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Original Research Paper

An Accurate Approach Based on Operational Matrices of Modified Hat Functions for Solving a System of Fractional Stochastic Integro-Differential Equations

E. Aryani
University of Mazandaran

A. Babaei*
University of Mazandaran

A. Valinejad
University of Mazandaran

Abstract. In this paper, a numerical method based on modified hat functions (MHFs) is investigated to find an approximate solution for a fractional-order system of stochastic integro-differential equations. The fractional and stochastic operational matrices of integration of these functions are employed to present the numerical approach. By using these operational matrices and properties of MHFs, the considered problem is transformed into a system of algebraic equations which can be easily solved by an iterative method. Also, error analysis of the proposed method is discussed. At the end, the accuracy and effectiveness of this approach are studied by some numerical examples.

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*Corresponding Author

1 Introduction

In recent years, fractional integral equations and fractional differential equations have gained popularity for modelling widespread fields of science such as control theory of dynamical systems, electrical networks, optics, signal processing, biology and so on [6, 8, 11, 12, 14, 16, 27, 30, 31]. Thus, numerous numerical methods have been proposed for solving these classes of equations, for example hybrid collocation method [18], least squares method [10], finite volume element method [5], Genocchi polynomials [32] and Green-Haar wavelets method [29].

Deterministic differential equations have an important role in application of mathematics to many branches of science. But in general, more realistic formulations of deterministic equations that have wide efficiency in sciences are encounter with random noise or uncertainties. This phenomenon generates stochastic equations [3, 17, 22, 26, 28]. In other words, stochastic differential equations arise when a random noise is introduced into deterministic differential equations.

For accurate describing these events with mathematical modelings, various kinds of stochastic differential equations or stochastic integro-differential equations have been utilized. A well known example of stochastic processes is Brownian motion that is a random process of particle when it encounters with fluid molecules [26]. Wide range of problems in sciences such as economics, medicine and physics have been simulated as fractional stochastic integral equations [7, 9, 19]. In many cases, analytical solution of fractional stochastic integro-differential and integral equations are not known. So, several researchers have been influenced to obtain numerical approaches for these equations, such as Monte-Carlo Galerkin Approximation [4], Galerkin method based on orthogonal polynomials [13], expansion method [15], Block pulse approximation [2], operational matrix of the Chebyshev wavelets [23], spectral collocation method [33], operational matrices of hybrid of block-pulse and parabolic functions method [20], computational scheme based on B-spline interpolation method [22] and so on.

In this work, we consider a system of fractional stochastic integro-differential equations (FSIDEs) as

$$\begin{aligned}\mathbf{D}_{0,t}^{\alpha}\mathbf{Y}(t) = \mathbf{F}(t) + \mu\mathbf{Y}(t) \quad &+ \quad \int_0^t \mathbf{k}(t,s)\mathbf{Y}(s)\mathrm{d}s \\ &+ \quad \sigma\int_0^t \hat{\mathbf{k}}(t,s)\mathbf{Y}(s)\mathrm{d}\mathcal{B}(s), \quad t \in \Omega, \quad (1)\end{aligned}$$

with the initial condition

$$\mathbf{Y}(0) = \Lambda_{\text{int}}, \tag{2}$$

where σ is a positive real constant, $\Omega := [0,T]$ and the vectors in the system (1) are defined as follows:

$$\begin{aligned}\mathbf{Y}(t) &= [y_1(t), \ldots, y_i(t), \cdots, y_m(t)]^T, \\ \mathbf{F}(t) &= [f_1(t), \ldots, f_i(t), \ldots, f_m(t)]^T,\end{aligned}$$

and the matrices of the system (1) are defined by

$$\begin{aligned}\mu &= \operatorname{diag}[\mu_1, \cdots, \mu_i, \ldots, \mu_m], \\ \mathbf{k}(t,s) &= \Big[k_{ij}(t,s)\Big]_{m\times m}, \qquad i,j = 1,2,\ldots,m, \\ \hat{\mathbf{k}}(t,s) &= \Big[\hat{k}_{ij}(t,s)\Big]_{m\times m}, \qquad i,j = 1,2,\ldots,m.\end{aligned}$$

The vector of initial conditions Λ_{int} in (2) is defined as:

$$\Lambda_{\text{int}} := [d_1, \ldots, d_i, \cdots, d_m]^T,$$

where d_i, $i = 1, ..., m$, are real constants. Furthermore, the operator $\mathbf{D}_{0,t}^{\alpha}[\cdot]$ denotes the vector of Caputo fractional derivative defined as [27]:

$$\mathbf{D}_{0,t}^{\alpha}[\cdot] := \Big[D_{0,t}^{\alpha_1}[\cdot], \cdots, D_{0,t}^{\alpha_i}[\cdot], \cdots, D_{0,t}^{\alpha_m}[\cdot]\Big]^T,$$

where

$$D_{0,t}^{\alpha_i} g(t) = \frac{1}{\Gamma(1-\alpha_i)}\int_0^t \frac{g'(s)}{(t-s)^{\alpha_i}}\mathrm{d}s, \qquad \alpha_i \in (0,1),$$

and $\Gamma(\cdot)$ represents the Gamma function. Let $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space with a normal filtration $(\mathcal{F}_t)_{t\geq 0}$, and $\mathcal{B} = \{\mathcal{B}(t),\ t \geq 0\}$ is a Brownian motion. Moreover, the functions $f_i(t),\ i = 1, ..., m,\ k_{ij}(t,s)$ and $\hat{k}_{ij}(t,s),\ i,j = 1, ..., m$, for $t,s \in \Omega$, are known stochastic processes defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $y_i(t)$ are the unknown functions to be found.

To our knowledge, a few works have been concerned with numerical solving for the problems related to the systems of fractional order stochastic Volterra-Fredholm integro-differential equations. Thus, in the rest, we will try to propose a numerical technique to solve the problem (1)-(2), based on the operational matrices of modified hat functions (MHFs). By using these operational matrices, this problem will be reduced into a system of algebraic equations. Some previous articles show that numerical methods of MHFs are convergence and very accurate [1, 21, 24, 25, 34]. Principal advantages of numerical approaches with this type of basis functions include the speed and simplicity of implementation, stability, and low computational costs.

The outline of the paper is as follows: In Section 2, some basic definitions of fractional calculus, the definition of modified hat functions, their properties and operational matrices of integration based on MHFs are introduced. The numerical method is described in Section 3. Error estimate of the proposed method is investigated in Section 4. Some numerical implementations are presented in Section 5 to illustrate the accuracy of our method. Finally, conclusion of this work is included in Section 6.

2 Preliminary concepts and tools

In this section, some useful concepts and tools that are used during this paper will be introduced.

2.1 Fractional calculus

First, we consider some definitions and properties of the Caputo derivative and the Riemann-Liouville integral.

Definition 2.1. [27] The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, is defined as

$$\mathcal{I}^{\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) \mathrm{d}s, \quad t > 0.$$

Lemma 2.2. *[27] Assume that $r \in \mathbb{N}$, $r-1 < \alpha \leq r$ and $g \in C^{r-1}[0,T]$. Then*

$$\mathcal{I}^{\alpha}(\mathcal{I}^{\beta} g(t)) = \mathcal{I}^{\beta}(\mathcal{I}^{\alpha} g(t)) = \mathcal{I}^{\alpha+\beta} g(t),$$
$$\mathcal{I}^{\alpha}(D_{0,t}^{\alpha} g(t)) = g(t) - \sum_{k=0}^{r-1} \frac{t^k}{k!} g^{(k)}(0^+),$$
$$D_{0,t}^{\alpha}(D_{0,t}^{r} g(t)) = D_{0,t}^{r}(D_{0,t}^{\alpha} g(t)) = D_{0,t}^{r+\alpha} g(t).$$

2.2 Properties of MHFs

Hat functions, sometimes called tent or triangular functions, are continuous functions defined on the interval $[0,T]$ as [34]

$$\psi_0(t) = \begin{cases} 1 - \frac{t}{h}, & t \in [0,h], \\ \\ 0, & otherwise, \end{cases}$$

$$\psi_i(t) = \begin{cases} \frac{t}{h} - (i-1), & t \in [(i-1)h, ih], \\ \\ (i+1) - \frac{t}{h}, & t \in [ih, (i+1)h], \qquad\qquad i = 1, 2, \ldots, n-1, \\ \\ 0, & otherwise, \end{cases}$$

$$\psi_n(t) = \begin{cases} \frac{t-T}{h} + 1, & t \in [T-h, T], \\ \\ 0, & otherwise, \end{cases}$$

where $h = \frac{T}{n}$ and $n \geq 2$ is an even integer.

