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Original Research Paper

FBSM Solution of Optimal Control Problems Using Hybrid Runge-Kutta Based Methods

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Abstract. Solving optimal control problems (OCP) with analytical methods has usually been difficult or not cost-effective. Therefore, solving these problems requires numerical methods. There are, of course, many ways to solve these problems. One of the methods available to solve OCP is a forward-backward sweep method (FBSM). In this method, the state variable is solved in a forward and co-state variable by a backward method where an explicit Runge–Kutta method (ERK) is often used to solve differential equations arising from OCP. In this paper, instead of the ERK method, three hybrid methods based on ERK method of order 3 and 4 are proposed for the numerical approximation of the OCP. Truncation errors and stability analysis of the presented methods are illustrated. Finally, numerical results of the four optimal control problems obtained by new methods, which shows that new methods give us more detailed results, are compared with those of ERK approaches of orders 3 and 4 for solving OCP.

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1 Introduction

Optimal control (OC) is an effective tool for using physical, economic, enginering, biological and other science models. Richard Bellman and Pontryagin in the 1950s were the first for solving OC problems with numerical techniques. Numerical methods can be applied to solve OC problems to three categories: 1-direct methods 2-indirect methods 3-dynamic programming. In recent years, numerical solution of fractional OC problems is also one of the topics of OC that many articles have been published in the context [24, 25, 26, 36]. Indirect approaches according to the pontryagin's maximum principles (PMP). In these methods, the OC problem becomes a two-point boundary value problem(TPBVP). Forward-backward sweep method(FBSM) is one of the indirect methods for solving OCP. In the second decade of 21th century, many people have used the pontryagin's maximum principles(PMP) in their articles [11, 15, 21, 27, 37]. In 2007 S. Lenhart and J.T. Workman first introduced the FBSM method in their book "optimal control applied to biological models". They used FBSM methodology for a variety of optimal control problems [19]. M.Siliva and colleagues in 2009 have been suggested and evaluated some methods for the segmentation of skin lesions for analyzing dermoscopic images. They conclude the FBSM methodology was the best fully automatic approach, with resultss just a bit worse than adaptive snake methodology (AS) and expectation-maximization level set methodology (EM-LS) [34]. Convergence of the FBSM methodology was proven by M. Mcasey and his colleaguess in 2012 [22]. D.P. Moule and colleagues in 2015 utilized the FBSM methodology to solve an OCP about a tuberculosis pattern with undetected instances in the country of Cameroon. They were able to reduce 80% this type of tuberculosis in the past ten years by combining education and chemotherapy [23]. G.R. Rose in 2015 in his thesis showed that the FBSM method is more accurate than direct shooting method and matlab optimization [31]. M. Sana and colleagues in 2015 applied FBSM method with trapezoidal and Euler methods and compared with Range-Kutta(RK) method of rank 4 and concluded that RK and trapezoidal methods for OCPs have the same functional performance, but better than the Euler method and with increasing steplength, the Euler method differnces with the rest of the methods [21, 33]. M.Lhous and colleagues in 2017 provided discrete mathematical pattern and OC of the marital condition. They used the pontryagin's maximum principles (PMP) and FBSM method. They made aware of the people of the benefits of marriage and disadvantages of divorce and they could reduce the amount of divorce in community [20]. N.D. Bianca and colleagues in 2018 have compared direct and indirect methodologies about minimum time OCPs. They selected the PINS numerical solver from the direct method and the next-generation of general purpose optimal control software (GPOPS-2) numerical solver from the indirect method, and compared them. They concluded that the calculation time in PINS is shorter and the calculation accuracy is higher. While GPOPS-2 is stronger and much more accurate [3]. M.Q. Duran and colleagues in 2019 provided an improved FBSM-based method for reconfiguration of unbalanced distribution networks [6]. Y. Kongjeen and colleagues in 2019 have provided an improved FBSM to analyze microgrid-load-flow, as mathematical models are different electric vehicle load [17]. NH. Sweilam and colleagues in 2019 introduced, different numerical techniques for solving OCP and tried to achieve the best accuracy for the OCP. The goal is to reduce the size of tumor-cells until the end of the treatment procedure. They claim that direct method can be used to get better results than indirect method and it is easy to implement [35]. S. Ouali and A. Cherkaoui in 2020 presented a modified FBSM power flow methodology according to a novel network information organization in the systems of radial distribution [28]. Chronic heart disease (CHD) is one of the greatest defies currently. I. Ameen and colleagues in 2020 proposed a mathematical model to study the connection between fish consumption and (CHD). They use the improved FBSM based on the predictor-corrector method. They concluded consumption of fish reduces the risk of hear disease (CHD) and reduces its mortality [1]. A. Kouider and colleagues in 2020 proposed a mathematical modeling for describing the dynamics of transmission of the novel corona virus (COVID-19), between potential people and infected people with-out symptoms. They proposed an optimal technique by carrying out a wareness campaigns about people with pratical measures for reduction of spread of the COVID-19-virus, and discernment and surveillance of airports space and the quarantine of infected cases. They used the pontryagin's maximum principles(PMP) to characterize and solve the OCs [18]. In 2020, A. Bhih and colleagues introduced a new model for spreading rumors in social networks And introduced three types of controls to minimize the number of fake pages, the number of publisher users and related costs; theoretically. They used the FBSM method to solve their own OCP [2].

