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Original Research Paper

Ricci and Scalar Curvatures of Hemi-Slant Submanifolds in 3-Sasakian Space Forms

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Abstract. In this paper, we introduce a hemi-slant submanifold of a 3-Sasakian manifold. First, we obtain some new results in terms of the operators T_i and f_i. By using Gauss, Codazzi and Ricci equations, we prove some results involving Ricci and scalar curvatures by using the slant angle and the mean curvature vector of the submanifold.

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1 Introduction

At the beginning of the last decade of twentieth century, Chen defined the concept of slant submanifolds [7] and proved many interesting fun-

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damental results for these submanifolds in his book [8]. After that, this notion has been investigated and generalized by many authors for some ambient manifolds which were equipped with various structures such as almost complex, contact and quaternionic structures [2, 3, 9, 14]. Hemi-slant submanifolds in contact structures were defined by Cabrerizo et al. [5]. Since one of the distributions of hemi-slant submanifolds are anti-invariant, the hemi-slant submanifold can be considered as particular class of the bi-slant submanifolds. Later, B. Sahin obtained some interesting results on warped products of hemi-slant submanifolds in Kaehler manifolds [15]. In [16], second author and Chen investigated warped product bi-slant submanifolds and as a generalization of these submanifolds, they also introduce the idea of pointwise bi-slant submanifolds [10], when the ambient manifold was a Kaehlerian manifold.

In [13, 14], first author and Malek defined 3-slant submanifolds of 3-structure manifolds. In this paper, we extend these subjects to the hemi-slant submanifolds of a special class of almost contact 3-structures which is called 3-Sasakian manifolds.

The paper is organised as follows: In Section 2, we review the basic notations about 3-Sasakian manifolds. In Section 3, we introduce hemi-slant submanifolds of 3-structures and Sasakian 3-hemi slant submanifolds. We prove that a hemi-slant submanifold M of a 3-Sasakian manifold $(\tilde{M}, \xi_i, \eta_i, \varphi_i)_{i\in\{1,2,3\}}$ which admits a 3-Sasakian structure T_i, is anti-invariant if and only if T_i is parallel. Moreover, we obtain some results in terms of the tangential and normal components of φ_i. In Section 4, we suppose the ambient manifold of the hemi-slant submanifold be a 3-Sasakian space form and obtain fundamental results by using Gauss, Codazzi and Ricci equations. We deduce some new results for Ricci tensor and scalar curvatures involving the mean curvature and slant angle of the submanifold.

2 Preliminaries

Definition 2.1. [4] A Riemannian manifold $(\tilde{M}, g)$ has an almost contact metric structure if it is endowed by the tensor fields ξ, η and φ of type $(1,0), (0,1)$ and $(1,1)$ respectively, such that

$$\varphi^2(X) = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \tag{1}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \ \forall X, Y \in T\tilde{M}. \tag{2}$$

Definition 2.2. [12] Let $\tilde{M}$ be a Riemannian manifold and for $i = 1, 2, 3$, $(\eta_i, \xi_i, \varphi_i)$, be three almost contact structures on $\tilde{M}$ satisfying

$$\eta_i(\xi_j) = 0, \quad \varphi_i \xi_j = -\varphi_j \xi_i = \xi_k, \ \ \eta_i(\varphi_j) = -\eta_j(\varphi_i) = \eta_k, \tag{3}$$

$$\varphi_i \circ \varphi_j - \eta_{j \otimes} \xi_i = -\varphi_j \circ \varphi_i + \eta_i \circ \xi_j = \varphi_k, \tag{4}$$

for a permutation (i, j, k) of $(1, 2, 3)$, then $\tilde{M}$ has an almost contact 3-structure $(\xi_i, \eta_i, \varphi_i)$. In addition, if there exists a compatible Riemannian metric g on $\tilde{M}$ which for all vector fields X, Y on $\tilde{M}$ the following relation holds

$$g(\varphi_i X, \varphi_i Y) = g(X, Y) - \eta_i(Y)\eta_i(X), \tag{5}$$

then $(\tilde{M}, \xi_i, \eta_i, \varphi_i, g)$ is an almost contact metric 3-structure manifold which in this paper shortly is called as *3-structure*.

It is well-known that this compatible metric is skew-symmetric with respect to the φ_i, i.e. $g(\varphi_i X, Y) = -g(X, \varphi_i Y)$. Moreover, the dimension of a 3-structure is $4k + 3$ [17].