MHFs are a modification of hat functions and defined as follows [25]:

$$\theta_0(t) = \begin{cases} \frac{1}{2h^2}(t-h)(t-2h), & t \in [0, 2h], \\ \\ 0, & otherwise, \end{cases}$$

when i is odd and $1 \leq i \leq n-1$,

$$\theta_i(t) = \begin{cases} \frac{-1}{h^2}(t-(i-1)h)(t-(i+1)h), & t \in [(i-1)h, (i+1)h], \\ \\ 0, & otherwise, \end{cases}$$

when i is even and $2 \leq i \leq n-2$,

$$\theta_i(t) = \begin{cases} \frac{1}{2h^2}(t-(i-1)h)(t-(i-2)h), & t \in [(i-2)h, ih], \\ \\ \frac{1}{2h^2}(t-(i+1)h)(t-(i+2)h), & t \in [ih, (i+2)h], \\ \\ 0, & otherwise, \end{cases}$$

and

$$\theta_n(t) = \begin{cases} \frac{1}{2h^2}(t-(T-h))(t-(T-2h)), & t \in [T-2h, T], \\ \\ 0, & otherwise. \end{cases}$$

According to the definition of MHFs, we have

$$\theta_i(jh) = \begin{cases} 1, & i = j, \\ \\ 0, & i \neq j, \end{cases} \tag{3}$$

$$\theta_i(t)\theta_j(t) = \begin{cases} 0, & i \text{ is even and } |i-j| \geq 3, \\ \\ 0, & i \text{ is odd and } |i-j| \geq 2, \end{cases}$$

and

$$\sum_{i=0}^{n} \theta_i(t) = 1.$$

An arbitrary function $g(t) \in L^2(\Omega)$ can be expanded in terms of MHFs as

$$g(t) \simeq g_n(t) = \sum_{i=0}^{n} c_i \theta_i(t) = \mathbf{C}^T \Theta(t) = \Theta^T(t)\mathbf{C}, \tag{4}$$

where $\Theta(t)$ is a $(n+1)$-vector of MHFs as

$$\Theta(t) = [\theta_0(t), \ldots, \theta_i(t), \ldots, \theta_n(t)]^T, \tag{5}$$

and

$$\mathbf{C} = [c_0, \ldots, c_i, \ldots, c_n]^T,$$

in which $c_i = g(ih),\ i = 0, 1, \ldots, n$.

Let $k(t, s)$ be an arbitrary function of two variables, defined on $L^2(\Omega \times \Omega)$. Similarly, it can be expanded in terms of MHFs as

$$k(t, s) \simeq k_n(t, s) = \Theta^T(t)\mathbf{K}\Theta(s) = \Theta^T(s)\mathbf{K}^T\Theta(t), \tag{6}$$

where

$$\mathbf{K} = \begin{pmatrix} \mathtt{k}_{0,0} & \cdots & \mathtt{k}_{0,n} \\ \vdots & \ddots & \vdots \\ \mathtt{k}_{n,0} & \cdots & \mathtt{k}_{n,n} \end{pmatrix}_{(n+1)\times(n+1)},$$

in which $\mathtt{k}_{i,j} = k(ih, jh),\ i, j = 0, 1, \ldots, n$. Also, $\Theta(s)$ and $\Theta(t)$ are MHFs vectors of dimension $(n+1)$.

According to Eq. (3), if $\Theta(t)\Theta^T(t)$ is expanded by MHFs, we get

$$\Theta(t)\Theta^T(t) \simeq \mathrm{diag}(\Theta(t)),$$

thus, we have

$$\Theta(t)\Theta^T(t)\mathbf{Q} \simeq \tilde{\mathbf{Q}}\Theta(t), \tag{7}$$

where $\mathbf{Q}$ is a $(n+1)$-vector and $\tilde{\mathbf{Q}}$ is a diagonal matrix with the elements of $\mathbf{Q}$. Also, if $\mathbf{G}$ be a $(n+1) \times (n+1)$ matrix, we attain

$$\Theta^T(t)\mathbf{G}\Theta(t) \simeq \hat{\mathbf{G}}^T\Theta(t), \tag{8}$$

where $\hat{\mathbf{G}}$ is a $(n+1)$-vector with components equal to the diagonal entries of the matrix $\mathbf{G}$.

Now, we review some operational matrices based on MHFs that will be used in our proposed method. Let $\Theta(t)$ be the MHFs vector defined in Eq. (5). Hence the Volterra integral of $\Theta(t)$ can be estimated as follows:

$$\int_0^t \Theta(s)\mathrm{d}s \simeq \mathcal{L}\Theta(t), \tag{9}$$

in which $\mathcal{L}$ is the $(n+1)\times(n+1)$ operational matrix of integration for MHFs defined as [24]

$$\mathcal{L} = \begin{pmatrix} 0 & \rho_1 & \rho_2 & \rho_2 & \rho_2 & \dots & \rho_2 & \rho_2 & \rho_2 & \rho_2 \\ 0 & \rho_3 & \rho_4 & \rho_4 & \rho_4 & \dots & \rho_4 & \rho_4 & \rho_4 & \rho_4 \\ 0 & \rho_5 & \rho_2 & \rho_6 & \rho_3 & \dots & \rho_3 & \rho_3 & \rho_3 & \rho_3 \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & & & \ddots & \ddots & \ddots & \ddots & \ddots & \\ 0 & 0 & 0 & 0 & 0 & \dots & \rho_5 & \rho_2 & \rho_6 & \rho_3 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \rho_3 & \rho_4 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \rho_5 & \rho_2 \end{pmatrix},$$

where $\rho_1 = \frac{5h}{12}$, $\rho_2 = \frac{h}{3}$, $\rho_3 = \frac{2h}{3}$, $\rho_4 = \frac{4h}{3}$, $\rho_5 = -\frac{h}{12}$ and $\rho_6 = \frac{3h}{4}$.

Theorem 2.3. *Let $\Theta(t)$ be the MHFs vector given by Eq. (5), then*

$$\mathcal{I}^{\alpha_i}\Theta(t) \simeq \mathcal{L}^{\alpha_i}\Theta(t), \tag{10}$$

where $\mathcal{L}^{\alpha_i}$ is the $(n+1)\times(n+1)$ operational matrix of fractional-order integration

$$\mathcal{L}^{\alpha_i} = \begin{pmatrix} 0 & \phi_1^i & \phi_2^i & \phi_3^i & \phi_4^i & \dots & \phi_{n-1}^i & \phi_n^i \\ 0 & \vartheta_0^i & \vartheta_1^i & \vartheta_2^i & \vartheta_3^i & \dots & \vartheta_{n-2}^i & \vartheta_{n-1}^i \\ 0 & \eta_{-1}^i & \eta_0^i & \eta_1^i & \eta_2^i & \dots & \eta_{n-3}^i & \eta_{n-2}^i \\ 0 & 0 & 0 & \vartheta_0^i & \vartheta_1^i & \dots & \vartheta_{n-4}^i & \vartheta_{n-3}^i \\ 0 & 0 & 0 & \eta_{-1}^i & \eta_0^i & \dots & \eta_{n-5}^i & \eta_{n-4}^i \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & \vartheta_0^i & \vartheta_1^i \\ 0 & 0 & 0 & 0 & 0 & \dots & \eta_{-1}^i & \eta_0^i \end{pmatrix},$$

