline{\mathcal{S}}_{M,j} \langle \varepsilon^j, \varepsilon^k \rangle \\ + \tau^\alpha \langle \mathcal{R}^k, \varepsilon^k \rangle,\end{aligned} \tag{14}$$

where $\mathcal{R}^M \leq C\mathcal{O}(\tau^{2-\alpha})$. Regarding with proof procedure of Theorem 3.1, we have

$$(1+\mathfrak{r}\tau^{\alpha})\langle \varepsilon^k, \varepsilon^k\rangle \leq \sum_{j=0}^{M-1} \overline{\mathcal{S}}_{M,j}\langle \varepsilon^j, \varepsilon^k\rangle + \tau^{\alpha}\langle \mathcal{R}^k, \varepsilon^k\rangle.$$

We can easily achieved the following result

$$\mathfrak{r}\tau^{\alpha}\|\varepsilon^k\| \leq \sum_{j=0}^{M-1} \overline{\mathcal{S}}_{M,j}\|\varepsilon^j\| + \tau^{\alpha}\|\mathcal{R}^k\|.$$

Now with the previous theorem result i.e. $\|\varepsilon^j\| \leq C\|\varepsilon^0\|, j = 1, 2, \ldots, M$, we have

$$\mathfrak{r}\tau^{\alpha}\|\varepsilon^k\| \leq C\|\varepsilon^0\| \sum_{j=0}^{M-1} \overline{\mathcal{S}}_{M,j} + \tau^{\alpha}\|\mathcal{R}^k\|.$$

Since $\|\varepsilon^0\| = 0$ and $\frac{1}{\mathfrak{r}} = \frac{1}{r\Gamma(2-\alpha)} < 1$, then we gain

$$\mathfrak{r}\tau^{\alpha}\|\varepsilon^k\| \leq \tau^{\alpha}\|\mathcal{R}^k\| \Longrightarrow \|\varepsilon^k\| \leq \frac{1}{\mathfrak{r}}\|\mathcal{R}^k\| \leq \|\mathcal{R}^k\|,$$

which completes the proof. □

4 Presentation Numerical Results

The section of numerical results contains the efficiency and accuracy of the developed method for the numerical scheme of TFBSM that the present form of price barrier choice regulated by a time fractional BSM model. The computational order is calculated by $\mathcal{C}_{\mathcal{O}} = \log_2(\frac{E_{i+1}}{E_i})$ where errors E_{i+1} and E_i correspond to mesh sizes $2M$ and M, respectively. The calculated results support the theoretical analysis. The authors calculated the numerical results applying Wolfram Mathematica v11.3.0 software on a Core i7, 2.8GHz device with 4 Gbyte of memory.

Example 4.1. The TFBSM with homogeneous boundary conditions consider as below

$$\begin{cases} {}_0D_t^{\alpha}u(x,t) = p\frac{\partial^2 u(x,t)}{\partial x^2} + q\frac{\partial u(x,t)}{\partial x} - ru(x,t) + f(x,t), \\ u(x,0) = x^2(1-x), \quad 0 < x < 1, \\ u(0,t) = u(1,t) = 0, \end{cases}$$

Table 1: The error and computational order for Example 4.1 with $N = 5$ at $T = 1$.

	$\alpha = 0.2$				$\alpha = 0.7$			
M	L_∞	$\mathcal{C}_\mathcal{O}$	L_2	$\mathcal{C}_\mathcal{O}$	L_∞	$\mathcal{C}_\mathcal{O}$	L_2	$\mathcal{C}_\mathcal{O}$
100	$5.49597E{-}6$		0.000011871		0.000193171		0.000415628	
200	$1.63901E{-}6$	1.74555	$3.54044E{-}6$	1.74546	0.000078748	1.29457	0.000169444	1.29449
400	$4.85842E{-}7$	1.75426	$1.04953E{-}6$	1.75419	0.000032053	1.29678	0.000068972	1.29673
800	$1.43310E{-}7$	1.76135	$3.09593E{-}7$	1.76129	0.000013035	1.29808	0.000028049	1.29805
1600	$4.21016E{-}8$	1.76719	$9.09555E{-}8$	1.76714	$5.29802E{-}6$	1.29885	0.000011401	1.29883
TOC		1.8		1.8		1.3		1.3

Table 2: The temporal convergence order for Example 4.1 at $T = 1$.