The goal of this work is to illustrate details of a new single step explicit Rung-Kutta (ERK) type method according to off-step-points for the numerical solution of OCPs. For mildly stiff and stiff problems of ODEs which may be appeared in OCP, we need to use numerical methods with wide stability regions and domains as well as good accuracy. In this research, we illustrate three implicit hybrid methods of orders 3 and 4 and then convert them into explicit methods using explicit Runge-Kutta methodologies of third and fourth orders as a predictor of the scheme. The stability and order of truncation error of the methods discussed indicating that novel strategies have wide stability regains by which more accurate results can be obtained compared to the FBSM based explicit Runge-Kutta methodologies of third and fourth orders.

This work is arranged as follows: Some basics about optimal control prblem, and Pontryagin Maximum Principle are presented in Section 2 and in Section 3 the FBSM method is introduced. In section 4 hybrid methods of orders 3 and 4 is described and their orders of truncation errors discussed. In Section 5, stability of the presented methods is analyzed. In section 6 convergence of FBSM and new methods are proven. Numerical results for solving some OCPs presented in Section 7. Finally, the conclusion is presented in Section 8.

2 Optimal control (OC) problems

In an OC problem, there is an objective function J in terms of the state variables $x \in X$ and the control variables $u \in U$. Solve an OCP, that is, obtain a piecewise continuous function $u(t)$ with $t_0 \leq t \leq t_f$ and the state variable $x(t)$, Which minimize the objective function $J(x, u)$. For more explanation we need a few definitions.

2.1 Basic optimal control problems

Definition 2.1 (Bolza problem)**.** Let us study the optimization of functional

$$J(x,u) = \int_{t_0}^{t_e} f(t, x(t), u(t))dt \tag{1}$$

By adding a payoff term to problem (1), the Bolza problem is obtained:

$$J(x,u) = h(t_1, x(t_1)) + \int_{t_0}^{t_e} f(t, x(t), u(t))dt$$

Definition 2.2 (OC problem, in Lagrangian form, in general)**.** The OC problem define as:

$$\max_u J(x,u) = \int_{t_0}^{t_e} f(t, x(t), u(t))dt \tag{2}$$

$$x'(t) = g(t, x(t), u(t))$$

$$x(t_0) = x_0$$

Note that $\min\{j\} = -\max\{-j\}$.

2.2 Pontryagin Maximum Principle

The Pontryagin's maximum principle is a good tool for solving OC problems, especially when the state variable is constrained. Indirect numerical approaches to solve OCP, are based on this principle.

Definition 2.3 (Hamiltonian)**.** Let have optimal control problem (2), then the

$$H(t, x, u, \lambda) = f(t, x, u) + \lambda g(t, x, u) \tag{3}$$

is supposed to be the hamiltonian function, and the λ is supposed adjoint variable.