in which

$$
\begin{aligned}
\phi_1^i &= \frac{h^{\alpha_i}\alpha_i(3+2\alpha_i)}{2\Gamma(\alpha_i+3)}, \qquad \vartheta_0^i = \frac{2h^{\alpha_i}(1+\alpha_i)}{\Gamma(\alpha_i+3)},\\
\eta_{-1}^i &= -\frac{h^{\alpha_i}\alpha_i}{2\Gamma(\alpha_i+3)}, \qquad \eta_0^i = \frac{h^{\alpha_i}2^{\alpha_i+1}(2-\alpha_i)}{2\Gamma(\alpha_i+3)},\\
\eta_1^i &= \frac{h^{\alpha_i}}{2\Gamma(\alpha_i+3)}\Big(3^{\alpha_i+1}(4-\alpha_i)-6(2+\alpha_i)\Big),\\
\phi_k^i &= \frac{h^{\alpha_i}}{2\Gamma(\alpha_i+3)}\Big(k^{\alpha_i+1}(2k-6-3\alpha_i)+2j^{\alpha_i}(1+\alpha_i)(2+\alpha_i)\\
&- (k-2)^{(\alpha_i+1)}(2k-2+\alpha_i)\Big), \qquad k=2,3,\dots,n,\\
\vartheta_k^i &= \frac{2h^{\alpha_i}}{\Gamma(\alpha_i+3)}\Big((k-1)^{\alpha_i+1}(k+1+\alpha_i)\\
&- (k+1)^{\alpha_i+1}(k-1-\alpha_i)\Big), \qquad k=1,2,\dots,n-1,
\end{aligned}
$$

and

$$
\begin{aligned}
\eta_k^i = \frac{h^{\alpha_i}}{2\Gamma(\alpha_i+3)}\Big(&(k+2)^{\alpha_i+1}(2k+2-\alpha_i)-6k^{\alpha_i+1}(2+\alpha_i)\\
&-(k-2)^{\alpha_i+1}(2k-2+\alpha_i)\Big), \qquad k=2,3,\dots,n-2.
\end{aligned}
$$

Proof. See [25]. $\square$

Theorem 2.4. *Let $\Theta(t)$ be the MHFs vector given by Eq. (5), then*

$$\int_0^t \Theta(s)\mathrm{d}\mathcal{B}(s) \simeq \mathcal{L}_s\Theta(t), \tag{11}$$

where $\mathcal{L}_s$ is the operational matrix of stochastic integration of MHFs as

$$
\mathcal{L}_s = \begin{pmatrix}
0 & \xi_1 & \xi_2 & \xi_2 & \xi_2 & \cdots & \xi_2 & \xi_2\\
0 & \mathcal{B}(h)+\beta_{1,1} & \beta_{2,1} & \beta_{2,1} & \beta_{2,1} & \cdots & \beta_{2,1} & \beta_{2,1}\\
0 & \gamma_{1,2} & \mathcal{B}(2h)+\gamma_{2,2} & \gamma_{3,2} & \gamma_{4,2} & \cdots & \gamma_{4,2} & \gamma_{4,2}\\
 & \ddots & \ddots & \ddots & \ddots & & \ddots & \ddots\\
0 & 0 & 0 & 0 & 0 & \cdots & \gamma_{3,n-2} & \gamma_{4,n-2}\\
0 & 0 & 0 & 0 & 0 & \cdots & \mathcal{B}(T-h)+\beta_{1,n-1} & \beta_{2,n-1}\\
0 & 0 & 0 & 0 & 0 & \cdots & \gamma_{1,n} & \mathcal{B}T+\gamma_{2,n}
\end{pmatrix},
$$

in which

$$
\begin{aligned}
\xi_1 &= \int_0^{h} \frac{-1}{2h^2}(2s-3h)\ \mathcal{B}(s)\mathrm{d}s,\\
\xi_2 &= \int_0^{2h} \frac{-1}{2h^2}(2s-3h)\ \mathcal{B}(s)\mathrm{d}s,\\
\beta_{1,i} &= \int_{(i-1)h}^{ih} \frac{2}{h^2}(s-ih)\ \mathcal{B}(s)\mathrm{d}s, \qquad i=1,3,\dots,n-1,\\
\beta_{2,i} &= \int_{(i-1)h}^{(i+1)h} \frac{2}{h^2}(s-ih)\ \mathcal{B}(s)\mathrm{d}s, \qquad i=1,3,\dots,n-1,\\
\gamma_{1,i} &= \int_{(i-2)h}^{(i-1)h} \frac{-1}{2h^2}(2s-(2i-3)h)\ \mathcal{B}(s)\mathrm{d}s, \qquad i=2,4,\dots,n-2,\\
\gamma_{2,i} &= \int_{(i-2)h}^{ih} \frac{-1}{2h^2}(2s-(2i-3)h)\ \mathcal{B}(s)\mathrm{d}s, \qquad i=2,4,\dots,n-2,\\
\gamma_{3,i} &= \int_{(i-2)h}^{ih} \frac{-1}{2h^2}(2s-(2i-3)h)\ \mathcal{B}(s)\mathrm{d}s\\
&\quad + \int_{ih}^{(i+1)h} \frac{-1}{2h^2}(2s-(2i+3)h)\ \mathcal{B}(s)\mathrm{d}s, \qquad i=2,4,\dots,n-2,\\
\gamma_{4,i} &= \int_{(i-2)h}^{ih} \frac{-1}{2h^2}(2s-(2i-3)h)\ \mathcal{B}(s)\mathrm{d}s\\
&\quad + \int_{ih}^{(i+2)h} \frac{-1}{2h^2}(2s-(2i+3)h)\ \mathcal{B}(s)\mathrm{d}s, \qquad i=2,4,\dots,n-2.
\end{aligned}
$$

Proof. See [24]. □

3 Description of the method

In this section, by using the operational matrices based on MHFs and other concepts defined in the previous section, a numerical method will be proposed to solve the problem (1)-(2). First, by Riemann-Liouville fractional integration of the equations of (1), we get

$$y_i(t) = d_i + \mathcal{I}^{\alpha_i}(f_i(t)) + \mu_i(\mathcal{I}^{\alpha_i} y_i(t)) + \sum_{j=1}^{m} \mathcal{I}^{\alpha_i}(\int_0^t k_{ij}(t,s) y_j(s) \mathrm{d}s)$$
$$+\sigma \sum_{j=1}^{m} \mathcal{I}^{\alpha_i}(\int_0^t \hat{k}_{ij}(t,s) y_j(s) \mathrm{d}\mathcal{B}(s)), \qquad i = 1, ..., m. \tag{12}$$

