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Original Research Paper

Fekete-Szegö Inequality for Certain Subclasses of Analytic Functions Related with Crescent-Shaped Domain and Application of Poison Distribution Series

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Abstract. The purpose of this paper is to define a new class of analytic, normalized functions in the open unit disk $\mathbb{D} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$ subordinating with crescent shaped regions, and to derive certain coefficient estimates a_2 , a_3 and Fekete-Szegö inequality for $f \in \mathcal{M}_q(\alpha, \beta, \lambda)$. A similar result have been done for the function f^{-1}. Further application of our results to certain functions defined by convolution products with a normalized analytic function is given, in particular we obtain Fekete-Szegö inequalities for certain subclasses of functions defined through Poisson distribution series.

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1 Introduction

Let $\mathcal{A}$ denote the class of all functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

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which are analytic in the open unit disk

$$\mathbb{D} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$$

and $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions. A function $f \in \mathcal{S}$ is said to be *starlike* in $\mathbb{D}$ if and only if

$$\Re\Big(\frac{zf'(z)}{f(z)}\Big) > 0, \qquad (z \in \mathbb{D})$$

and on the other hand , a function $f \in \mathcal{S}$ is said to be *convex* in $\mathbb{D}$ if and only if

$$\Re\Big(1 + \frac{zf''(z)}{f'(z)}\Big) > 0, \quad (z \in \mathbb{D})$$

denoted by $\mathcal{S}^*$ and $\mathcal{C}$ respectively.

Let f and g be functions analytic in $\mathbb{D}$. Then we say that the function f is subordinate to g if there exists a Schwarz function $w(z)$, analytic in $\mathbb{D}$ with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \quad (z \in \mathbb{D}),$$

such that

$$f(z) = g(\omega(z)) \quad (z \in \mathbb{D}).$$

We denote this subordination by

$$f \prec g \quad \text{or} \quad f(z) \prec g(z) \quad (z \in \mathbb{D}).$$

In particular, if the function g is univalent in $\mathbb{D}$, the above subordination is equivalent to

$$f(0) = g(0) \quad \text{and} \quad f(\mathbb{D}) \subset g(\mathbb{D}).$$

Definition 1.1. [17] Let $\mathcal{S}^*(q)$ denote the class of analytic functions f in the unit disc $\mathbb{D}$ normalized by $f(0) = f'(0) - 1 = 0$ and satisfying the condition that

$$\frac{zf'(z)}{f(z)} \prec \sqrt{1+z^2} + z =: q(z), \quad z \in \mathbb{D}, \tag{2}$$

where the branch of the square root is chosen to be $q(0) = 1$.

It may be noted from (2) of Definition 1 that the set $q(\mathbb{D})$ lies in the right half-plane and it is not a starlike domain with respect to the origin, see Fig. 1 (below).

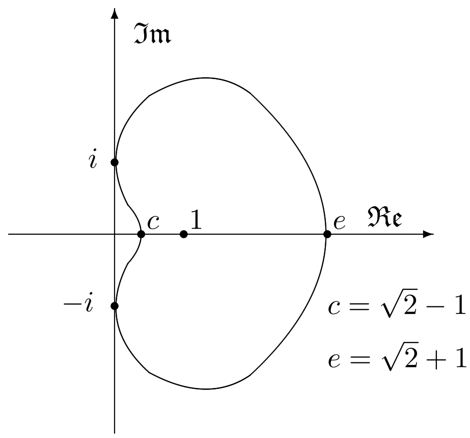

Fig. 1. The boundary of the set $q(\mathbb{D})$.

In [19, 17], initial coefficient estimates $|a_2|$ and $|a_3|$ are obtained for functions in the classes $\mathcal{S}^*(q)$ and $\mathcal{C}(q)$ and the Fekete-Szegö inequality , were also obtained. For a brief history of Fekete-Szegö problem[4] for the class of starlike, convex and various other subclasses of analytic functions, one may refer to [21].

In this paper,motivated essentially by the aforementioned works, in [17, 19] and [6], we define the new function class $\mathcal{M}_q(\alpha, \beta, \lambda)$ which unifies the class $\mathcal{S}^*(q)$ $\mathcal{C}(q)$ and $\mathcal{M}_\lambda(q)$. First,we shall find estimations of first few coefficients of functions f of the form (1) belonging to $\mathcal{M}_q(\alpha, \beta, \lambda)$ and we prove the Fekete-Szegö inequality for a more general class of analytic functions which we define below in Definition 1.2. Also we give applications of our results to certain functions defined through Hadamard product and in particular we consider a class defined through Poisson distribution .