	Method of [6] for $N = 150$ and $\alpha = 0.7$		Method of [13] for $N = 150$ and $\alpha = 0.7$		Current method for $N = 5$ and $\alpha = 0.7$		Current method for $N = 5$ and $\alpha = 0.2$	
M	L_∞	$\mathcal{C}_\mathcal{O}$	L_∞	$\mathcal{C}_\mathcal{O}$	L_∞	$\mathcal{C}_\mathcal{O}$	L_∞	$\mathcal{C}_\mathcal{O}$
10	$3.5000E{-}3$		$5.821E{-}3$		$3.68335E{-}3$		$2.86016E{-}4$	
20	$1.4400E{-}3$	1.3300	$2.304E{-}3$	1.3372	$1.53054E{-}3$	1.26698	$8.82980E{-}5$	1.69564
40	$5.9000E{-}4$	1.3150	$9.081E{-}4$	1.3421	$6.29816E{-}4$	1.28104	$2.68851E{-}5$	1.71558
80	$2.4000E{-}4$	1.3400	$3.572E{-}4$	1.3461	$2.57742E{-}4$	1.28900	$8.10114E{-}6$	1.73061
160	$9.5000E{-}5$	1.3600	$1.411E{-}4$	1.3400	$1.05144E{-}4$	1.29356	$2.42134E{-}6$	1.74232
320	$3.8000E{-}5$	1.3800	$5.387E{-}5$	1.3892	$4.28146E{-}5$	1.29619	$7.19043E{-}7$	1.75166
TOC		1.3		1.3		1.3		1.8

with $\sigma = 0.25, p = \frac{1}{2}\sigma^2, q = r - p, r = 0.05, \alpha = 0.7$ and the source term $f(x,t) = (\frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{2t^{1-\alpha}}{\Gamma(2-\alpha)})x^2(1-x) - (t+1)^2[p(2-6x) + q(2x - 3x^2) - rx^2(1-x)]$. For this problem, the actual answer is $u(x,t) = (t+1)^2x^2(1-x)$.

The obtained order is shown in Table 1 with $N = 5$ at $T = 1$. We can see that the computational order is $\mathcal{O}(\tau^{2-\alpha})$ and it is close with time order of convergence (TOC). Focused on the thorough similarities in Table 2, it is possible to conclude that the results are in an absolute agreement with [6] and [13]. In addition, highly accurate results are given with very low space size. In Figure 1, the absolute error and approximate solution are shown at $T = 1$.

Example 4.2. As the second example, the following TFBSM with non-

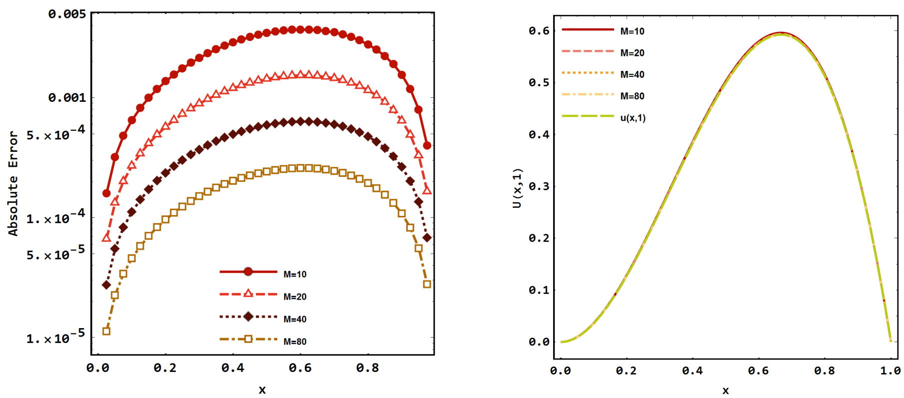

Figure 1: The error (left-side) and approximate solution (right-side) for Example 4.1 at $T = 1$.

homogeneous boundary conditions be considered:

$$
\begin{cases}
{}_0D_t^{\alpha} u(x,t) = p\frac{\partial^2 u(x,t)}{\partial x^2} + q\frac{\partial u(x,t)}{\partial x} - ru(x,t) + f(x,t), \\
u(x,0) = x^3 + x^2 + 1, \quad 0 < x < 1, \\
u(0,t) = (t+1)^2, \quad u(1,t) = 3(t+1)^2,
\end{cases}
$$

in which the source term $f(x,t)$ is obtained from the exact solution $u(x,t) = (t+1)^2(x^3+x^2+1)$. It is possible to pick dependent parameters with values such as $p = 1, q = r - p, r = 0.5$ and $\alpha = 0.7$.

Comparisons of compact finite difference way [6] and radial base functions relying on finite difference design [13] with the current method are shown in Table 3 to provide better results for our method. Furthermore, the convergence order is seen in Table 4 with $N = 5$ at $T = 1$, indicating that the theoretical results are confirmed by the computational order. The numerical simulation and absolute error in which the approximation solution is compared to the exact solution are shown in Figure 2.

5 Conclusion

This study is presented to evaluate a numerical scheme of TFBSM that the nature of the model's fractional-order derivative leads to compli-

Table 3: The obtained errors and temporal convergence order at $T = 1$ for Example 4.2.