Now, we approximate the functions $y_i(t)$, d_i, $f_i(t)$, $k_{ij}(t,s)$, $\hat{k}_{ij}(t,s)$, for $i, j = 1, 2, \ldots, m$ in terms of MHFs as

$$\begin{aligned}
y_i(t) \simeq& \mathbf{C}_i^T \Theta(t) = \Theta^T(t) \mathbf{C}_i, \\
d_i \simeq& \mathbf{d}_i^T \Theta(t) = \Theta^T(t) \mathbf{d}_i, \\
f_i(t) \simeq& \mathbf{f}_i^T \Theta(t) = \Theta^T(t) \mathbf{f}_i, \\
k_{ij}(t,s) \simeq& \Theta^T(t) \mathbf{K}_{ij} \Theta(s) = \Theta^T(s) \mathbf{K}_{ij}^T \Theta(t), \\
\hat{k}_{ij}(t,s) \simeq& \Theta^T(t) \hat{\mathbf{K}}_{ij} \Theta(s) = \Theta^T(s) \hat{\mathbf{K}}_{ij}^T \Theta(t),
\end{aligned}$$

where $\Theta(t)$ is MHFs vector and $\mathbf{C}_i, \mathbf{d}_i, \mathbf{f}_i$, are MHFs coefficients vectors of $y_i(t)$, d_i, and $f_i(t)$. Also, $\mathbf{K}_{ij}$ and $\hat{\mathbf{K}}_{ij}$ are MHFs coefficients matrices of $k_{ij}(t,s)$ and $\hat{k}_{ij}(t,s)$, respectively. Substituting these relations in Eq. (12) results

$$\begin{aligned}
\mathbf{C}_i^T \Theta(t) = \mathbf{d}_i^T \Theta(t) + \mathcal{I}^{\alpha_i}(\mathbf{f}_i^T \Theta(t)) + \mu_i \mathcal{I}^{\alpha_i}(\mathbf{C}_i^T \Theta(t)) \\
+ \sum_{j=1}^{m} \mathcal{I}^{\alpha_i} \left(\Theta^T(t) \mathbf{K}_{ij} \int_0^t \Theta(s) \Theta^T(s) \mathbf{C}_j \mathrm{d}s \right) \\
+\sigma \sum_{j=1}^{m} \mathcal{I}^{\alpha_i} \left(\Theta^T(t) \hat{\mathbf{K}}_{ij} \int_0^t \Theta(s) \Theta^T(s) \mathbf{C}_j \mathrm{d}\mathcal{B}(s) \right).
\end{aligned} \tag{13}$$

The integral part of Eq. (13) can be estimated according to Eqs. (7)-(11) as

$$\begin{aligned}
&\mathcal{I}^{\alpha_i}\left(\Theta^T(t)\mathbf{K}_{ij}\int_0^t \Theta(s)\Theta^T(s)\mathbf{C}_j \mathrm{d}s\right)\\
&\qquad\qquad = \mathcal{I}^{\alpha_i}\left(\Theta^T(t)\mathbf{K}_{ij}\int_0^t \tilde{\mathbf{C}}_j\Theta(s)\mathrm{d}s\right)\\
&\qquad\qquad \simeq \mathcal{I}^{\alpha_i}\left(\Theta^T(t)\mathbf{K}_{ij}\tilde{\mathbf{C}}_j\mathcal{L}\Theta(t)\right),\\
&\qquad\qquad = \mathcal{I}^{\alpha_i}\left(\hat{\mathcal{A}}_j^T\Theta(t)\right) \simeq \hat{\mathcal{A}}_j^T\mathcal{L}^{\alpha_i}\Theta(t),
\end{aligned} \tag{14}$$

and for stochastic integral, we have

$$\begin{aligned}
&\mathcal{I}^{\alpha_i}\left(\Theta^T(t)\hat{\mathbf{K}}_{ij}\int_0^t \Theta(s)\Theta^T(s)\mathbf{C}_j \mathrm{d}\mathcal{B}(s)\right)\\
&\qquad\qquad = \mathcal{I}^{\alpha_i}\left(\Theta^T(t)\hat{\mathbf{K}}_{ij}\int_0^t \tilde{\mathbf{C}}_j\Theta(s)\mathrm{d}\mathcal{B}(s)\right)\\
&\qquad\qquad \simeq \mathcal{I}^{\alpha_i}\left(\Theta^T(t)\hat{\mathbf{K}}_{ij}\tilde{\mathbf{C}}_j\mathcal{L}_s\Theta(t)\right),\\
&\qquad\qquad = \mathcal{I}^{\alpha_i}\left(\hat{\mathcal{S}}_j^T\Theta(t)\right) \simeq \hat{\mathcal{S}}_j^T\mathcal{L}^{\alpha_i}\Theta(t),
\end{aligned} \tag{15}$$

where $\tilde{\mathbf{C}}_j = diag(Y_j)$. Also, $\hat{\mathcal{A}}_j$ and $\hat{\mathcal{S}}_j$ are $(n+1)$-vectors and their elements consist of diagonal entries of matrices $\mathcal{A}_j = \mathbf{K}_{ij}\tilde{\mathbf{C}}_j\mathcal{L}$ and $\mathcal{S}_j = \hat{\mathbf{K}}_{ij}\tilde{\mathbf{C}}_j\mathcal{L}_s$. By using Eqs. (14) and (15), for $i = 1, 2, \ldots, m$, we get

$$\mathbf{C}_i^T = \bar{\mathbf{T}}_i^T + \mu_i\mathbf{C}_i^T\mathcal{L}^{\alpha_i} + \sum_{j=1}^{m}\left(\hat{\mathcal{A}}_j^T + \sigma\hat{\mathcal{S}}_j^T\right)\mathcal{L}^{\alpha_i}, \tag{16}$$

where

$$\bar{\mathbf{T}}_i^T = \mathbf{d}_i^T + \mathbf{f}_i^T\mathcal{L}^{\alpha_i}.$$

For each $i = 1, 2\ldots, m$, the relation (16) is a system of $(n+1)\times(n+1)$ linear algebraic equations. Solving this system leads to an approximate solution for the problem (1)-(2), in the form

$$y_i(t) \simeq \mathbf{C}_i^T\Theta(t), \qquad i = 1, 2, \ldots, m.$$

4 Error estimate

In this section, an error estimate of the proposed algorithm is discussed. Here, we consider the norm

$$\|g\| = \mathbb{E}[\sup_{t\in\Omega} |g(t)|],$$

where $\mathbb{E}[.]$ is the mathematical expectation.

Theorem 4.1. *([21]) Suppose $g(t) \in C^3(\Omega)$ and $g_n(t)$ be the MHFs expansion of $g(t)$ as defined in Eq. (4). Then, we have*

$$\|g(t) - g_n(t)\| \simeq O(h^3).$$

Theorem 4.2. *([21]) Suppose $k(t,s) \in C^3(\Omega \times \Omega)$ and $k_n(t,s)$ is the MHFs expansion of $k(t,s)$ as defined in Eq. (6). Then*

$$\|k(t,s) - k_n(t,s)\| \simeq O(h^3).$$

Theorem 4.3. *Let $\mathbf{Y}_n(t)$ is the numerical solution of (1) obtained by the proposed method, $\mathbf{Y}(t)$ is its exact solution and $\mathbf{R}_n(t)$ is the residual error for this numerical solution. Also, assume that $\|\mathbf{Y}(t)\| \leq \lambda$, $\|\mathbf{k}(t,s)\| \leq \delta$ and $\|\hat{\mathbf{k}}(t,s)\| \leq \hat{\delta}$, in which λ, δ , $\hat{\delta}$ are real positive constants. Then, $\|\mathbf{R}_n(t)\|$ tends to zero, when $n \to \infty$.*

Proof. From (12) subject to (2), we can rewrite the system (1) in the following form

$$\begin{aligned}
\mathbf{Y}(t) = \Lambda_{\text{int}} &+ \int_0^t \varpi(t,s)\mathbf{F}(s)\mathrm{d}s + \mu \int_0^t \varpi(t,s)\mathbf{Y}(s)\mathrm{d}s \\
&+ \int_0^t \varpi(t,s)\left(\int_0^s \mathbf{k}(s,r)\mathbf{Y}(r)\mathrm{d}r\right)\mathrm{d}s \\
&+\sigma \int_0^t \varpi(t,s)\left(\int_0^s \hat{\mathbf{k}}(r,s)\mathbf{Y}(r)\mathrm{d}\mathcal{B}(r)\right)\mathrm{d}s,
\end{aligned}$$

where

$$\varpi(t,s) = \operatorname{diag}\left(\frac{(t-s)^{\alpha_1-1}}{\Gamma(\alpha_1)}, \dots, \frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)}, \dots, \frac{(t-s)^{\alpha_m-1}}{\Gamma(\alpha_m)}\right).$$