The 3-structure $(\tilde{M}, \xi_i, \eta_i, \varphi_i, g)$ is called a 3-Sasakian manifold if the following conditions satisfy

$$(\tilde{\nabla}_X \varphi_i)Y = g(X, Y)\xi_i - \eta_i(Y)X, \tag{6}$$

$$\tilde{\nabla}_X \xi_i = -\varphi_i X, \tag{7}$$

for any $X \in T\tilde{M}$ where $\tilde{\nabla}$ is the Riemannian connection on $\tilde{M}$.

Let the Ricci tensor of the Riemannian manifold $(\tilde{M}, g)$ satisfies $S(X, Y) = ag(X, Y) + bA(X)A(Y)$, for an 1-form A and $a, b \in C^\infty(\tilde{M})$, then $\tilde{M}$ is a *quasi Einstein manifold* [6]. If $b = 0$, then $(\tilde{M}, g)$ is called an Einstein manifold. 3-Sasakian manifolds are famous examples of the Einstein manifolds.

3 Hemi-slant submanifolds of 3-structures

For a submanifold M of a Riemannian manifold $\tilde{M}$, we have the Gauss and Weingarten formulas as follows [17]

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y), \quad \forall X, Y \in TM \tag{8}$$

$$\tilde{\nabla}_X V = -A_V X + D_X V, \quad \forall V \in (TM)^{\perp}. \tag{9}$$

Here, $\tilde{\nabla}$ and ∇ are the Riemannian connections of $(\tilde{M}, g)$ and (M, g), σ and A are the second fundamental form and the associated operator of σ, respectively. Also, the connection on the normal bundle is denoted by D.

The curvature tensors $\tilde{R}$ and R with respect to the connections $\tilde{\nabla}$ and ∇ satisfy

$$\begin{aligned} g(\tilde{R}(X,Y)Z,W) = g(R(X,Y)Z,W) + g(\sigma(X,Z),\sigma(Y,W)) \\ - g(\sigma(X,W),\sigma(Y,Z)), \end{aligned} \tag{10}$$

for all vector fields X, Y, Z, W on M, which is called *equation of Gauss*. Moreover, for the normal component $(R(X,Y)Z)^{\perp}$ of $\tilde{R}(X,Y)Z$ the *equation of Codazzi* states

$$(\tilde{R}(X,Y)Z)^{\perp} = (\tilde{\nabla}_X\sigma)(Y,Z) - (\tilde{\nabla}_Y\sigma)(X,Z). \tag{11}$$

Let M be a submanifold of $(\tilde{M}, \xi_i, \eta_i, \varphi_i, g)$. For any $X \in TM$ and $V \in T^{\perp}M$, we put

$$\varphi_i X = T_i X + F_i X, \ \varphi_i V = t_i V + f_i V, \tag{12}$$

where T_i, t_i are tangential and F_i, f_i are normal projections of φ_i.

One can verify that

$$g(t_i V, Y) = -g(V, F_i Y), \tag{13}$$

$$g(f_i V, X) = -g(V, T_i X). \tag{14}$$

The submanifold M is invariant if at any point $p \in M$, $\varphi_i(T_pM) \subset T_pM$ and is anti-invariant if $\varphi_i(T_pM) \subset T_p^{\perp}M$. Therefore, based on the notation of Equation (14), on invariant (resp. anti-invariant) submanifolds we get $F_i = 0$ (resp. $T_i = 0$).

Motivated by Chen [7, 8] works on slant immersions and submanifolds which are interesting generalizations of invariant (complex) and anti-invariant (totally real) submanifolds, the first author and Malek defined the following type of submanifolds.

Definition 3.1. [13] A submanifold M of a 3-structure $(\tilde{M}, \varphi_i, \xi_i, \eta_i, g)$, $i \in \{1, 2, 3\}$ is a 3-slant submanifold, if for all $p \in M$ and for any $X \in T_pM$, $(X \neq 0)$, the angel $\alpha = \widehat{\varphi_i X, T_j X}$ has a constant value, for all $i, j \in \{1, 2, 3\}$.

So, in 3-slant submanifolds, the choice of point p and X does not effect on the value of the slant angle α. For example, on anti-invariant and invariant submanifolds the slant angle α between $\varphi_i X$ and the tangent space T_pM are equal to $\frac{\pi}{2}$ and 0, respectively.