	Method of [6] for $N = 150$ and $\alpha = 0.7$		Method of [13] for $N = 150$ and $\alpha = 0.7$		Present method for $N = 5$ and $\alpha = 0.7$		Present method for $N = 5$ and $\alpha = 0.2$	
M	L_∞	$\mathcal{C}_\mathcal{O}$	L_∞	$\mathcal{C}_\mathcal{O}$	L_∞	$\mathcal{C}_\mathcal{O}$	L_∞	$\mathcal{C}_\mathcal{O}$
10	$5.2000E{-}3$		$6.345E{-}3$		$5.46926E{-}3$		$4.40763E{-}4$	
20	$2.0700E{-}3$	1.3300	$2.507E{-}3$	1.3372	$2.23904E{-}3$	1.28846	$1.35306E{-}4$	1.70378
40	$8.3000E{-}4$	1.3150	$9.957E{-}4$	1.3421	$9.13691E{-}4$	1.29310	$4.10373E{-}5$	1.72122
80	$3.3000E{-}4$	1.3400	$4.011E{-}4$	1.3461	$3.72148E{-}4$	1.29583	$1.23299E{-}5$	1.73478
160	$1.3000E{-}4$	1.3600	$1.591E{-}4$	1.3400	$1.51405E{-}4$	1.29746	$3.67703E{-}6$	1.74554
320	$5.0000E{-}4$	1.3800	$6.274E{-}5$	1.3892	$6.15558E{-}5$	1.29845	$1.09000E{-}6$	1.75422
TOC		1.3		1.3		1.3		1.8

Table 4: The temporal order, L_∞ and L_2 with $N = 5$ at $T = 1$ for Example 4.2.

	$\alpha = 0.2$				$\alpha = 0.9$			
M	L_∞	$\mathcal{C}_\mathcal{O}$	L_2	$\mathcal{C}_\mathcal{O}$	L_∞	$\mathcal{C}_\mathcal{O}$	L_2	$\mathcal{C}_\mathcal{O}$
15	$2.21261E{-}4$		$5.11673E{-}4$		$7.74056E{-}3$		$1.78842E{-}2$	
30	$6.74186E{-}5$	1.71453	$1.55906E{-}4$	1.71454	$3.61591E{-}3$	1.09808	$8.35434E{-}3$	1.09809
60	$2.03299E{-}5$	1.72954	$4.70128E{-}5$	1.72955	$1.68801E{-}3$	1.09903	$3.90004E{-}3$	1.09904
120	$6.08040E{-}6$	1.74136	$1.40608E{-}5$	1.74137	$7.87759E{-}4$	1.09950	$1.82006E{-}3$	1.09950
240	$1.80667E{-}6$	1.75083	$4.17788E{-}6$	1.75084	$3.67569E{-}4$	1.09974	$8.49241E{-}4$	1.75084
TOC		1.8		1.8		1.1		1.1

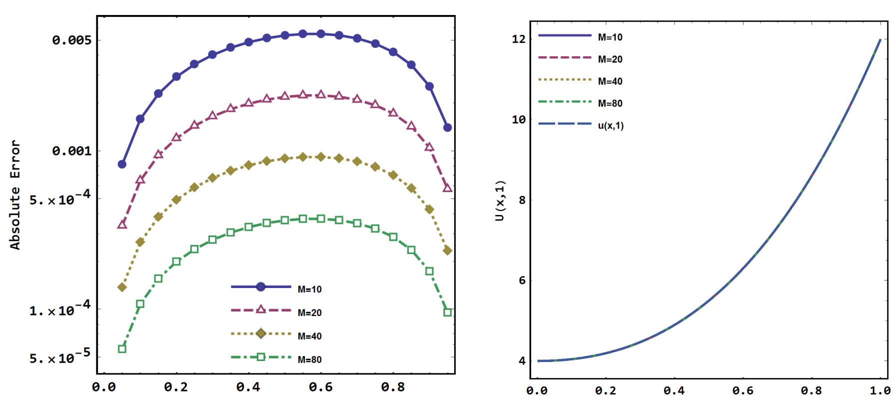

Figure 2: The error (left-side) and approximate solution (right-side) for Example 4.2 at $T = 1$.

cated precise and numerical solutions. For this reason, the modified Riemann–Liouville fractional derivative is already replaced with the Caputo fractional derivative in the TFBSM. The first step of the method in discretizing the equation is the discretization of the time variable that is discretized with linear interpolation (accuracy order of $\mathcal{O}(\tau^{2-\alpha})$). This resulted in a semi-discrete scheme for TFBSM. By applying the Chebyshev collocation manner based on the fourth form, we will then illustrate how to achieve the full-discrete scheme. Moreover, by applying the energy method, the unconditional stability of the time-discrete structure and the convergence order of the time-discrete were proved. To show the precision and convergence order of the numerical method, two numerical instances with analytical solutions are selected that the numerical conclusion has demonstrated the preciseness of the new scheme.

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