Also, the numerical solution for system (12) satisfies the following matrix form

$$\begin{aligned}\mathbf{Y}_n(t) = \Lambda^n_{\text{int}} + \int_0^t \varpi(t,s)\mathbf{F}_n(s)\mathrm{d}s + \mu\int_0^t \varpi(t,s)\mathbf{Y}_n(s)\mathrm{d}s \\ + \int_0^t \varpi(t,s)\left(\int_0^s \mathbf{k}_n(s,r)\mathbf{Y}_n(r)\mathrm{d}r\right)\mathrm{d}s \\ +\sigma\int_0^t \varpi(t,s)\left(\int_0^s \hat{\mathbf{k}}_n(r,s)\mathbf{Y}_n(r)\mathrm{d}\mathcal{B}(r)\right)\mathrm{d}s + \mathbf{R}_n(t).\end{aligned}$$

The residual function $\mathbf{R}_n(t)$ can be obtained from the following relation

$$\mathbf{R}_n(t) = -\mathbf{e}_n(t) + \mathbf{e}_{int} + \mathbf{e}[\mathbf{F}](t) + \mathbf{e}[\mathbf{Y}](t) + \mathbf{e}[\mathbf{k}](t) + \mathbf{e}[\hat{\mathbf{k}}](t), \tag{17}$$

in which

$$\begin{aligned}\mathbf{e}_n(t) = \mathbf{Y}(t) - \mathbf{Y}_n(t), \quad \mathbf{e}_{int} = \Lambda_{\text{int}} - \Lambda^n_{\text{int}}, \\ \mathbf{e}[\mathbf{F}](t) = \int_0^t \varpi(t,s)\left[\mathbf{F}(s) - \mathbf{F}_n(s)\right]\mathrm{d}s, \quad \mathbf{e}[\mathbf{Y}](t) = \mu\int_0^t \varpi(t,s)\mathbf{e}_n(s)\mathrm{d}s, \\ \mathbf{e}[\mathbf{k}](t) = \int_0^t \varpi(t,s)\mathcal{J}_1(s)\mathrm{d}s, \qquad \mathbf{e}[\hat{\mathbf{k}}](t) = \sigma\int_0^t \varpi(t,s)\mathcal{J}_2(s)\mathrm{d}s.\end{aligned}$$

where

$$\mathcal{J}_1(t) = \int_0^t \left(\mathbf{k}(t,s)\mathbf{Y}(s) - \mathbf{k}_n(t,s)\mathbf{Y}_n(s)\right)\mathrm{d}s, \tag{18}$$

and

$$\mathcal{J}_2(t) = \int_0^t \left(\hat{\mathbf{k}}(t,s)\mathbf{Y}(s) - \hat{\mathbf{k}}_n(t,s)\mathbf{Y}_n(s)\right)\mathrm{d}\mathcal{B}(s). \tag{19}$$

Eq. (17) results

$$\begin{aligned}\|\mathbf{R}_n(t)\| \leq \Big(\|\mathbf{e}_{int}\| + \|\mathbf{e}_n(t)\| + \|\mathbf{e}[\mathbf{F}](t)\| \\ +\|\mathbf{e}[\mathbf{Y}](t)\| + \|\mathbf{e}[\mathbf{k}](t)\| + \|\mathbf{e}[\hat{\mathbf{k}}](t)\|\Big).\end{aligned} \tag{20}$$

Now, using Theorem 4.1 results

$$\|\mathbf{e}_{int}\| \leq c_1 h^3, \tag{21}$$
$$\|\mathbf{e}_n(t)\| \leq c_2 h^3. \tag{22}$$

$$\begin{aligned}
\|\mathbf{e}[\mathbf{F}](t)\| &\leq \left(\int_0^t \|\varpi(t,s)\|\|\mathbf{F}(s)-\mathbf{F}_n(s)\|\mathrm{d}s\right)\\
&\leq \gamma\left(\int_0^t \|\mathbf{F}(s)-\mathbf{F}_n(s)\|\mathrm{d}s\right)\\
&\leq \gamma T\|\mathbf{F}(s)-\mathbf{F}_n(s)\| \leq c_3\gamma T h^3. \qquad (23)
\end{aligned}$$

where $\gamma = \|\varpi(t,s)\|_\infty$, and

$$\begin{aligned}
\|\mathbf{e}[\mathbf{Y}](t)\| &\leq \left(\int_0^t \|\varpi(t,s)\|\|\mathbf{e}_n(s)\|\mathrm{d}s\right)\\
&\leq \gamma\bar{\mu}\left(\int_0^t \|\mathbf{e}_n(s)\|\mathrm{d}s\right)\\
&\leq \gamma T\bar{\mu}\|\mathbf{e}_n(t)\| \leq c_2\gamma T\bar{\mu}h^3. \qquad (24)
\end{aligned}$$

where $\bar{\mu} = \|\mu\|_\infty$. Moreover, we have

$$\begin{aligned}
\|\mathbf{e}[\mathbf{k}](t)\| &\leq \left(\int_0^t \|\varpi(t,s)\|\|\mathcal{J}_1(s)\|\mathrm{d}s\right)\\
&\leq \gamma\left(\int_0^t \|\mathcal{J}_1(s)\|\mathrm{d}s\right)\\
&\leq \gamma T\|\mathcal{J}_1(t)\|,
\end{aligned}$$

in which by using Theorem 4.2 and Eq. (18), we get

$$\begin{aligned}
\|\mathcal{J}_1(t)\| &\leq \left(\int_0^t \|\mathbf{k}(t,s)\mathbf{Y}(s)-\mathbf{k}_n(t,s)\mathbf{Y}_n(s)\|\mathrm{d}s\right)\\
&\leq T\|\mathbf{k}(t,s)\mathbf{Y}(s)-\mathbf{k}_n(t,s)\mathbf{Y}_n(s)\|\\
&\leq T\Big(\|\mathbf{k}(t,s)\|\|\mathbf{e}_n(t)\| + \|\mathbf{k}(t,s)-\mathbf{k}_n(t,s)\|(\|\mathbf{Y}(t)\|+\|\mathbf{e}_n(t)\|\Big)\\
&\leq T(c_2\delta + c_4\lambda)h^3 + c_2c_4Th^6,
\end{aligned}$$

thus

$$\|\mathbf{e}[\mathbf{k}](t)\| \leq \upsilon_1 h^3 + \upsilon_2 h^6, \qquad (25)$$

where $\upsilon_1 = \gamma T^2(c_2\delta + c_4\lambda)$ and $\upsilon_2 = \gamma c_2 c_4 T^2$.

In a similar manner, we have

$$\begin{aligned}\|\mathbf{e}[\hat{\mathbf{K}}](t)\| &\leq \sigma \left(\int_0^t \|\varpi(t,s)\| \|\mathcal{J}_2(s)\| \mathrm{d}s \right) \\ &\leq \gamma\sigma \left(\int_0^t \|\mathcal{J}_2(s)\| \mathrm{d}s \right) \\ &\leq \gamma\sigma T \|\mathcal{J}_2(t)\|.\end{aligned}$$

By employing Theorem 4.2 and Eq. (19), we get

$$\begin{aligned}\|\mathcal{J}_2(t)\| &\leq \left(\int_0^t \|\hat{\mathbf{k}}(t,s)\mathbf{Y}(s) - \hat{\mathbf{k}}_n(t,s)\mathbf{Y}_n(s)\| \mathrm{d}\mathcal{B}(s) \right) \\ &\leq \|\mathcal{B}(t)\| \|\hat{\mathbf{k}}(t,s)\mathbf{Y}(s) - \hat{\mathbf{k}}_n(t,s)\mathbf{Y}_n(s)\| \\ &\leq \|\mathcal{B}(t)\| \Big(\|\hat{\mathbf{k}}(t,s)\| \|\mathbf{e}_n(t)\| + \|\hat{\mathbf{k}}(t,s) - \hat{\mathbf{k}}_n(t,s)\| (\|\mathbf{Y}(t)\| + \|\mathbf{e}_n(t)\| \Big) \\ &\leq \|\mathcal{B}(t)\| (c_2\hat{\delta} + c_5\lambda) h^3 + \|\mathcal{B}(t)\| c_2 c_5 h^6,\end{aligned}$$

thus

$$\|\mathbf{e}[\hat{\mathbf{K}}](t)\| \leq \upsilon_3 h^3 + \upsilon_4 h^6, \tag{26}$$

where

$$\upsilon_3 = \|\mathcal{B}(t)\| \gamma\sigma T (c_2\hat{\delta} + c_5\lambda), \quad \upsilon_3 = \|\mathcal{B}(t)\| \gamma\sigma c_2 c_5 T.$$