Now, we define the following class $\mathcal{M}_q(\alpha, \beta, \lambda)$:

Definition 1.2. For $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1, 0 \leq \lambda \leq 1$ a function $f \in \mathcal{A}$ is in the class $\mathcal{M}_q(\alpha, \beta, \lambda)$ if

$$\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left[(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right)\right]^{\beta} \prec \quad z + \sqrt{1+z^2} = q(z); \; z = re^{i\theta} \in \mathbb{D}. \tag{3}$$

By suitably specializing the parameter we state the following subclasses:

Remark 1.3. 1. $\mathcal{M}_q(1,0,0) \equiv \mathcal{S}^*(q)$[17], the class of starlike functions satisfying the condition

$$\left(\frac{zf'(z)}{f(z)}\right) \prec z + \sqrt{1+z^2} = q(z)$$

2. $\mathcal{M}_q(0,1,1) \equiv \mathcal{C}(q)$ [20], the class of convex functions satisfying the condition

$$\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec z + \sqrt{1+z^2} = q(z)$$

3. $\mathcal{M}_q(0,1,\lambda) \equiv \mathcal{M}_\lambda(q)$[19], the class of $\lambda-$ convex functions satisfying the condition

$$(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec z + \sqrt{1+z^2} = q(z)$$

4. $\mathcal{M}_q(\alpha,\beta,1) \equiv \mathcal{M}_q(\alpha,\beta)$ the class of $\alpha-$ starlike functions satisfying the condition

$$\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1 + \frac{zf''(z)}{f'(z)}\right)^{\beta} \prec z + \sqrt{1+z^2} = q(z).$$

To prove our main result, we recall the following lemmas:

Lemma 1.4. *[10] If $p_1(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is a function with positive real part in $\mathbb{D}$, then*

$$|c_2 - vc_1^2| \leqq \begin{cases} -4v + 2, & \text{if} \quad v \leqq 0, \\ 2, & \text{if} \quad 0 \leqq v \leqq 1, \\ 4v - 2, & \text{if} \quad v \geqq 1. \end{cases}$$

When $v < 0$ or $v > 1$, the equality holds if and only if $p_1(z)$ is $\dfrac{1+z}{1-z}$ or one of its rotations. If $0 < v < 1$, then equality holds if and only if $p_1(z)$ is $\dfrac{1+z^2}{1-z^2}$ or one of its rotations. If $v = 0$, the equality holds if and only if

$$p_1(z) = \left(\frac{1}{2} + \frac{1}{2}\eta\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}\eta\right)\frac{1-z}{1+z} \quad (0 \leqq \eta \leqq 1)$$

or one of its rotations. If $v = 1$, the equality holds if and only if p_1 is the reciprocal of one of the functions such that the equality holds in the case of $v = 0$.

Although the above upper bound is sharp, when $0 < v < 1$, it can be improved as follows:

$$|c_2 - vc_1^2| + v|c_1|^2 \leqq 2 \quad (0 < v \leqq 1/2)$$

and

$$|c_2 - vc_1^2| + (1-v)|c_1|^2 \leqq 2 \quad (1/2 < v \leqq 1).$$

We also need the following:

Lemma 1.5. *[5] If $p_1(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is a function with positive real part in $\mathbb{D}$, then*

$$|\, c_n \,| \leq 2 \ \ for\ all\ n \geq 1 \qquad \text{and} \qquad |c_2 - \frac{c_1^2}{2}| \leq 2 - \frac{|c_1|^2}{2}.$$

The class of all such functions with positive real part are denoted by $\mathcal{P}$.

Lemma 1.6. *[9] If $p_1(z) = 1+c_1z+c_2z^2+\cdots$ is a function with positive real part in $\mathbb{D}$, then*

$$|c_2 - vc_1^2| \leqq 2\max(1, |2v-1|).$$

The result is sharp for the functions

$$p(z) = \frac{1+z^2}{1-z^2}, \quad p(z) = \frac{1+z}{1-z}.$$

Due to Keogh and Merkes[8], We note that if ω is of the form

$$\omega(z) = \sum_{n=1}^{\infty} w_n z^n, \quad z \in \mathbb{D},$$

then for $\nu \in \mathbb{C}$,

$$|w_2 - \nu w_1^2| \leq \max\{1, |\nu|\}. \tag{4}$$

2 Coefficient Estimates

In this section to start with we obtain the the initial coefficient estimates for $f \in \mathcal{M}_q(\alpha, \beta, \lambda)$. By suitably specializing the parameters α, β, λ as mentioned in Remark 1.3 we deduce the results for the function classes stated in Remark1.3.

Theorem 2.1. *Let $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$, and $0 \leq \lambda \leq 1$. If $f(z)$ given by (1) belongs to $\mathcal{M}_q(\alpha, \beta, \lambda)$, then*

$$\begin{aligned} |a_2| &\leqq \frac{1}{\alpha + (1+\lambda)\beta}, \\ |a_3| &\leqq \frac{1}{2[\alpha + (1+2\lambda)\beta]} \max\{1, \frac{1}{2}\left|\frac{\Phi(\alpha,\beta,\lambda)}{[\alpha+(1+\lambda)\beta]^2} - 1\right|\} \end{aligned}$$

where

$$\Phi(\alpha, \beta, \lambda) = \alpha(\alpha-3) + \beta(\beta-1)(1+\lambda)^2 + 2\alpha\beta(1+\lambda) - 2(1+3\lambda)\beta.$$

Proof. If $f \in \mathcal{M}_{\alpha,\lambda}(q)$, then there is a Schwarz function $w(z)$, analytic in $\mathbb{D}$ with $w(0) = 0$ and $|w(z)| < 1$ in $\mathbb{D}$ such that

$$\begin{aligned} &\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left[(1-\lambda)\frac{zf'(z)}{f(z)}+\lambda\left(1+\frac{zf''(z)}{f'(z)}\right)\right]^{\beta} \\ &= q(w(z)) \\ = \quad & w(z)+\sqrt{1+[w(z)]^2}. \end{aligned} \tag{5}$$