Definition 3.2. Suppose that M is a submanifold of a 3-structure $(\tilde{M}, \varphi_i, \xi_i, \eta_i, g)$ and admits orthogonal distributions $\mathfrak{D}^\alpha, \mathfrak{D}^\perp$ and $< \xi_i >$, where $TM = \mathfrak{D}^\alpha \oplus \mathfrak{D}^\perp \oplus < \xi_i >$. We say M is a 3-hemi slant submanifold, if
(a) $\mathfrak{D}^\perp$ is anti-invariant, this means, $\varphi_i(\mathfrak{D}^\perp) \subset T^\perp M$.
(b) $\mathfrak{D}^\alpha$ is a 3-slant distribution with respect to the φ_i's and has slant angle $\alpha \neq 0$, i.e. $\forall X \in \mathfrak{D}^\alpha$, the angle $\alpha = \widehat{\varphi_i X, \mathfrak{D}^\alpha}$ is constant.

Lemma 3.3. *[11] Let M be a 3-slant submanifold of 3-structure $(\tilde{M}, g)$. Then for any $X \in TM$ which is normal to ξ_i, we have $T_i T_j (X) = -\cos^2 \alpha X$, where α is the slant angle and $i, j \in \{1, 2, 3\}$.*

Theorem 3.4. *Let for $i \in \{1, 2, 3\}$, $(\tilde{M}, \varphi_i, \xi_i, \eta_i, g)$ be a 3-Sasakian manifold and M be its 3-hemi slant submanifold. Then the structure vector fields ξ_i's are parallel with respect to the connection ∇ if and only if $\alpha = \frac{\pi}{2}$ on M.*

Proof. From Eq. (7), we have

$$\tilde{\nabla}_X \xi_i = -\varphi_i X, \quad \forall X \in TM. \tag{15}$$

On the other hand, Gauss formula implies

$$\tilde{\nabla}_X \xi_i = \nabla_X \xi_i + \sigma(X, \xi_i). \tag{16}$$

Therefore, if ξ_i is parallel, then we obtain $\sigma(X, \xi_i) = -\varphi_i X$. Thus $\varphi_i X \in T^\perp M$, so M is an anti-invariant submanifold.

Conversely, if M is an anti-invariant submanifold then for any $X \in TM$, we have $\varphi_i X \in T^\perp M$. Thus from (15) and (16), we get $\varphi_i X = -\sigma(X, \xi_i)$ and $\nabla_X \xi_i = 0$, i.e., ξ_i is parallel. □

Definition 3.5. Let M be a 3-hemi slant submanifold of a 3-structure $(\tilde{M}, \varphi_i, \xi_i, \eta_i, g), i \in \{1, 2, 3\}$. Then M is called a Sasakian 3-hemi slant submanifold if

$$(\nabla_X T_i)Y = g(X, Y)\xi_i - \eta_i(Y)X, \quad \forall X, Y \in TM.$$

Theorem 3.6. *Let M be a Sasakian 3-hemi slant submanifold of 3-structure $(\tilde{M}, \xi_i, \eta_i, \varphi_i, g)$. Then the tensor field T_i is parallel if and only if the submanifold is anti-invariant.*

Proof. Since M is a Sasakian 3-hemi slant submanifold, if any $X, Y \in \mathfrak{D}^\alpha$ and then we have

$$(\nabla_X T_i)Y = g(X, Y)\xi_i - \eta_i(Y)X.$$

If T is parallel, then by taking $X = T_i X, Y = T_j X$, we obtain

$$0 = g(T_i X, T_j X)\xi_i - \eta_i(T_j X)T_i X = -g(T_i T_j X, X)\xi_i. \tag{17}$$

From Lemma 3.3, we get

$$\cos^2 \alpha \, g(X, X)\xi_i = 0. \tag{18}$$

Since g is a positive definite metric, from (18), we find that $\alpha = \frac{\pi}{2}$,thus M is an anti-invariant submanifold. The converse is trivial. □

Theorem 3.7. *For a 3-hemi slant submanifold M of a 3-Sasakian manifold $(\tilde{M}, \xi_i, \eta_i, \varphi_i, g), i \in \{1, 2, 3\}$, the following relation holds*

$$g((\nabla_X T_i)Y, Z) = -g((\nabla_X T_i)Z, Y), \quad \forall X, Y \in TM.$$

Proof. On a 3-Sasakian manifold $(\tilde{M}, \xi_i, \eta_i, \varphi_i, g)$, we have

$$(\tilde{\nabla}_X \varphi_i)Y = g(X, Y)\xi_i - \eta_i(Y)X = R(\xi_i, X)Y.$$

Therefore, by using (12) and Gauss-Weingarten formulas, we see

$$R(\xi_i, X)Y = \nabla_X T_i Y + \sigma(X, T_i Y) + D_X F_i Y - A_{F_i Y} X - \\ T_i(\nabla_X Y) - t_i \sigma(X, Y) - F_i(\nabla_X Y) - f_i \sigma(X, Y).$$