Then, from the relations (20)-(26), it can be concluded that

$$\|\mathbf{R}_n(t)\| \leq \xi h^3 + \hat{\xi} h^6,$$

where

$$\xi = c_1 + c_2 + c_3\gamma T + c_2\gamma T\bar{\mu} + \upsilon_1 + \upsilon_3, \quad \hat{\xi} = \upsilon_2 + \upsilon_4.$$

Therefore, it is clear that $\|\mathbf{R}_n(t)\|$ tends to zero, when $h \to 0$ (or $n \to \infty$). $\square$

5 Numerical implementation

To clarify the accuracy and efficiency of the proposed scheme in the previous sections, some test problems will be considered. The computations have been executed on a personal computer using a 2.20 GHz processor and the codes are written in Matlab 2017. For testing the numerical solution obtained from the presented algorithm, we run our algorithm for $\hat{M}$ iterations. Also, the error of numerical solution is computed as

$$\|y_{exact}(t) - y_{num}(t)\|.$$

Example 1. Consider the FSIDE

$$D_{0,t}^{\frac{1}{2}} y(t) + y(t) = t^2 + 2\frac{t^{1.5}}{\Gamma(2.5)} + \int_0^t s d\mathcal{B}(s), \quad t \in [0,1),$$

with the initial condition $y(0) = 0$. Here, we do not have exact solution, thus we use the approximate solution with $n = 60$ and $\hat{M} = 700$ iterations as a replacement of the exact solution. A comparison between our algorithm and the spectral collocation method in [33] are shown in Table 1. This table confirms that the presented numerical technique gives better results than the method in [33]. Also, numerical solutions for $\hat{M} = 1$ and $\hat{M} = 100$ iterations are depicted in Figure 1. This figure shows the convergence of MHFs method.

Example 2. Consider the system of FSIDEs

$$\begin{cases} D_{0,t}^{\alpha_1} y_1(t) = f_1(t) + y_1(t) + \int_0^t s y_2(s) ds - \sigma \int_0^t y_2(s) d\mathcal{B}(s), \\ \\ D_{0,t}^{\alpha_2} y_2(t) = f_2(t) - y_2(t) + \int_0^t \frac{s^2}{2} y_2(s) ds - \frac{1}{2}\sigma \int_0^t y_1(s) d\mathcal{B}(s), \end{cases}$$

where

$$\begin{aligned} f_1(t) =& \frac{t^{1-\alpha_1}}{\Gamma(2-\alpha_1)} - t - 1 - \frac{t^3}{6} + \frac{1}{2}\sigma \left(t\mathcal{B}(t) - \int_0^t \mathcal{B}(s) ds \right), \\ f_2(t) =& \frac{1}{2}\frac{t^{1-\alpha_2}}{\Gamma(2-\alpha_2)} + \frac{t}{2} - \frac{t^4}{16} + \frac{1}{2}\sigma \left((t+1)\mathcal{B}(t) - \int_0^t \mathcal{B}(s) ds \right). \end{aligned}$$

For this system the exact solution is

$$\begin{aligned} y_1(t) =& t + 1, \\ y_2(t) =& \frac{t}{2}. \end{aligned}$$

Table 1: Absolute errors of the numerical solution of Example 1.

t	Method in [33] $n = 4$	Our algorithm $n = 4$	Method in [33] $n = 8$	Our algorithm $n = 8$
0.01	0.0008	1.2580×10^{-4}	0.0003	6.2431×10^{-5}
0.08	0.0022	9.0286×10^{-4}	0.0005	3.9007×10^{-4}
0.16	0.0006	1.4203×10^{-3}	0.0018	3.8136×10^{-4}
0.30	0.0061	1.4378×10^{-3}	0.0037	1.7454×10^{-5}
0.43	0.0061	2.9194×10^{-4}	0.0048	2.0843×10^{-5}
0.78	0.0019	1.3033×10^{-3}	0.0007	2.7080×10^{-3}
0.84	0.0035	8.4564×10^{-4}	0.0027	2.0699×10^{-3}
0.94	0.0046	4.8891×10^{-4}	0.0035	2.9305×10^{-3}
1.00	0.0016	2.3861×10^{-3}	0.0009	6.4688×10^{-3}

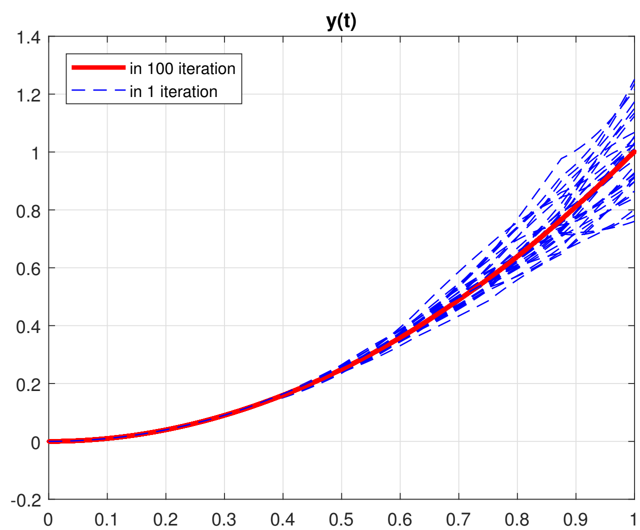

Figure 1: The numerical solution of Example 1 for $\hat{M} = 1$ and $\hat{M} = 100$ iterations.

Table 2: The absolute error of Example 2 for $n = 16$ and $n = 32$.

	$n = 16$		$n = 32$	
t	$y_1(t)$	$y_2(t)$	$y_1(t)$	$y_2(t)$
0.1	6.6629×10^{-3}	1.2115×10^{-3}	1.4689×10^{-3}	7.6930×10^{-3}
0.2	4.6417×10^{-3}	1.5385×10^{-2}	1.7993×10^{-3}	7.4140×10^{-3}
0.3	6.4087×10^{-3}	1.6887×10^{-2}	2.8465×10^{-3}	9.4791×10^{-3}
0.4	7.6164×10^{-3}	1.6728×10^{-2}	4.0157×10^{-3}	1.0582×10^{-2}
0.5	1.0276×10^{-2}	1.8492×10^{-2}	5.3916×10^{-3}	1.1561×10^{-2}
0.6	1.3725×10^{-2}	2.0832×10^{-2}	7.2755×10^{-3}	1.2592×10^{-2}
0.7	1.7016×10^{-2}	2.3725×10^{-2}	9.4031×10^{-3}	1.3189×10^{-2}
0.8	2.1748×10^{-2}	2.5051×10^{-2}	1.1836×10^{-2}	1.4717×10^{-2}
0.9	2.5247×10^{-2}	2.1826×10^{-2}	1.4597×10^{-2}	1.6389×10^{-2}
1.0	3.1081×10^{-2}	2.7405×10^{-2}	1.8378×10^{-2}	1.7842×10^{-2}

The absolute error of Example 2 for $\sigma = 1$ and different value of n is illustrated in Table 2. The exact and numerical solutions when $\alpha_1 = \alpha_2 = 0.5$, $\sigma = 1$ and $n = 16$ for $\hat{M} = 100$ iterations, are shown in Figure 2. The absolute error for $n = 32$, $\sigma = 1$ and $\hat{M} = 500$ iterations is depicted in Figure 3. Table 2 and Figures 2 and 3 confirm that the obtained numerical results are in a good agreement with the exact solution. Also, in Figure 4, the approximate solutions of $y_1(t)$ and $y_2(t)$ for different values of σ with $n = 16$ and $\hat{M} = 100$ iterations are displayed. This figure shows the impact of σ on the accuracy of numerical scheme.