Theorem 2.4 (Pontryagin's Maximum principle in the OC problem (2))**.** *Let u^* and x^* be the optimal pair for (OCP) (2), thus one can find a piecewise differentiable adjoint variable λ as*

$$H(t, x^*(t), u(t), \lambda(t)) \leq H(t, x^*(t), u(t), \lambda(t))$$

in any controlers $u(t)$ at every time t, in which H shows Hamiltonian (3), also

$$\lambda'(t) = -\frac{\partial H(t, x^*(t), u^*(t), \lambda(t))}{\partial x}$$
$$\lambda(t_f) = 0.$$

Proof. We refer the reader to [19]. □

The condition of $\lambda(t_f) = 0$ called the transversality condition,and The condition $\frac{\partial H}{\partial u} = 0$ at u^* for each t is called the optimality condition.

Theorem 2.5. *When all the controls of problem (2) be lebesgue integrable functions and $t_0 \leq t \leq t_e$ in R . Consider that $f(t, x, u)$ is concave in u and there are variables $d_1, d_2, d_3 > 0$, d_4 and $b > 1$ as*

$$\begin{aligned} g(t, x, u) &= a(t, x) + b(t, x)u, \\ |g(t, x, u)| &d_1(1 + |x| + |u|), \\ |g(t, x_0, u) - g(t, x, u)| &\leq d_2|x_1 - x_2|(1 + |u|), \\ f(t, x, u) &\leq d_3|u|^b - d_4, \end{aligned}$$

for any t with $t_0 \leq t \leq t_e$, x_1, x_2, u in R. Thus the optimal pair (x^*, u^*) can be found that, maximizing J , with $J(x^*, u^*)$ finite.

Proof. We refer the reader to [19]. □

3 Forward-backward sweep method (FBSM)

Consider the OC problem (2). To solve such problems numerically, an algorithm is developed that produces an approximation for the piecewise continiuous control u^*. In this algorithm, the time interval $[t_0, t_1]$ is broken into pieces with specific points of interest $t_0 = b_1, b_2, ..., b_N, b_{N+1} = t_1$; these points will usually be equally spaced. The approximation will be a vector $\overrightarrow{u} = (u_1, u_2, ..., u_{N+1})$, where $u_i \approx u(b_i)$. any solution to the above OC problem should satisfies:

$$x'(t) = g(t, x(t), u(t)), \quad x(t_0) = x_0,$$
$$\lambda' = -\frac{\partial H}{\partial x}, \quad \lambda(t_1) = 0,$$
$$\frac{\partial H}{\partial u} = 0 \quad at \quad u^*.$$

The third equation, the optimality conditions can usally be retouched to find a showing of u^* in terms of $t, x, and \quad \lambda$. Then the first two equations form a problem of two point boundary values. The generalized problem can be solved by using indirect methods which are numerical techniques to solve them. The forward-backward sweep methodology (FBSM) is one of these methods. In [22], convergence analysis of the FBSM has been done. In fact, by using FBSM, the differential equations arising from the maximum principle are numerically solved. Euler, Trapezoidal and Runge-Kutta methods can be used for the numerical solution of OCP by using FBSM where we are faced with initial value problems(IVPS) arising from the state and adjoint equations. A straightforward algorithm for this method is as follows. Let $\overrightarrow{x} = (x_1, x_2, ..., x_{N+1})$ and $\overrightarrow{\lambda} = (\lambda_1, \lambda_2, ..., \lambda_{N+1})$ are the vector approximations for the state and adjoint.

1. Guess an initial value for $\overrightarrow{u}$ in the given time interval.
2. Obtain the $\overrightarrow{x}$ by solving forward in time, the differential equation system, using the initial condition $x(t_0) = x_1 = a$ and the value $\overrightarrow{u}$.
3. Obtain the $\overrightarrow{\lambda}$, by solving backward in time, of the differential equation system, using the transfer condition $\lambda_{N+1} = \lambda(t_e) = 0$.
4. Obtain the updated value $\overrightarrow{u}$ by substituting the new values $\overrightarrow{x}$ and $\overrightarrow{\lambda}$.
5. Test convergence. In the case that, the difference between the values of the variables of latest iteration, and the previous iteration, was too small, select these values as the answer. If there is a bigger difference, go back to step 2.