We consider the tangential components of the previous equation and obtain

$$R(\xi_i, X)Y = (\nabla_X T_i)Y - A_{F_i Y} X - t_i \sigma(X, Y).$$

Thus, we have

$$g((\nabla_X T_i)Y, Z) = g(R(\xi_i, X)Y, Z) + g(A_{F_i Y} X, Z) + g(t_i \sigma(X, Y), Z).$$

From (14), self-adjoint property of A and symmetry properties of R and σ, previous equation gives

$$g((\nabla_X T_i)Y, Z) = g(\sigma(X, Z), F_i Y) - g(R(\xi_i, X)Z, Y) - g(\sigma(X, Y), F_i Z) \\ = -g(R(\xi_i, X)Z, Y) - g(t_i \sigma(X, Z), Y) - g(A_{F_i Z} X, Y) = -g(R(\xi_i, X)Z \\ + t_i \sigma(X, Z) + A_{F_i Z} X, Y) = -g((\nabla_X T_i)Z, Y).$$

□

Theorem 3.8. *For a* 3*-hemi slant submanifold* M *of a* 3*-Sasakian manifold* $(\tilde{M}, \xi_i, \eta_i, \varphi_i, g), i \in \{1, 2, 3\}$*, the operator* f_i *is an anti-symmetric tensor field with respect to the normal covariant derivative, this means, for all* $X \in TM$ *and* $U, V \in T^{\perp}M$*, we have*

$$g((D_X f_i)V, U) = -g((D_X f_i)U, V).$$

Proof. For any $X \in TM$ and $V \in T^{\perp}M$, we have

$$\tilde{\nabla}_X \varphi_i V - \varphi_i \tilde{\nabla}_X V = (\tilde{\nabla}_X \varphi_i)V = g(X, V)\xi_i - \eta_i(V)X = 0, \qquad (19)$$

Then, from (12), we get

$$\tilde{\nabla}_X t_i V + \tilde{\nabla}_X f_i V - \varphi_i \left(-A_V X + D_X V\right) = 0.$$

By using Gauss-Weingarten formulas and (12) we obtain

$$\begin{aligned}&\nabla_X t_iV + \sigma(X, t_iV) + D_X f_iV - A_{f_iV}X - t_iD_XV + T_iA_VX\\ &- f_iD_XV + F_iA_VX = (\nabla_X t_i)V + (D_X f_i)V + \sigma(X, t_iV)-\\ &A_{f_iV}X + \varphi_iA_VX = 0. \end{aligned} \tag{20}$$

For any $U \in T^{\perp}M$, by taking the inner product of U and Equation (20), we get

$$\begin{aligned} g(D_X f_i)V, U) &= g(-(\nabla_X t_i)V - \sigma(X, t_iV) - \varphi_iA_VX + A_{f_iV}X, U)\\ &= g(\nabla_X t_i)U, V) - g(\sigma(X, t_iV), U) - g(\varphi_iA_VX, U)\\ &= g(\nabla_X t_i)U, V) - g(A_UX, t_iV) + g(A_VX, t_iU)\\ &= g(\nabla_X t_i)U, V) + g(F_iA_UX, V) + g(\sigma(X, t_iU), V)\\ &= g(\nabla_X t_i)U, V) + g(\varphi_iA_UX, V) + g(\sigma(X, t_iU), V)\\ &= -g(-(\nabla_X t_i)U - \varphi_iA_UX - \sigma(X, t_iU) - A_{f_iU}X, V)\\ &= -g((D_X f_i)U, V). \end{aligned}$$

Therefore, f_i is an anti-symmetric tensor field. $\square$

Theorem 3.9. *For a 3-hemi slant submanifold M of a 3-Sasakian manifold $(\tilde{M}, \xi_i, \eta_i, \varphi_i, g)$, If f_i and t_i are parallel operators, then the shape operator A vanishes on M.*

Proof. For any $X \in TM, V \in T^{\perp}M$, from (19), we have

$$(\tilde{\nabla}_X \varphi_i)V = (\tilde{\nabla}_X t_i)V + (\tilde{\nabla}_X f_i)V + \varphi_iA_VX = 0.$$