Define the function $p_1(z)$ by

$$p_1(z) := \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + \cdots . \tag{6}$$

Since $w(z)$ is a Schwarz function, we see that $\Re(p_1(z)) > 0$ and $p_1(0) = 1$. Let us define the function $p(z)$ by

$$\begin{aligned} p(z) : &= \left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left[(1-\lambda)\frac{zf'(z)}{f(z)}+\lambda\left(1+\frac{zf''(z)}{f'(z)}\right)\right]^{\beta} \\ &= 1 + b_1 z + b_2 z^2 + \cdots . \end{aligned} \tag{7}$$

In view of the equations (5), (6), (7), we have

$$p(z) = \phi\left(\frac{p_1(z)-1}{p_1(z)+1}\right). \tag{8}$$

$$\begin{aligned} \sqrt{1+\Big(\frac{P(z)-1}{P(z)+1}\Big)^2}+\frac{P(z)-1}{P(z)+1} &= 1+\frac{c_1}{2}z+\Big(\frac{c_2}{2}-\frac{c_1^2}{8}\Big)z^2 \\ &\quad +\Big(\frac{c_3}{2}-\frac{c_1c_2}{4}\Big)z^3+\cdots . \end{aligned} \tag{9}$$

Using (6) in (8), we get,

$$b_1 = \frac{c_1}{2} \qquad \text{and} \qquad b_2 = \frac{c_2}{2} - \frac{c_1^2}{8}.$$

A computation shows that

$$\frac{zf'(z)}{f(z)} = 1 + a_2 z + (2a_3 - a_2^2)z^2 + (3a_4 + a_2^3 - 3a_3 a_2)z^3 + \cdots .$$

Similarly we have

$$1+\frac{zf''(z)}{f'(z)}=1+2a_2z+(6a_3-4a_2^2)z^2+\cdots.$$

An easy computation shows that
$\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left[(1-\lambda)\frac{zf'(z)}{f(z)}+\lambda\left(1+\frac{zf''(z)}{f'(z)}\right)\right]^{\beta}$

$$\begin{aligned}&=1+[\alpha+(1+\lambda)\beta]a_2z+2[\alpha+(1+2\lambda)\beta]a_3z^2\\&+\left(\frac{\alpha(\alpha-3)}{2}+\frac{\beta(\beta-1)}{2}(1+\lambda)^2+\alpha\beta(1+\lambda)\right.\\&\left.-(1+3\lambda)\beta\right)a_2^2z^2+\cdots.\end{aligned}$$

In view of the equation (7), we see that

$$\begin{aligned}b_1&=[\alpha+(1+\lambda)\beta]a_2\\b_2&=2[\alpha+(1+2\lambda)\beta]a_3\\&+\left(\frac{\alpha(\alpha-3)}{2}+\frac{\beta(\beta-1)}{2}(1+\lambda)^2+\alpha\beta(1+\lambda)\right.\\&-\left.(1+3\lambda)\beta\right)a_2^2\end{aligned}$$

or equivalently, we have

$$\begin{aligned}a_2&=\frac{c_1}{2[\alpha+(1+\lambda)\beta]},\qquad(10)\\a_3&=\frac{1}{2[\alpha+(1+2\lambda)\beta]}\times\left(\frac{c_2}{2}-\frac{c_1^2}{8}-\right.\\&\left(\frac{\alpha(\alpha-3)}{2}+\frac{\beta(\beta-1)}{2}(1+\lambda)^2+\alpha\beta(1+\lambda)-(1+3\lambda)\beta\right)\\&\times\left.\frac{c_1^2}{4[\alpha+(1+\lambda)\beta]^2}\right).\end{aligned}$$

Let

$$\Phi(\alpha,\beta,\lambda)=\alpha(\alpha-3)+\beta(\beta-1)(1+\lambda)^2+2\alpha\beta(1+\lambda)-2\beta(1+3\lambda).\qquad(11)$$

Therefore, we have

$$\begin{aligned} a_3 &= \frac{1}{2[\alpha+(1+2\lambda)\beta]}\left(\frac{c_2}{2}-\frac{c_1^2}{8}-\frac{\Phi(\alpha,\beta,\lambda)c_1^2}{8[\alpha+(1+\lambda)\beta]^2}\right) \\ &= \frac{1}{4[\alpha+(1+2\lambda)\beta]}\left(c_2-\frac{c_1^2}{4}\left(1+\frac{\Phi(\alpha,\beta,\lambda)}{[\alpha+(1+\lambda)\beta]^2}\right)\right) \qquad (12) \end{aligned}$$

where $\Phi(\alpha,\beta,\lambda)$ is given by (11).