4 Hybrid methods and order of truncation errors

To obtains the numerical answer of initial-valued-problems (IVPs), in the following form

$$x' = f(t,x), \quad x \in \mathbb{R}^n, \quad x(t_0) = x_0, \quad t_0 \leq t \leq t_1, \tag{4}$$

where $f : [t_0, t_1] \times \mathbb{R}^n \to \mathbb{R}^n$, one can use an explicit or implicit method. Methods based of off-step points, such as backward differential forward (BDF), hybrid backward differential forward (HBDF), new class of HBDFs and class$-2+1$ hybrid BDF-like schemes have wide stability regions and higher order compared to some Runge-Kutta method and implicit BDF methods [7, 8, 9, 10, 16]. Let us consider the IVP of the form (4). Linear k-step methods of the

$$x_{n+1} = \alpha_1 x_n + \alpha_2 x_{n-1} + ... + \alpha_k x_{n-k+1} + h\{\beta_0 f_{n+1} + \beta_1 f_n + ... + \beta_k f_{n-k+1}\} \tag{5}$$

has $2k+1$ arbitrary parameter and we can write it as

$$\rho(E)x_{n-k+1} - h\sigma(E)f_{n-k+1} = 0$$

in which E is the shift operator as $E(x(t)) = x(t+h)$, with the step length h and ρ and σ are first and second characteristic polynomials defined by

$$\rho(\xi) = \xi^k - \alpha_1\xi^{k-1} - \alpha_2\xi^{k-2} - ... - \alpha_k,$$
$$\sigma(\xi) = \beta_0\xi^k + \beta_1\xi^{k-1} + ... + \beta_k.$$

To increase the order of k-step methods of the form (5), at several points between t_n and t_{n+1}, we use a linear combination of the slopes, where $t_{n+1} = t_n + h$ and h is the step length on $[t_0, t_1]$. Then, the modified form of (5) with m slops is given by

$$x_{n+1} = \sum_{j=1}^{k} \alpha_j x_{n-j+1} + h\sum_{j=0}^{k} \beta_j f_{n-j+1} + h\sum_{j=1}^{m} \gamma_j f_{n-\theta_j+1} \tag{6}$$

where α_j, β_j, γ_j and θ_j are $2k+2m+1$ arbitrary parameters [16]. Methods of the form (6) with m off-step points are called hybrid methods where $0 < \theta_j < 1, \quad j = 1, 2, ..., m$. In this work, we set $\beta_0 = 0, \quad k = 1$ and $m = 1$. Hence, we write (6) as

$$x_{n+1} = \alpha_1 x_n + h\{\beta_0 f_{n+1} + \beta_1 f_n\} + h\gamma_1 f_{n-\theta_1+1}$$

where $\alpha_1, \beta_0, \beta_1, \gamma_1$ and θ_1 are arbitrary parameters and $\theta_1 \neq 0$ or 1. Expanding terms y_{n+1}, $f_{n+1}, f_{n-\theta_1+1}$ in Taylor's series about t_n, we can obtain a family of third order methods if the equations

$$\begin{aligned}
&\alpha_1 = 1, \\
&\beta_1 + \beta_0 + \gamma_1 = 1, \\
&\beta_0 + (1 - \theta_1)\gamma_1 = \tfrac{1}{2}, \\
&\tfrac{1}{2}\beta_0 + \tfrac{1}{2}(1 - \theta_1)^2\gamma_1 = \tfrac{1}{6}.
\end{aligned}$$

are satisfied where the principal provision of the truncation error will be

$$\frac{1}{4!} c_4 h^4 x^{(4)}(t_n) + o(h^5), \qquad c_4 \ = 1 - 4\beta_2 - 4\gamma_1(1 - \theta_1)^3.$$

For more details, one can see the [16].
Consider the following three cases:

1. $\beta_1 = 0, \quad \alpha_1 = 1, \quad \beta_0 = \frac{1}{4}, \quad \gamma_1 = \frac{3}{4}, \quad \theta_1 = \frac{2}{3}, \quad c_4 = -\frac{1}{9},$
2. $\beta_1 = \frac{1}{4}, \quad \alpha_1 = 1, \quad \beta_0 = 0, \quad \gamma_1 = \frac{3}{4}, \quad \theta_1 = \frac{1}{3}, \quad c_4 = \frac{1}{9},$
3. $\beta_1 = \frac{1}{6}, \quad \alpha_1 = 1, \quad \beta_0 = \frac{1}{6}, \quad \gamma_1 = \frac{2}{3}, \quad \theta_1 = \frac{1}{2}, \quad c_4 = 0.$