Since f_i and t_i are parallel, then we get $\varphi_iA_VX = 0$, for any $X \in TM, V \in T^{\perp}M$. This means the shape operator $A = 0$. $\square$

4 Submanifolds of 3-Sasakian space forms

A 3-Sasakian space form $\tilde{M}(c)$ is a 3-Sasakian manifold $(\tilde{M}, \xi_i, \eta_i, \varphi_i, g)$, $i \in \{1, 2, 3\}$, such that for a constant $c \in \mathbb{R}$, its φ_i-holomorphic sectional curvature is equal to c at any point of $\tilde{M}$. Since all the Sasakian structures $(\xi_i, \eta_i, \varphi_i, g), i \in \{1, 2, 3\}$ are Sasakian space forms (cf. [17], p314), so the curvature tensor $\tilde{R}$ satisfies in the following relation (cf. [1], p342)

$$\tilde{R}(X,Y)Z = \left(\frac{c+3}{4}\right)[g(Y,Z)X - g(X,Z)Y] + \left(\frac{c-1}{4}\right)\sum_{i=1}^{3}[(\eta_i(X)Y - \eta_i(Y)X)\eta_i(Z) + (g(X,Z)\eta_i(Y) - g(Y,Z)\eta_i(X))\xi_i + g(\varphi_i Y,Z)\varphi_i X - g(\varphi_i X,Z)\varphi_i Y + 2g(\varphi_i Y,X)\varphi_i Z]. \tag{21}$$

By using Lemma 3.3 and (5) we can prove the following lemma.

Lemma 4.1. *Let M be a 3-hemi slant submanifold of a 3-Sasakian manifold $(\tilde{M},\xi_i,\eta_i,\varphi_i,g), i\in\{1,2,3\}$. Then for all $X,Y\in\mathfrak{D}^\alpha$*

$$g(F_iX,F_iY) = \sin^2\alpha g(X,Y), \quad g(T_iX,T_iY) = \cos^2\alpha g(X,Y). \tag{22}$$

Theorem 4.2. *There is no proper 3-hemi slant submanifold of a 3-Sasakian space form $(\tilde{M}(c),\xi_i,\eta_i,\varphi_i,g)$, $i\in\{1,2,3\}$, such that the second fundamental form be parallel and $c\neq 1$.*

Proof. For all $X,Y\in\mathfrak{D}^\alpha$ and $Z\in\mathfrak{D}^\perp$, the equation of Codazzi implies

$$g(\tilde{R}^\perp(X,Y)Z,\varphi_i Z) = g((\tilde{\nabla}_X\sigma)(Y,Z) - (\tilde{\nabla}_Y\sigma)(X,Z),\varphi_i Z).$$

By the hypothesis of the theorem that σ is parallel, we find

$$g(\tilde{R}^\perp(X,Y)Z,\varphi_i Z) = 0. \tag{23}$$

On the other hand, from (21) we conclude

$$g(\tilde{R}^\perp(X,Y)Z,\varphi_i Z) = \left(\frac{c-1}{4}\right)\sum_{i=1}^{3}2g(X,\varphi_i Y)g(\varphi_i Z,\varphi_i Z). \tag{24}$$

Thus (23) and (24) imply

$$(c-1)\sum_{i=1}^{3}g(X,\varphi_i Y)g(\varphi_i Z,\varphi_i Z) = 0. \tag{25}$$

Since $c\neq 1$ then by taking $Y = T_jX$ and using (22), we obtain

$$\sum_{i,j=1}^{3}g(X,\varphi_i T_jX)g(Z,Z) = \cos^2\alpha g(X,X)g(Z,Z) = 0.$$

Since g is a Riemannian metric, from the last relation we conclude that $\alpha = \frac{\pi}{2}$, that is, M is an anti-invariant submanifold. □

Theorem 4.3. *Suppose M is a 3-hemi slant submanifold of a 3-Sasakian space form $(\tilde{M}(c), \xi_i, \eta_i, \varphi_i, g), i \in \{1,2,3\}$. If $R^\perp = 0$ and $c \neq 1$ and for any $V \in T^\perp M$, $A_V A_{f_i V} = A_{f_i V} A_V$, then the slant angle α is either equal to $\frac{\pi}{2}$ or 0.*