Example 3. For the last example, we consider the following system of FSIDEs:

$$\begin{cases} D_{0,t}^{\alpha_1} y_1(t) = f_1(t) - 2y_1(t) + \int_0^t e^{t-s} y_2(s)ds + \sigma \int_0^t sty_3(s)d\mathcal{B}(s), \\ \\ D_{0,t}^{\alpha_2} y_2(t) = f_2(t) + y_2(t) - \int_0^t ts^2 y_3(s)ds + \sigma \int_0^t sin(s)y_2(s)d\mathcal{B}(s), \\ \\ D_{0,t}^{\alpha_3} y_3(t) = f_3(t) - y_3(t) + \int_0^t (t-s)y_2(s)ds + \sigma \int_0^t y_1(s)d\mathcal{B}(s), \end{cases}$$

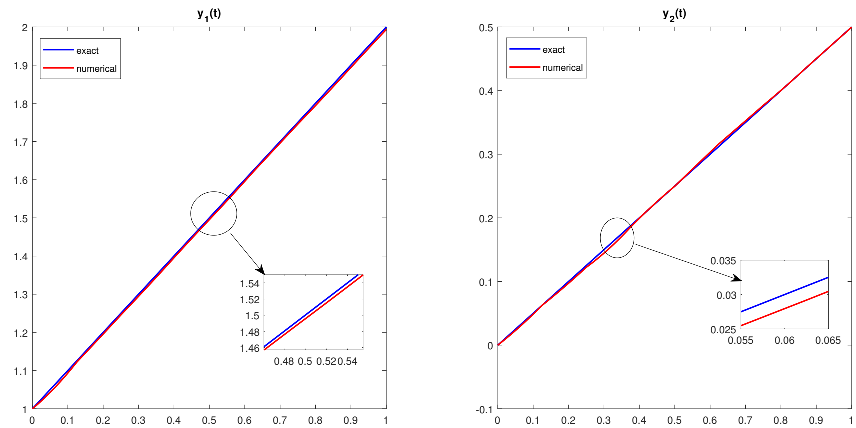

Figure 2: The exact and numerical solution for $y_1(t)$ and $y_2(t)$ in Example 2.

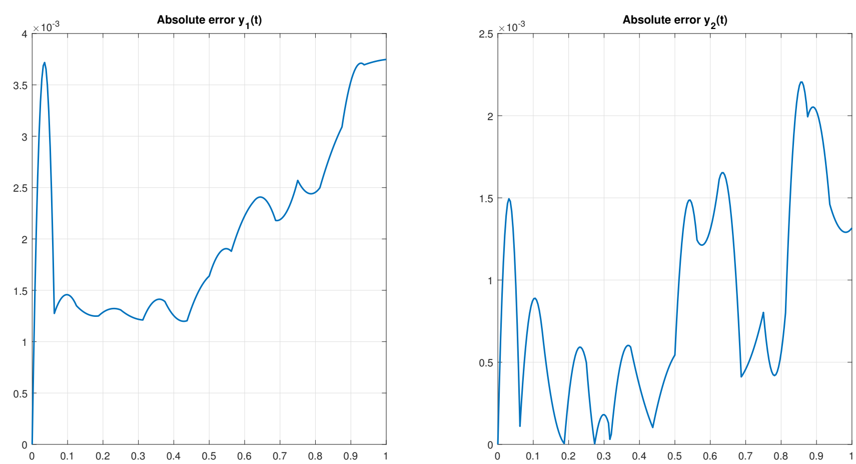

Figure 3: The absolute error of $y_1(t)$ and $y_2(t)$ for Example 2 when $\alpha_1 = \alpha_2 = 0.5$, $n = 32$ and $\hat{M} = 500$.

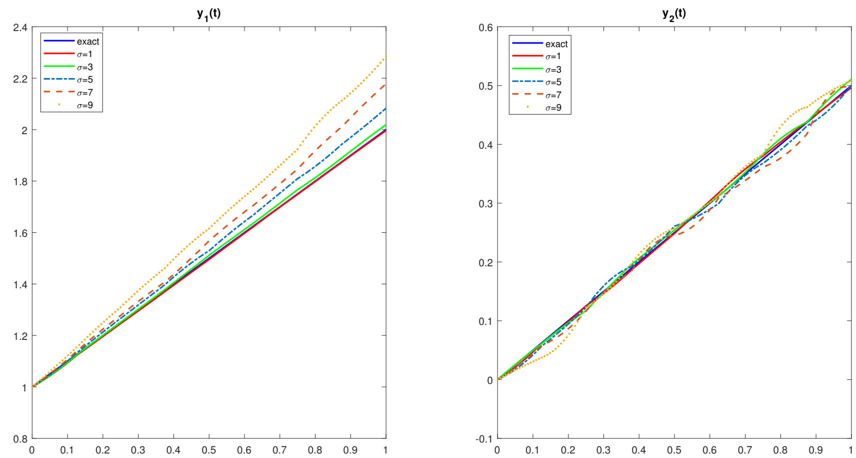

Figure 4: Approximate solutions of $y_1(t)$ and $y_2(t)$ in Example 2 for different values of σ with $n = 16$, $\hat{M} = 100$ iterations.

in which

$$
\begin{aligned}
f_1(t) &= \frac{\Gamma(4)t^{(3-\alpha_1)}}{\Gamma(4-\alpha_1)} - \frac{\Gamma(5)t^{(4-\alpha_1)}}{\Gamma(5-\alpha_1)} + 2t^3(1-t) + 4 - 4e^t + 4t + 2t^2 \\
&\quad -\sigma t\left(te^t\mathcal{B}(t) - \int_0^t e^s(1+s)\mathcal{B}(s)ds\right), \\
f_2(t) &= \frac{2\Gamma(3)t^{(2-\alpha_2)}}{\Gamma(3-\alpha_2)} - 2t^2 - 2t + te^t\left(2 - 2t + t^2\right) \\
&\quad -\sigma\left(2t^2 sin(t)\mathcal{B}(t) - 2\int_0^t \left(-2 - (s^2-2)cos(s) + 2s\ sin(s)\right)\mathcal{B}(s)ds\right), \\
f_3(t) &= \frac{e^t\Gamma(1-\alpha_3) - \Gamma(1-\alpha_3, t)}{\Gamma(1-\alpha_3)} + e^t - \frac{t^4}{6} - \sigma t^3(1-t)\mathcal{B}(t) \\
&\quad -\sigma\int_0^t (3s^2 - 4s^3)\mathcal{B}(s)ds,
\end{aligned}
$$

Table 3: The absolute error of Example 3 for $n = 16$ and $n = 32$.