Gives us the following methodologies of third and fourth orders, respectively (Method1, Method2 and Method3 in this work):

$$x_{n+1} = x_n + \frac{h}{4}\{f_{n+1} + 3f_{n+\frac{1}{3}}\}, \tag{7}$$

$$x_{n+1} = x_n + \frac{h}{4}\{f_n + 3f_{n+\frac{2}{3}}\}, \tag{8}$$

$$x_{n+1} = x_n + \frac{h}{4}\{f_{n+1} + 4f_{n+\frac{1}{2}} + f_n\}, \tag{9}$$

where $f_{n+1} = f(t_n, x_{n+1}), \quad f_{n+m} = f(t_n+mh, x_{n+m})$ and $f_n = f(t_n, x_n)$ for $m = \frac{1}{3}, \frac{2}{3}$ and $\frac{1}{2}$. Note that, x_{n+1}, x_{n+m} and x_n are numerical approximations according to the exact values of the solution $x(t)$ at

$t_{n+1} = t_n + h, \quad t_{n+m} = t_n + mh,$ for $m = \frac{1}{3}, \frac{2}{3}$ and $\frac{1}{2}$ respectively. In order to convert methods (7)–(9) into explicit methods at each step, we predict the values of x_{n+1} and x_{n+m} used on the right hand side of the new methods using fourth or third order explicit Runge-Kutta method as follows, respectively:

$$\begin{aligned}
&x_{n+1} = x_n + \tfrac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4),\\
&k_1 = f(t_n, x_n),\\
&k_2 = f(t_n + \tfrac{1}{2}h, x_n + \tfrac{1}{2}k_1 h),\\
&k_3 = f(t_n + \tfrac{1}{2}h, x_n + \tfrac{1}{2}k_2 h),\\
&k_4 = f(t_n + h, x_n + k_3 h).
\end{aligned}$$

or

$$\begin{aligned}
&x_{n+1} = x_n + \tfrac{h}{6}(k_1 + 4k_2 + k_3),\\
&k_1 = f(t_n, x_n),\\
&k_2 = f(t_n + \tfrac{1}{2}h, x_n + \tfrac{1}{2}k_1 h),\\
&k_3 = f(t_n + h, x_n - k_1 h + 2k_2 h).
\end{aligned} \tag{10}$$

In general, we rewrite methods (7)–(9) using RK4 method as a predictor as follows:

$$\bar{x}_{n+1} = x_n + \tfrac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4), \tag{11}$$

$$\bar{x}_{n+m} = x_n + \tfrac{mh}{6}(k_1 + 2k_{2m} + 2k_{3m} + k_{4m}), m = \tfrac{1}{3}, \tfrac{2}{3} \text{ or } \tfrac{1}{2}, \tag{12}$$

$$x_{n+1} = x_n + h\{\beta_0 \bar{f}_{n+1} + \gamma_1 \bar{f}_{n+m} + \beta_1 f_n\}, \tag{13}$$

where

$$\begin{aligned}
&k_1 = f(t_n, x_n),\\
&k_{2m} = f(t_n + mh, x_n + mk_1 h),\\
&k_{3m} = f(t_n + mh, x_n + mk_2 h),\\
&k_{4m} = f(t_n + mh, x_n + mk_3 h),
\end{aligned}$$

and $f_{n+1} = f(t_n, x_{n+1}), \quad f_{n+m} = f(t_n + mh, x_{n+m}), \quad f_n = f(t_n, x_n)$. Now, suppose that the order of stage equation (10) is $p_1, \quad p_1 = 4$, as like as (11). Thus, the difference of exact and numerical answer at $t = t_{n+m} = t_n + mh,\ m = \frac{1}{3}, \frac{2}{3}, \frac{1}{2}$ and 1 is

$$y(t_{n+m}) - y_{n+m} = C_m h^{p_1} y^{(p_1)}(t_n) + O(h^{p_1+1}) \tag{14}$$

where C_m is the error constant of the methodology (11) or (13) with corresponding m which can take only one of the values $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{2}$, together with the value 1 related to methods (10) and (11) respectively. The difference operator associated to methodology (12), of the *pth* order ($p = 3$ or 4), can be written as

$$y(t_{n+1}) - y_{n+1} = Ch^p y^{(p)}(t_n) + O(h^{p+1}) \tag{15}$$

which C is the error constant of the methodology (13). Therefore, we obtain a theorem as:

Theorem 4.1. *Suppose that*

1. *the Equation (11) be of order p_1,*
2. *the Equation(12) be of order p_1 too,*
3. *the Equation (13) be of order p,*

thus, the order of (11) –(12) is p.