Proof. For any $X, Y \in TM$ and $U, V \in T^\perp M$, from the equation of Ricci, we have

$$g(\tilde{R}(X,Y)V,U) = g(R^\perp(X,Y)V,U) - g([A_V, A_U]X, Y). \tag{26}$$

By considering $U = f_i V$ and $X = T_i Y$ in (26) and using the hypothesis of the theorem that is $R^\perp = 0$ and $A_V A_{f_i V} = A_{f_i V} A_V$, we obtain

$$g(\tilde{R}(T_i Y, Y)V, f_i V) = 0. \tag{27}$$

Also, from (21), we derive

$$\begin{aligned} g(\tilde{R}(X,Y)V,U) = \frac{(c-1)}{4} \sum_{i=1}^{3} \{ & g(\varphi_i Y, V) g(\varphi_i X, U) - \\ & g(\varphi_i X, V) g(\varphi_i Y, U) + 2g(X, \varphi_i Y) g(\varphi_i V, U)\}. \end{aligned} \tag{28}$$

Interchanging $U = f_i V$ and $X = T_i Y$ in (28) and using Lemma 4.1, we deduce that

$$g(\tilde{R}(T_i Y, Y)V, f_i V) = \cos^2\alpha \sin^2\alpha \left(\frac{c-1}{2}\right) g(Y,Y) g(V,V). \tag{29}$$

From (27) and (29), we find

$$\cos^2\alpha \sin^2\alpha \left(\frac{c-1}{2}\right) g(Y,Y) g(V,V) = 0. \tag{30}$$

Since g is a Riemannian metric and $c \neq 1$, then from (30), we conclude that either $\alpha = \frac{\pi}{2}$ or $\alpha = 0$. □

Theorem 4.4. *Let $(\tilde{M}(c), \xi_i, \eta_i, \varphi_i, g), i \in \{1,2,3\}$ be a 3-Sasakian space form and M be a 3-hemi slant submanifold of $\tilde{M}(c)$ with slant angle α*

and dimension of n*. Then for any* $X, W \in TM$*, the Ricci tensor* S *of submanifold* M *satisfies in the following relation*

$$S(X,W) = \left\{ \left(\frac{c+3}{4}\right)(4p+q) + 6\left(\frac{c-1}{4}\right)(3\cos^2\alpha + 1) \right\} g(X,W)$$
$$+ \left\{ -\left(\frac{c+3}{4}\right) + \left(\frac{c-1}{4}\right)(6\cos^2\alpha - 4p - q + 2) \right\} \sum_{i=1}^{3} \eta_i(X)\eta_i(W) +$$
$$ng(\sigma(X,W),H) - \sum_{j=1}^{n} g(\sigma(X,e_j),\sigma(e_j,W)),$$

where $n = 4p+q+3$ *such that* $p = \frac{1}{4}\dim(\mathfrak{D}^\alpha)$ *and* $q = \dim(\mathfrak{D}^\perp)$ *and* H *is the mean curvature.*

Proof. For any $X, Y, Z, W \in TM$, from Equations (10) and (21), we get

$$g(R(X,Y)Z,W) = \left(\frac{c+3}{4}\right)\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\}$$
$$+ \left(\frac{c-1}{4}\right)\sum_{i=1}^{3}\{\eta_i(Z)\eta_i(X)g(Y,W) - \eta_i(Z)\eta_i(Y)g(X,W) +$$
$$g(X,Z)\eta_i(Y)\eta_i(W) - g(Y,Z)\eta_i(X)\eta_i(W) + g(\varphi_i Y,Z)g(\varphi_i X,W) -$$
$$g(\varphi_i X,Z)g(\varphi_i Y,W) + 2g(\varphi_i Y,X)g(\varphi_i Z,W)\} - g(\sigma(X,Z),\sigma(Y,W))$$
$$+ g(\sigma(X,W),\sigma(Y,Z)). \tag{31}$$

Let $TM = \mathfrak{D}^\alpha \oplus \mathfrak{D}^\perp \oplus < \xi_i >$ and $\{e_1, \ldots, e_p, e_{p+1} = \sec\alpha T_1 e_1, \ldots, e_{4p} = \sec\alpha T_3 e_p\}$, $\{e_{4p+1}, e_{4p+3}, ...e_{4p+q}\}$, $\{e_{4p+q+i} = \xi_i\}$ be local orthonormal frames of $\mathfrak{D}^\alpha$, $\mathfrak{D}^\perp$ and $< \xi_i >$, respectively. By using these adapted frames for contracting $g(R(X,Y)Z,W)$ on Y, Z, we should compute the following components.