Using the estimate given in Lemma 1.6, we have

$$|c_2 - vc_1^2| \leqq 2\max(1, |2v-1|).$$

we get

$$\begin{aligned} |a_3| &\leqq \frac{1}{2[\alpha+(1+2\lambda)\beta]}\max\{1, \left|2\times\frac{1}{4}\left(1+\frac{\Phi(\alpha,\beta,\lambda)}{[\alpha+(1+\lambda)\beta]^2}\right)-1\right|\} \\ &= \frac{1}{2[\alpha+(1+2\lambda)\beta]}\max\{1, \left|\frac{\Phi(\alpha,\beta,\lambda)}{2[\alpha+(1+\lambda)\beta]^2}-\frac{1}{2}\right|\}. \end{aligned}$$

To show that the bounds are sharp, we define the functions K_{ϕ_n} $(n=2,3,\ldots)$ with $K_{\phi_n}(0)=0=[K_{\phi_n}]'(0)-1$, by

$$\left(\frac{z(K_{\phi_n})'(z)}{K_{\phi_n}(z)}\right)^{\alpha}\left[(1-\lambda)\frac{z(K_{\phi_n})'(z)}{K_{\phi_n}(z)}+\lambda\left(1+\frac{z(K_{\phi_n})''(z)}{(K_{\phi_n})'(z)}\right)\right]^{\beta} = q(z^{n-1}).$$

Clearly the functions $K_{q_n}\in\mathcal{M}_q(\alpha,\beta,\lambda)$. we write $K_q := K_{q_2}$. That is ,when $n=3$ we get$q(z^2)=z^2+\sqrt{1+z^4}=z^2+z^4/2-z^8/8+\cdots$. □

By suitably specializing the parameters α,β,λ as mentioned in Remark 1.3, we get the following estimates for the classes studied [17, 19, 20] and or new:

Remark 2.2. Let $0\le\alpha\le 1$, $0\le\beta\le 1$, and $\lambda=1$. If $f(z)$ given by (1) belongs to $\mathcal{M}_q(\alpha,\beta)$, then

$$|a_2| \leqq \frac{1}{\alpha+2\beta},$$

and

$$|a_3| \leqq \frac{1}{2[\alpha+3\beta]}\max\{1, \frac{1}{2}\left|\frac{\Phi(\alpha,\beta)}{[\alpha+2\beta]^2}-1\right|\}$$

where $\Phi(\alpha,\beta)=\alpha(\alpha-3)+4\beta(\beta-1)+4\alpha\beta-8\beta$.

Remark 2.3. Let $\alpha = 0$ and $0 \leq \lambda \leq 1$. If $f(z)$ given by (1) belongs to $\mathcal{M}_q(0, \beta, \lambda) = \mathcal{M}_{\beta,\lambda}(q)$, then

$$|a_2| \leqq \frac{1}{(1+\lambda)\beta},$$

and

$$|a_3| \leqq \frac{1}{2(1+2\lambda)\beta} \max\{1, |\frac{\Phi(\beta,\lambda)}{2[(1+\lambda)\beta]^2} - \frac{1}{2}|\}$$

where $\Phi(\beta, \lambda) = \beta(\beta - 1)(1 + \lambda)^2 - 2\beta(1 + 3\lambda)$.

Remark 2.4. [19] Let $\alpha = 0; \beta = 1,$ and $0 \leq \lambda \leq 1$. If $f(z)$ given by (1) belongs to $\mathcal{M}_q(0, 1, \lambda) = \mathcal{M}_\lambda(q)$, then

$$|a_2| \leqq \frac{1}{1+\lambda}, \qquad \text{and} \qquad |a_3| \leqq \frac{1}{2(1+2\lambda)} \max\{1, |\frac{1+3\lambda}{(1+\lambda)^2} + \frac{1}{2}|\}.$$

Remark 2.5. [17] Let $\alpha = 0; \beta = 1$ and $\lambda = 0$. If $f(z)$ given by (1) belongs to $\mathcal{M}_q(0, 1, 0) = \mathcal{S}^*(q)$, then

$$|a_2| \leqq 1, \qquad \text{and} \qquad |a_3| \leqq \frac{1}{2} \max\{1, |\frac{3}{2}|\} = \frac{3}{4}.$$

Remark 2.6. [20] Let $\alpha = 0; \beta = 1$ *and* $\lambda = 1$. If $f(z)$ given by (1) belongs to $\mathcal{M}_q(0, 1, 1) = \mathcal{C}(q)$, then

$$|a_2| \leqq \frac{1}{2}, \qquad \text{and} \qquad |a_3| \leqq \frac{1}{6} \max\{1, |\frac{12}{8}|\} = \frac{1}{4}.$$

By making use of the Lemma 1.4, we obtain the upper bound for $|a_3 - \mu a_2^2|$ in the the following theorems.