Proof. Suppose that m can only take one of the values $\frac{1}{3}, \frac{2}{3}$ or $\frac{1}{2}$ and y_n is exact. From (15) and (13) one can write

$$\begin{aligned} y(t_{n+1}) - y_{n+1} &= h\beta_m[f(t_{n+m}, y(t_{n+m})) - f(t_{n+m}, \overline{y}_{n+m})] \\ &\quad + h\beta_1[f(t_{n+1}, y(t_{n+1})) - f(t_{n+1}, \overline{y}_{n+1})] + Ch^p y^{(p)}(t_n) + O(h^{p+1}). \end{aligned}$$

Considering properties of the IVPs of the form (4), for some values such as η_m and η_1 belong to intervals $(\overline{y}_{n+m}, y(t_{n+m}))$ and $(\overline{y}_{n+1}, y(t_{n+1}))$ respectively, we can write

$$\begin{aligned} f(t_{n+m}, y(t_{n+m})) - f(t_{n+m}, \overline{y}_{n+m}) &= \tfrac{\partial f}{\partial y}(t_{n+m}, \eta_{n+m})(y(t_{n+m}) - \overline{y}_{n+m}), \\ f(t_{n+1}, y(t_{n+1})) - f(t_{n+1}, \overline{y}_{n+1}) &= \tfrac{\partial f}{\partial y}(t_{n+1}, \eta_{n+1})(y(t_{n+1}) - \overline{y}_{n+1}). \end{aligned}$$

Therefore, by using (14), we have

$$\begin{aligned} y(t_{n+1}) - y_{n+1} &= h\beta_m \left[\tfrac{\partial f}{\partial y}(t_{n+m}, \eta_{n+m})(y(t_{n+m}) - \overline{y}_{n+m})\right] \\ &\quad + h\beta_1 \left[\tfrac{\partial f}{\partial y}(t_{n+1}, \eta_{n+1})(y(t_{n+1}) - \overline{y}_{n+1})\right] + Ch^p y^{(p)}(t_n) + O(h^{p+1}). \end{aligned}$$

Applying equation (13) to this gives us

$$y(t_{n+1}) - y_{n+1} = h\beta_m \left[\frac{\partial f}{\partial y}(t_{n+m}, \eta_{n+m}) C_m h^{p_1} y^{(p_1)}(t_n) + O(h^{p_1+1})\right]$$
$$+ h\beta_1 \left[\frac{\partial f}{\partial y}(t_{n+1}, \eta_{n+1}) C_1 h^{p_1} y^{(p_1)}(t_n) + O(h^{p_1+1})\right] + Ch^p y^{(p)}(t_n) + O(h^{p+1})$$
$$= h^p \left\{\beta_m \left[\frac{\partial f}{\partial y}(t_{n+m}, \eta_{n+m}) C_m h^{p_1-p+1} y^{(p_1)}(t_n)\right]\right\}$$
$$+ h^p \beta_1 \left\{\left[\frac{\partial f}{\partial y}(t_{n+1}, \eta_{n+1}) C_1 h^{p_1-p+1} y^{(p_1)}(t_n)\right] + Cy^{(p)}(t_n)\right\} + O(h^{p+1})$$

where $p_1 \geq p$. Thus, it can be concluded that the methodology (11)–(13) is of order p and so the proof is completed. □