$$S(X,W) = \sum_{i=1}^{p} g(R(X,e_i)e_i,W) + \sum_{i=1}^{3}\sum_{j=p+1}^{4p} g(R(X,\sec\alpha T_i e_j)\sec\alpha T_i e_j,$$
$$W) + \sum_{k=4p+1}^{4p+q} g(R(X,e_k)e_k,W) + \sum_{i=1}^{3} g(R(X,\xi_i)\xi_i,W). \tag{32}$$

Now, we use (31) to find the terms of the right side of Equation (32). For the first term, since $\sum_{i=1}^{p} g(e_i, e_i) = p$, $\eta_i(e_j) = 0, g(e_i, \varphi_j e_i) = 0$ and $\sum_{i=1}^{p} g(X, e_i) g(e_i, W) = g(X, Y)$, we have

$$\sum_{i=1}^{p} g(R(X, e_i)e_i, W) = \left(\frac{c+3}{4}\right) \sum_{i=1}^{p} \{g(e_i, e_i) g(X, W) -$$
$$g(X, e_i) g(e_i, W)\} + \left(\frac{c-1}{4}\right) \sum_{j=1}^{3} \sum_{i=1}^{p} \{\eta_j(X)\eta_j(e_i) g(e_i, W) -$$
$$\eta_j(e_i)\eta_j(e_i) g(X, W) + \eta_j(e_i)\eta_j(W) g(X, e_i) - \eta_j(X)\eta_j(W) g(e_i, e_i)$$
$$+ g(X, \varphi_j e_i) g(\varphi_j e_i, W) - g(e_i, \varphi_j e_i) g(\varphi_j X, W) + 2g(X, \varphi_j e_i)$$
$$g(\varphi_j e_i, W)\} - \sum_{i=1}^{p} \{g(\sigma(e_i, W), \sigma(X, e_i)) + g(\sigma(X, W), \sigma(e_i, e_i))\}$$
$$= \left(\frac{c+3}{4}\right)(p-1) g(X, W) + \left(\frac{c-1}{4}\right) \{\sum_{j=1}^{3} (-p\eta_j(X)\eta_j(W) +$$
$$3g(T_j X, T_j W))\} - \sum_{i=1}^{p} \{g(\sigma(e_i, W), \sigma(X, e_i)) - g(\sigma(X, W), \sigma(e_i, e_i))\}. \tag{33}$$

Also, from (22) we have $g(\sec\alpha T_i e_j, \sec\alpha T_i e_j) = 1$, so the second term of (32) implies

$$\sum_{i=1}^{3} \sum_{j=p+1}^{4p} g(R(X, \sec\alpha T_i e_j) \sec\alpha T_i e_j, W) = \left(\frac{c+3}{4}\right) \{3p\, g(X, W) -$$
$$g(X, W)\} + \left(\frac{c-1}{4}\right) \sum_{i=1}^{3} \{-3p\, \eta_i(X)\eta_i(W) + 3g(T_i X, T_i W)\} -$$
$$\sum_{j=p+1}^{4p} \{g(\sigma(\sec\alpha T_i e_j, W), \sigma(X, \sec\alpha T_i e_j)) - g(\sigma(X, W),$$
$$\sigma(\sec\alpha T_i e_j, \sec\alpha T_i e_j))\}. \tag{34}$$

Since $\mathfrak{D}^\perp$ is anti-invariant, we get $g(X, \varphi_i e_k) = g(X, \varphi_i Y) = 0$. So, for the third term we obtain

$$\sum_{k=4p+1}^{4p+q} g(R(X,e_k)e_k,W) = \left(\frac{c+3}{4}\right)(q-1)g(X,W)+$$
$$\left(\frac{c-1}{4}\right)\sum_{i=1}^{3}\{-q\eta_i(X)\eta_i(W)+3g(\varphi_i X,\varphi_i W)\}-$$
$$\sum_{k=4p+1}^{4p+q}\{g(\sigma(e_k,W),\sigma(X,e_k))-g(\sigma(X,W),\sigma(e_k,e_k)\}. \tag{35}$$