Theorem 2.7. *Let $0 \leqq \alpha \leqq 1, 0 \leqq \beta \leqq 1, 0 \leqq \lambda \leqq 1$ and μ is a real number. If $f(z)$ given by (1) belongs to $\mathcal{M}_q(\alpha, \beta, \lambda)$, then*

$$|a_3 - \mu a_2^2| \leqq \begin{cases} \frac{1}{2\xi}\left(1 - \frac{\Phi(\alpha,\beta,\lambda) + 2\mu\xi}{\tau^2}\right), & \text{if} \quad \mu \leqq \sigma_1, \\ \frac{1}{\xi}, \ \text{if } \sigma_1 \leqq \mu \leqq \sigma_2, & \\ \frac{1}{2\xi}\left(-1 + \frac{\Phi(\alpha,\beta,\lambda) + 2\mu\xi}{\tau^2}\right), & \text{if} \quad \mu \geqq \sigma_2, \end{cases}$$

where, for convenience,

$$\begin{aligned}\sigma_1 &:= -\frac{\Phi(\alpha,\beta,\lambda)+\tau^2}{2\xi}, \qquad \sigma_2 := \frac{3\tau^2+\Phi(\alpha,\beta,\lambda)}{2\xi},\\ \sigma_3 &:= \frac{\tau^2-\Phi(\alpha,\beta,\lambda)}{2\xi},\end{aligned}$$

$$\xi := \alpha+(1+2\lambda)\beta \quad \textit{and} \quad \tau := \alpha+(1+\lambda)\beta. \tag{13}$$

Further, if $\sigma_1 \leqq \mu \leqq \sigma_3$, then

$$|a_3-\mu a_2^2| + \frac{\tau^2}{2\xi}\left(1+\frac{\Phi(\alpha,\beta,\lambda)+2\mu\xi}{\tau^2}\right)|a_2|^2 \leqq \frac{1}{\xi}.$$

If $\sigma_3 \leqq \mu \leqq \sigma_2$, then

$$|a_3-\mu a_2^2| + \frac{\tau^2}{2\xi}\left(3-\frac{\Phi(\alpha,\beta,\lambda)+2\mu\xi}{\tau^2}\right)|a_2|^2 \leqq \frac{1}{\xi}.$$

These results are sharp.

Proof. Now by making use of (10) and (12) , we get

$$\begin{aligned}& a_3-\mu a_2^2\\ =\ & \frac{1}{4[\alpha+(1+2\lambda)\beta]}\left(c_2-\frac{c_1^2}{4}\left(1+\frac{\Phi(\alpha,\beta,\lambda)}{[\alpha+(1+\lambda)\beta]^2}\right)\right)\\ -\ & \frac{\mu c_1^2}{4[\alpha+(1+\lambda)\beta]^2}\\ =\ & \frac{1}{4[\alpha+(1+2\lambda)\beta]}\\ \times\ & \left(c_2-\frac{c_1^2}{2}\left(\frac{1}{2}+\frac{\Phi(\alpha,\beta,\lambda)+2\mu[\alpha+(1+2\lambda)\beta]}{2\left(\alpha+(1+\lambda)\beta\right)^2}\right)\right)\end{aligned}$$

where $\Phi(\alpha,\beta,\lambda)$ is given by (11). Letting

$$v := \frac{1}{2}\left(\frac{1}{2}+\frac{\Phi(\alpha,\beta,\lambda)+2\mu[\alpha+(1+2\lambda)\beta]}{2\left(\alpha+(1+\lambda)\beta\right)^2}\right).$$

the assertion of Theorem 2.1 now follows by an application of Lemma 1.4.

To show that the bounds are sharp, we define the functions the functions F_η and G_η $(0 \leqq \eta \leqq 1)$, respectively, with $F_\eta(0) = 0 = F'_\eta(0) - 1$ and $G_\eta(0) = 0 = G'_\eta(0) - 1$ by

$$\left(\frac{z(F_\eta)'(z)}{F_\eta(z)}\right)^\alpha \left[(1-\lambda)\left(\frac{z(F_\eta)'(z)}{F_\eta(z)}\right) + \lambda\left(1 + \frac{z(F_\eta)''(z)}{(F_\eta)'(z)}\right)\right]^\beta$$
$$= \phi\left(\frac{z(z+\eta)}{1+\eta z}\right)$$

and

$$\left(\frac{z(G_\eta)'(z)}{G_\eta(z)}\right)^\alpha \left[(1-\lambda)\frac{z(G_\eta)'(z)}{G_\eta(z)} + \lambda\left(1 + \frac{z(G_\eta)''(z)}{(G_\eta)'(z)}\right)\right]^\beta$$
$$= \phi\left(-\frac{z(z+\eta)}{1+\eta z}\right)$$

respectively.

Clearly the functions $K_{\phi_n}, F_\eta, G_\eta \in \mathcal{M}_\phi(\alpha,\beta,\lambda)$. Also we write $K_\phi := K_{\phi_2}$.

If $\mu < \sigma_1$ or $\mu > \sigma_2$, then the equality holds if and only if f is K_ϕ or one of its rotations. When $\sigma_1 < \mu < \sigma_2$, then the equality holds if and only if f is K_{ϕ_3} or one of its rotations. If $\mu = \sigma_1$ then the equality holds if and only if f is F_η or one of its rotations. If $\mu = \sigma_2$ then the equality holds if and only if f is G_η or one of its rotations. $\square$

By making use of Lemma 1.6, we immediately obtain the following:

Theorem 2.8. *Let* $0 \leqq \alpha \leqq 1, 0 \leqq \beta \leqq 1$ *and* $0 \leqq \lambda \leqq 1$. *If* $f \in \mathcal{M}_q(\alpha,\beta,\lambda)$, *then for complex* μ, *we have*

$$\begin{aligned} |a_3 - \mu a_2^2| &= \frac{1}{2[\alpha + (1+2\lambda)\beta]} \\ &\times \max\left\{1, \frac{1}{2}\left|-1 + \frac{\Phi(\alpha,\beta,\lambda) + 2\mu[\alpha + (1+2\lambda)\beta]}{2\left(\alpha + (1+\lambda)\beta\right)^2}\right|\right\}. \end{aligned}$$

The result is sharp.