By following the same way as presented above, it can be proved that the methods (7) –(9) using RK3 method as a predictor (Runge-kutta of order 3) of the form

$$\bar{x}_{n+1} = x_n + \tfrac{h}{6}(k_1 + 4k_2 + k_3),$$
$$\bar{x}_{n+m} = x_n + \tfrac{mh}{6}(k_1 + 2k_{2m} + 2k_{3m}), \quad m = \tfrac{1}{3}, \tfrac{2}{3} \text{ or } \tfrac{1}{2},$$
$$x_{n+1} = x_n + h\{\beta_1 \bar{f}_{n+1} + \gamma_1 \bar{f}_{n+m} + \beta_0 f_n\},$$

where

$$k_1 = f(t_n, x_n),$$
$$k_{2m} = f(t_n + mh, x_n + mk_1 h),$$
$$k_{3m} = f(t_n + mh, x_n + mk_2 h),$$

and

$f_{n+1} = f(t_n, x_{n+1}), \quad f_{n+m} = f(t_n + mh, x_{n+m}), \quad f_n = f(t_n, x_n).$

5 Stability analysis of the new methods

Now we want to examine the stability analysis of new methods. We consider Dahlquist test problem $x' = \lambda x$ to investigate the stability region of the methods presented in this study. Using the Dahlquist test problem to the methods (11)–(13) inserting $p_1 = 4$, the following equations can be obtained:

$$\bar{x}_{n+1} = \left(1 + \bar{h} + \frac{(\bar{h})^2}{2!} + \frac{(\bar{h})^3}{3!} + \frac{(\bar{h})^4}{4!}\right) x_n, \tag{16}$$

$$\bar{x}_{n+m} = \left(1 + \frac{m\bar{h}}{1!} + \frac{(m\bar{h})^2}{2!} + \frac{(m\bar{h})^3}{3!} + \frac{(m\bar{h})^4}{4!}\right) x_n, \quad m = \tfrac{1}{3}, \tfrac{2}{3} \text{ or } \tfrac{1}{2}, \tag{17}$$

$$x_{n+1} = x_n + h\{\beta_0 \bar{f}_{n+1} + \gamma_1 \bar{f}_{n+m} + \beta_1 f_n\}, \tag{18}$$

where $\bar{h} = h\lambda$. By substituting (16) and (17) into (18), the following equation is obtained:

$$\begin{aligned} x_{n+1} = x_n + h&\left\{\beta_0 \left(1 + \bar{h} + \frac{(\bar{h})^2}{2!} + \frac{(\bar{h})^3}{3!}\frac{(\bar{h})^4}{4!}\right) x_n\right\} \\ &+ h\left\{\gamma_1 \left(1 + \frac{m\bar{h}}{1!} + \frac{(m\bar{h})^2}{2!} + \frac{(m\bar{h})^3}{3!} + \frac{(m\bar{h})^4}{4!}\right) x_n + \beta_1 x_n\right\}. \end{aligned} \tag{19}$$

By inserting $x_n = r^n$ into (19) and dividing by r^n we can obtain:

$$\begin{aligned} r^{n+1} = r^n &\left\{1 + \bar{h}(\beta_0 + \beta_1 + \gamma_1) + (\beta_0 + \gamma_1 m)\bar{h}^2 + \frac{(\beta_0 + \gamma_1 m^2)\bar{h}^3}{2}\right\} \\ &+ r^n \left\{\frac{(\beta_0 + \gamma_1 m^3)\bar{h}^4}{6} + \frac{(\beta_0 + \gamma_1 m^4)\bar{h}^5}{24}\right\} \end{aligned}$$

$$\begin{aligned} \Rightarrow r = 1 + \bar{h}(\beta_0 + \beta_1 + \gamma_1) + (\beta_0 + \gamma_1 m)\bar{h}^2 + \frac{(\beta_0 + \gamma_1 m^2)\bar{h}^3}{2} + \frac{(\beta_0 + \gamma_1 m^3)\bar{h}^4}{6} \\ + \frac{(\beta_0 + \gamma_1 m^4)\bar{h}^5}{24}. \end{aligned}$$

which is the stability polynomial of the methods (11) − (13) for $m = \frac{1}{3}, \frac{2}{3}$ or $\frac{1}{2}$ where $p_1 = 4$. By following the same way for $p_1 = 3$, we can obtain:

$$\bar{x}_{n+1} = \left(1 + \bar{h} + \frac{(\bar{h})^2}{2!} + \frac{(\bar{h})^3}{3!}\right) x_n, \tag{20}$$

$$\bar{x}_{n+m} = \left(1 + \frac{m\bar{h}}{1!} + \frac{(m\bar{h})^2}{2!} + \frac{(m\bar{h})^3}{3!}\right) x_n, \quad m = \tfrac{1}{3}, \tfrac{2}{3} \text{ , or } \tfrac{1}{2}, \tag{21}$$