From (7) and (8), we conclude $0=\tilde{\nabla}_{\xi_i}\xi_i=\nabla_{\xi_i}\xi_i+\sigma(\xi_i,\xi_i)$, thus $\sigma(\xi_i,\xi_i)=0$. Therefore, the last term of (32) gives

$$\sum_{i=1}^{3} g(R(X,\xi_i)\xi_i,W) = 3\left(\frac{c+3}{4}\right)g(X,W)-$$
$$\left(\frac{c+3}{4}\right)\sum_{i=1}^{3}\eta_i(X)\eta_i(W)-3\left(\frac{c-1}{4}\right)g(X,W)-$$
$$\left(\frac{c-1}{4}\right)\sum_{i=1}^{3}\eta_i(X)\eta_i(W)-\sum_{i=1}^{3}g(\sigma(X,e_i),\sigma(e_i,W)). \tag{36}$$

Finally, by taking in to account of (32), the definition of mean curvature $H=\frac{1}{n}\sum_{i=1}^{n}\sigma(e_i,e_i)$ and sum of Equations (33), (34), (35) and (36) imply

$$S(X,W) = (\frac{c+3}{4})(4p+q)g(X,W)+6\left(\frac{c-1}{4}\right)(3\cos^2\alpha+1)g(X,W)$$
$$-\left(\frac{c+3}{4}\right)\sum_{i=1}^{3}\eta_i(X)\eta_i(W)+\left(\frac{c-1}{4}\right)(6\cos^2\alpha-4p-q+2)$$
$$\sum_{i=1}^{3}\eta_i(X)\eta_i(W)+ng(\sigma(X,W),H)-\sum_{j=1}^{n}g(\sigma(X,e_j),\sigma(e_j,W)), \tag{37}$$

which is the required result. □

Corollary 4.5. *Let M be a totally geodesic n-dimensional 3-hemi slant submanifold of a 3-Sasakian space form $\tilde{M}(c)$. Then M is a quasi Einstein manifold.*

Proof. Since $\sigma = 0$, if we consider

$$\kappa = \left(\frac{c+3}{4}\right)(4p+q) + 6\left(\frac{c-1}{4}\right)(3\cos^2\alpha + 1),$$

$$\mu = \left(\frac{c-1}{4}\right)(6\cos^2\alpha - 4p - q + 2) - \left(\frac{c+3}{4}\right);$$

then from (37), we derive

$$S(X,W) = \kappa g(X,W) + \mu \sum_{i=1}^{3} \eta_i(X)\eta_i(W).$$

Hence, M is a quasi-Einstein manifold. □

Theorem 4.6. *Let $(\tilde{M}(c), \xi_i, \eta_i, \varphi_i, g), i \in \{1,2,3\}$ be a 3-Sasakian space form and M be an n-dimensional 3-hemi slant submanifold of $\tilde{M}(c)$ with scalar curvature r and slant angle α, then*

$$r \leq n\left\{\left(\frac{c-3}{4}\right)(4p+q) + 6\left(\frac{c-1}{4}\right)(3\cos^2\alpha + 1)\right\} + 3\left(\frac{c-1}{4}\right)(6\cos^2\alpha - 4p - q + 2) - 3\left(\frac{c+3}{4}\right) + n^2\|H\|^2. \quad (38)$$

Proof. By contracting (37) on X, W, we get

$$r = n\left\{\left(\frac{c-3}{4}\right)(4p+q) + 6\left(\frac{c-1}{4}\right)(3\cos^2\alpha + 1)\right\} + 3\left(\frac{c-1}{4}\right)(6\cos^2\alpha - 4p - q + 2) - 3\left(\frac{c+3}{4}\right) + n^2\|H\|^2 - \|\sigma\|^2.$$

Since $\|\sigma\|^2 \geq 0$, the inequality is satisfied. □

Corollary 4.7. *Let* $(\tilde{M}(c), \xi_i, \eta_i, \varphi_i, g), i \in \{1,2,3\}$ *be a* 3*-Sasakian space form and* M *be a minimal* 3*-semi invariant submanifold of* $\tilde{M}(c)$ *with scalar curvature* r*, then*

$$r \leq n\{(\frac{c+3}{4})(2p+q-\frac{3}{n}) + 24\left(\frac{c-1}{4}\right)\} + 3\left(\frac{c-1}{4}\right)(8-2p-q).$$

where $n = 2p+q+3$ *is the dimension of* M*.*

Proof. Put the dimension of invariant distribution $\dim \mathfrak{D} = 2p$, $\dim\langle\xi_i\rangle$ $=3$ and the dimension of anti-invariant distribution $\mathfrak{D}^{\perp} = q$. Then, the result is easily obtained from (38) by considering $H = 0$ and $\alpha = 0$. □

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