Remark 2.9.

1. For the choices $\alpha = 0, \beta = 1$ and $\lambda = 0$, Theorem 2.8 reduces to the result for the class $\mathcal{S}^*(q)$[17].

2. For the choice $\alpha = 0, \beta = 1$ and $\lambda = 1$, Theorem 2.8, coincides with the result obtained for the class $\mathcal{C}(q)$ Sokol and Thomas [20].

3. For the choices $\alpha = 0$ and $\beta = 1$, Theorem 2.8 reduces to the result for the class $\mathcal{M}_q(\lambda)$[19].

3 Coefficient Inequalities for the Function f^{-1}

In section we obtain the Fekete-Szegö inequality results for f^{-1} in the new subclass $\mathcal{M}_q(\alpha, \beta, \lambda)$.

Theorem 3.1. *Let* $0 \leqq \alpha \leqq 1, 0 \leqq \beta \leqq 1$ *and* $0 \leqq \lambda \leqq 1$. *If* $f \in \mathcal{M}_q(\alpha, \beta, \lambda)$ *and* $f^{-1}(w) = w + \sum\limits_{n=2}^{\infty} d_n w^n$ *is the inverse function of* f *with* $|w| < r_0$ *where* r_0 *is greater than the radius of the Koebe domain of the class* $f \in \mathcal{M}_q(\alpha, \beta, \lambda)$, *then for any complex number* μ, *we have*

$$| d_3 - \mu d_2^2 | \leq \frac{1}{4\xi} max \left\{ 1, \frac{1}{2} \mid \frac{\Phi(\alpha, \beta, \lambda) + 4\xi(2 - \mu)}{2\tau^2} - 1 \mid \right\} \tag{14}$$

where $\Phi(\alpha, \beta, \lambda)$ *and* ξ, τ *are as defined in*(11)*and* (13)*respectively and the result is sharp.*

Proof. As

$$f^{-1}(w) = w + \sum_{n=2}^{\infty} d_n w^n \tag{15}$$

is the inverse function of f, it can be seen that

$$f^{-1}(f(z)) = f\{f^{-1}(z)\} = z \tag{16}$$

From equations (1) and (16), it can be reduced to

$$f^{-1}(z + \sum_{n=2}^{\infty} a_n z^n) = z \tag{17}$$

From (15) and (17), one can obtain

$$z + (a_2 + d_2)z^2 + (a_3 + 2a_2 d_2 + d_3)z^3 + ... = z \tag{18}$$

By comparing the coefficients of z and z^2 from relation (18), it can be seen that

$$d_2 = -a_2 \tag{19}$$
$$d_3 = 2a_2^2 - a_3 \tag{20}$$

From relations (10),(12),(19) and (20)

$$d_2 = -\frac{c_1}{2[\alpha+(1+\lambda)\beta]} = \frac{c_1}{2\tau}; \tag{21}$$

$$\begin{aligned} d_3 &= \frac{c_1^2}{2[\alpha+(1+\lambda)\beta]^2} - \frac{1}{2[\alpha+(1+2\lambda)\beta]} \\ &\quad \times \left(\frac{c_2}{2} - \frac{c_1^2}{8} - \frac{\Phi(\alpha,\beta,\lambda)c_1^2}{8[\alpha+(1+\lambda)\beta]^2}\right) \\ &= \frac{c_1^2}{2\tau^2} - \frac{1}{2\xi}\left(\frac{c_2}{2} - \frac{c_1^2}{8} - \frac{\Phi(\alpha,\beta,\lambda)c_1^2}{8\tau^2}\right) \\ &= \frac{1}{4\xi}\left(-c_2 + \frac{c_1^2}{2}\left[\frac{1}{2} + \frac{\Phi(\alpha,\beta,\lambda)+8\xi}{2\tau^2}\right]\right) \end{aligned} \tag{22}$$

where $\Phi(\alpha,\beta,\lambda)$ and ξ,τ are as defined in (11) and (13) respectively. Using (21) and (22) for any complex number μ, consider

$$\begin{aligned} d_3 - \mu d_2^2 &= \frac{1}{4\xi}\left(-c_2 + \frac{c_1^2}{2}\left[\frac{1}{2} + \frac{\Phi(\alpha,\beta,\lambda)+8\xi}{2\tau^2}\right]\right) - \frac{\mu c_1^2}{4\tau^2} \\ &= -\frac{1}{4\xi}\left(c_2 - \frac{c_1^2}{2}\left[\frac{1}{2} + \frac{\Phi(\alpha,\beta,\lambda)+4\xi(2-\mu)}{2\tau^2}\right]\right) \end{aligned} \tag{23}$$

Taking modulus on both sides and by applying Lemma 1.6 on the right hand side of (23), one can obtain the result as in (14). Hence this completes the proof. □