t	$n = 16$			$n = 32$		
	$y_1(t)$	$y_2(t)$	$y_3(t)$	$y_1(t)$	$y_2(t)$	$y_3(t)$
0.1	1.6212×10^{-4}	2.8120×10^{-4}	9.0697×10^{-3}	1.0337×10^{-4}	3.8442×10^{-4}	2.0847×10^{-3}
0.2	7.9713×10^{-4}	5.0715×10^{-4}	3.4573×10^{-3}	5.9030×10^{-4}	3.0112×10^{-4}	1.4435×10^{-3}
0.3	2.6256×10^{-3}	2.2863×10^{-3}	2.6693×10^{-3}	1.124×10^{-3}	9.4339×10^{-3}	1.1056×10^{-3}
0.4	5.1580×10^{-3}	5.4707×10^{-3}	2.1148×10^{-3}	3.3416×10^{-3}	2.9722×10^{-3}	8.8180×10^{-4}
0.5	8.2723×10^{-3}	1.0649×10^{-2}	2.0397×10^{-3}	4.9457×10^{-3}	5.4758×10^{-3}	9.0121×10^{-4}
0.6	1.1210×10^{-2}	1.7771×10^{-2}	2.2670×10^{-3}	9.1619×10^{-3}	1.1250×10^{-2}	1.2668×10^{-3}
0.7	2.0038×10^{-2}	3.1139×10^{-2}	2.7843×10^{-3}	1.1784×10^{-2}	1.5897×10^{-2}	1.4935×10^{-3}
0.8	3.4178×10^{-2}	5.0417×10^{-2}	3.1719×10^{-3}	1.9795×10^{-2}	2.6911×10^{-2}	1.8437×10^{-3}
0.9	3.8820×10^{-2}	7.0928×10^{-2}	3.4214×10^{-3}	2.5971×10^{-2}	3.8187×10^{-2}	1.8367×10^{-3}
1.0	5.8870×10^{-2}	9.2068×10^{-2}	3.4609×10^{-3}	3.5885×10^{-2}	5.2688×10^{-2}	1.8613×10^{-3}

and $\Gamma(a, z)$ is the incomplete Gamma function. The exact solution of this system is

$$\begin{aligned} y_1(t) =& t^3(1-t), \\ y_2(t) =& 2t^2, \\ y_3(t) =& e^t. \end{aligned}$$

The numerical results for $\alpha_1 = 0.45, \alpha_2 = 0.5, \alpha_3 = 0.7$, $\sigma = 1$, $n = 16$ and $n = 32$ are reported in Table 3. It is observed that by increasing n, we get more accurate results. The exact solution and the numerical solution for $\alpha_1 = 0.24$, $\alpha_2 = 0.45, \alpha_3 = 0.33$, $n = 16$, $\sigma = 1$ and $\hat{M} = 200$ iterations are depicted in Figure 5. Figure 6 displays the absolute error of the obtained numerical solution. These figures confirm that the approximate solutions obtained by presented MHFs scheme have high accuracy. Also, in Figure 7, the approximate values of $y_1(t)$, $y_2(t)$ and $y_3(t)$ are illustrated for different values of σ when $n = 16$ and $\hat{M} = 150$.

6 Conclusion

In this work, a numerical approach based on modified hat functions has been used for solving a system of fractional stochastic integro-differential equations. For this purpose, the operational matrices of fractional integral and stochastic integral of MHFs have been employed. By using these matrices and the properties of MHFs, the problem is reduced into a linear system of algebraic equations. The great feature of this algorithm is that, without any integration, unknown coefficients have been

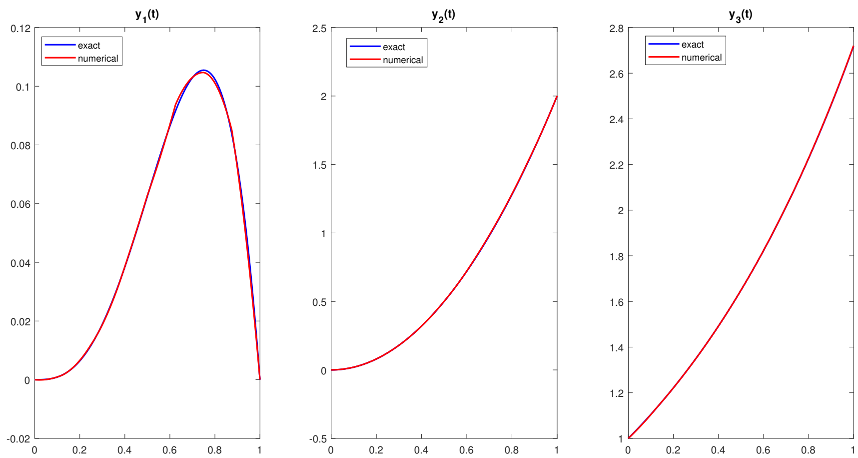

Figure 5: The exact and numerical solution of Example 3 when $\alpha_1 = 0.24, \alpha_2 = 0.45, \alpha_3 = 0.33$, $n = 16$ and $\hat{M} = 200$.

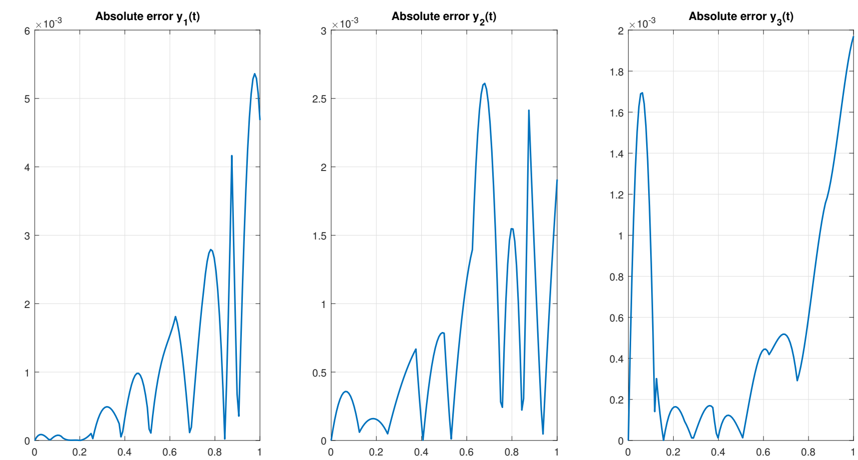

Figure 6: The absolute error of $y_1(t)$, $y_2(t)$, and $y_3(t)$ in Example 3 when $\alpha_1 = 0.24, \alpha_2 = 0.45, \alpha_3 = 0.33$, $n = 16$ and $\hat{M} = 500$.

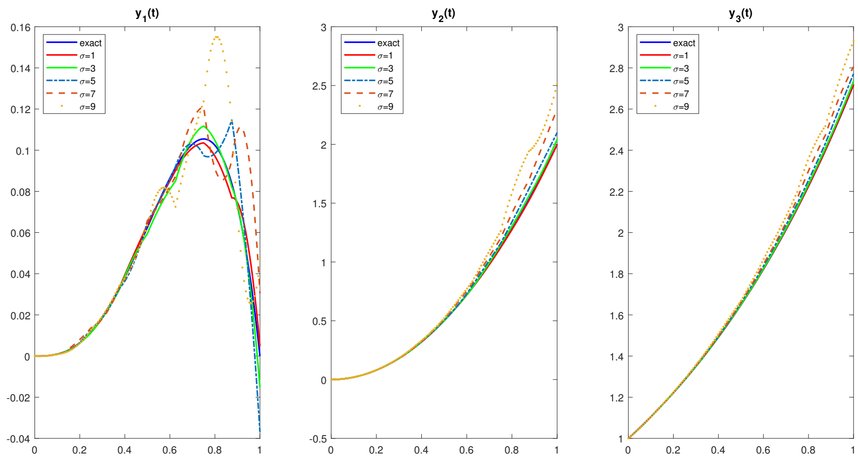

Figure 7: Approximate solutions of $y_1(t)$, $y_2(t)$, and $y_3(t)$ in Example 3 for different value of σ with $n = 16$ and $\hat{M} = 150$.

calculated. Error analysis of the method was discussed. Furthermore, several numerical examples have been investigated to illustrate the effectiveness and accuracy of this algorithm. Using the proposed method for solving nonlinear fractional stochastic integro-differential equations with delay could be the subject of some future research works. Also, this scheme may be employed to obtain the solution of some biological systems presented as stochastic integro-differential equations.

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Elnaz Aryani
Department of Applied Mathematics
Ph.D. Student of Applied Mathematics
University of Mazandaran
Babolsar, Iran
E-mail: aryani@stu.umz.ac.ir

Afshin Babaei
Department of Applied Mathematics
Associate Professor of Applied Mathematics
University of Mazandaran
Babolsar, Iran
E-mail: babaei@umz.ac.ir

Ali Valinejad
Department of Computer Science
Assistant Professor of Computer Science
University of Mazandaran
Babolsar, Iran
E-mail: valinejad@umz.ac.ir