4 Application to Functions Defined by Poisson Distribution

In this section we define a new function class based on convolution operator and we discuss the application of Poisson distribution series to

the function class.
We define the class $\mathcal{M}_q^{\psi}(\alpha,\beta,\lambda)$ in the following way:

$$\mathcal{M}_q^{\psi}(\alpha,\beta,\lambda) := \{f\in\mathcal{A} \quad \text{and} \quad f*\psi(z)\in\mathcal{M}_q(\alpha,\beta,\lambda)\}$$

where $(f*\psi)(z) = z+\sum\limits_{n=2}^{\infty}\psi_n a_n z^n$. That is equivalently satisfying the subordination condition

$$\left(\tfrac{z(f*\psi)'(z)}{(f*\psi)(z)}\right)^{\alpha}\left[(1-\lambda)\tfrac{z(f*\psi)'(z)}{(f*\psi)(z)}+\lambda\left(1+\tfrac{z(f*\psi)''(z)}{(f*\psi)'(z)}\right)\right]^{\beta}$$
$$\prec z+\sqrt{1+z^2}=q(z), \quad z=re^{i\theta}\in\mathbb{D}.$$

A variable $\mathcal{X}$ is said to be Poisson distributed if it takes the values $0,1,2,3,\cdots$ with probabilities e^{-m}, $m\frac{e^{-m}}{1!}$, $m^2\frac{e^{-m}}{2!}$, $m^3\frac{e^{-m}}{3!}$,... respectively, where m is called the parameter. Thus

$$P(\mathcal{X}=r)=\frac{m^r e^{-m}}{r!},\ r=0,1,2,3,\cdots.$$

In [15],Porwal introduced a power series whose coefficients are probabilities of Poisson distribution

$$\mathcal{K}(m,z)=z+\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!}e^{-m}z^n, \qquad z\in\mathbb{D},$$

where $m>0$. By ratio test the radius of convergence of above series is infinity. Using the Hadamard product, Porwal[15] (see also, [3, 11, 12, 16] introduced a new linear operator $\mathcal{I}^m(z):\mathcal{A}\to\mathcal{A}$ defined by

$$\begin{aligned}\mathcal{I}^m f=\mathcal{K}(m,z)*f(z) &= z+\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!}e^{-m}a_n z^n\\ &= z+\sum_{n=2}^{\infty}\psi_n(m)a_n z^n,\end{aligned}$$

where $\psi_n(m)=\frac{m^{n-1}}{(n-1)!}e^{-m}$, and $*$ denote the convolution or Hadamard product of two series. We define the class $\mathcal{M}_q^m(\alpha,\beta,\lambda)$ in the following way:

$$\mathcal{M}_q^m(\alpha,\beta,\lambda) := \{f\in\mathcal{A} \quad \text{and} \quad \mathcal{I}^m f\in\mathcal{M}_q(\alpha,\beta,\lambda)\}$$

where $\mathcal{M}_q(\alpha,\beta,\lambda)$ is given by Definition 1.2.

First we obtain the Fekete-Szegö inequality for $f \in \mathcal{M}_q^{\psi}(\alpha,\beta,\lambda)$, from the corresponding inequality for $f \in \mathcal{M}_q(\alpha,\beta,\lambda)$.
Applying Theorem 2.7 and 2.8 for the function

$$(f*\psi)(z) = z + \psi_2 a_2 z^2 + \psi_3 a_3 z^3 + \cdots,$$

we get the following Theorems 4.1 and 4.2 after an obvious change of the parameter μ.

Theorem 4.1. *Let* $0 \leqq \alpha \leqq 1, 0 \leqq \beta \leqq 1,$ *and* $0 \leqq \lambda \leqq 1$. *If* $f \in \mathcal{M}_q^{\psi}(\alpha,\beta,\lambda)$, *then for complex* μ, *we have*

$$\begin{aligned} & |a_3 - \mu a_2^2| \\ \leq\ & \frac{1}{2[\alpha+(1+2\lambda)\beta]\psi_3} \\ \times\ & \max\left\{1, \frac{1}{2}\left|-1+\frac{\Phi(\alpha,\beta,\lambda)}{2[\alpha+(1+\lambda)\beta]^2}+\frac{4\mu[\alpha+(1+2\lambda)\beta]\psi_3}{[(\alpha+(1+\lambda)\beta)\,\psi_2]^2}\right|\right\} \\ =\ & \frac{1}{2\xi\psi_3}\max\left\{1, \frac{1}{2}\left|-1+\frac{\Phi(\alpha,\beta,\lambda)}{2\tau^2}+\frac{4\mu\xi\psi_3}{[\tau\psi_2]^2}\right|\right\} \end{aligned}$$

where $\Phi(\alpha,\beta,\lambda)$ *and* ξ,τ *are given by*(11)*and* (13)*respectively .The result is sharp.*

Theorem 4.2. *Let* $0 \leqq \alpha \leqq 1, 0 \leqq \beta \leqq 1, 0 \leqq \lambda \leqq 1,\mu$ *a real number and* $\psi_n > 0$. *If* $f(z)$ *given by* (1) *belongs to* $\mathcal{M}_q^m(\alpha,\beta,\lambda)$, *then*

$$|a_3 - \mu a_2^2| \leqq \begin{cases} \dfrac{1}{2\xi\psi_3}\left(1-\dfrac{\Phi(\alpha,\beta,\lambda)}{\tau^2}-\dfrac{2\mu\xi\psi_3}{\tau^2\psi_2^2}\right), & \textit{if } \ \mu \leqq \sigma_1, \\ \frac{1}{\xi\psi_3}, \ \textit{if } \ \sigma_1 \leqq \mu \leqq \sigma_2, & \\ \dfrac{1}{2\xi\psi_3}\left(-1+\dfrac{\Phi(\alpha,\beta,\lambda)}{\tau^2}+\dfrac{2\mu\xi\psi_3}{\tau^2\psi_2^2}\right), & \textit{if } \ \mu \geqq \sigma_2, \end{cases}$$

where, for convenience,

$$\sigma_1 := -\ \frac{\psi_2^2}{\psi_3}\,\frac{\Phi(\alpha,\beta,\lambda)+\tau^2}{2\xi}, \qquad \sigma_2 = \frac{\psi_2^2}{\psi_3}\,\frac{3\tau^2+\Phi(\alpha,\beta,\lambda)}{2\xi},$$

where $\Phi(\alpha,\beta,\lambda)$ *and* ξ,τ *are as defined in* (11)*and* (13) *respectively. These results are sharp.*

Now we obtain the Fekete-Szegö inequality for $f \in \mathcal{M}_q^m(\alpha,\beta,\lambda)$, from the corresponding estimate for $f \in \mathcal{M}_q^{\psi}(\alpha,\beta,\lambda)$. Applying Theorem 2.7 and 2.8 and also Theorem 4.1 and 4.2 for the function $\mathcal{I}^m f = z + \sum_{n=2}^{\infty} \psi_n(m) a_n z^n, \qquad z \in \mathbb{D}$, in particular we have

$$\psi_2 = me^{-m} \quad \text{and} \quad \psi_3 = \frac{m^2}{2} e^{-m}. \tag{24}$$

By using the values of ψ_2 and ψ_3 given by (24) in Theorems 4.1 and 4.2 we get the following results:

Theorem 4.3. *Let* $0 \leqq \alpha \leqq 1, 0 \leqq \beta \leqq 1$, *and* $0 \leqq \lambda \leqq 1$. *If* $f \in \mathcal{M}_q^m(\alpha,\beta,\lambda)$, *then for complex* μ, *we have*

$$\begin{aligned} &|a_3 - \mu a_2^2| \\ = \quad &\frac{1}{\xi m^2 e^{-m}} \max\left\{1, \frac{1}{2}\left|-1 + \frac{\Phi(\alpha,\beta,\lambda)}{2\tau^2} + \frac{2\mu\xi}{\tau^2 e^{-m}}\right|\right\} \end{aligned}$$

where $\Phi(\alpha,\beta,\lambda)$ *and* ξ,τ *are given by*(11)*and* (13)*respectively .The result is sharp.*

Theorem 4.4. *Let* $0 \leqq \alpha \leqq 1, 0 \leqq \beta \leqq 1, 0 \leqq \lambda \leqq 1$, μ *a real number and* $\psi_n > 0$. *If* $f(z)$ *given by* (1) *belongs to* $\mathcal{M}_q^m(\alpha,\beta,\lambda)$, *then*

$$|a_3 - \mu a_2^2| \leqq \begin{cases} \dfrac{1}{\xi m^2 e^{-m}}\left(1 - \dfrac{\Phi(\alpha,\beta,\lambda)}{\tau^2} - \dfrac{\mu\xi}{\tau^2 e^{-m}}\right), & \text{if } \mu \leqq \sigma_1, \\ \dfrac{2}{\xi m^2 e^{-m}}, & \text{if } \sigma_1 \leqq \mu \leqq \sigma_2, \\ \dfrac{1}{\xi m^2 e^{-m}}\left(-1 + \dfrac{\Phi(\alpha,\beta,\lambda)}{\tau^2} + \dfrac{\mu\xi}{\tau^2 e^{-m}}\right), & \text{if } \mu \geqq \sigma_2, \end{cases}$$

where, for convenience,

$$\sigma_1 \quad := - \quad e^{-m}\frac{\Phi(\alpha,\beta,\lambda) + \tau^2}{\xi}, \qquad \sigma_2 = e^{-m}\ \frac{3\tau^2 + \Phi(\alpha,\beta,\lambda)}{\xi},$$

where $\Phi(\alpha, \beta, \lambda)$ *and* ξ, τ *are as defined in* (11)*and* (13) *respectively. These results are sharp.*

Concluding Remark: Suitably specializing the parameters in Theorems 4.1 and 4.2 and by various choices of ψ one can easily deduce the results for the function classes listed in Remark 1.3 involving differential and integral operators studied in [1, 2, 7, 14, 18, 22]. Further from Theorems 4.3 and 4.4 one can easily state applications of Poisons distribution for the classes listed in Remark 1.3 which are new and not yet been discussed.Further this study can be extended to a class of analytic functions associated with Mittag-Leffler-type functions based on Borel distribution[13].

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