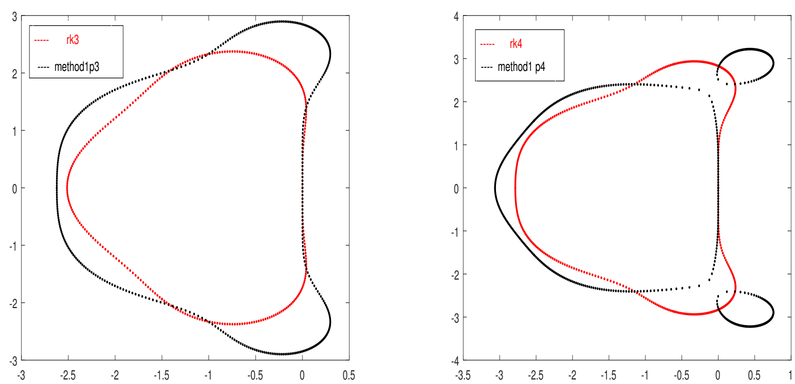
$$x_{n+1} = x_n + h\{\beta_0 \bar{f}_{n+1} + \gamma_1 \bar{f}_{n+m} + \beta_1 f_n\}, \tag{22}$$

where $\bar{h} = h\lambda$. By substituting (20) and (21) into (22), the following equation is obtained:

$$x_{n+1} = x_n + h\left\{\beta_0\left(1+\bar{h}+\frac{(\bar{h})^2}{2!}+\frac{(\bar{h})^3}{3!}\frac{(\bar{h})^4}{4!}\right)x_n\right\}$$
$$+h\left\{\gamma_1\left(1+\frac{m\bar{h}}{1!}+\frac{(m\bar{h})^2}{2!}+\frac{(m\bar{h})^3}{3!}+\right)x_n+\beta_1 x_n\right\}.$$

By inserting $x_n = r^n$ into (19) and dividing by r^n we can obtain:

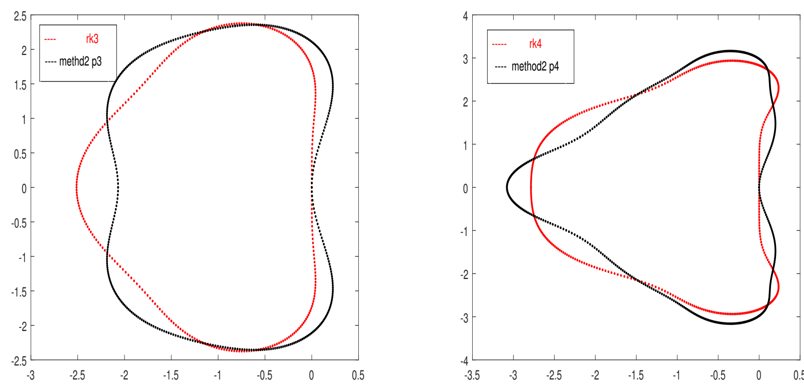
$$r^{n+1} = r^n\left\{1+\bar{h}(\beta_0+\beta_1+\gamma_1)+(\beta_0+\gamma_1 m)\bar{h}^2\right\}$$
$$+r^n\left\{\frac{(\beta_0+\gamma_1 m^2)\bar{h}^3}{2}+\frac{(\beta_0+\gamma_1 m^3)\bar{h}^4}{6}\right\}$$
$$\Rightarrow r = 1+\bar{h}(\beta_0+\beta_1+\gamma_1)+(\beta_0+\gamma_1 m)\bar{h}^2$$
$$+\frac{(\beta_0+\gamma_1 m^2)\bar{h}^3}{2}+\frac{(\beta_0+\gamma_1 m^3)\bar{h}^4}{6}.$$



(a) (b)

Figure 1: (a) Stability region of rk3 and Method1 where $p_1 = 3$. (b) Stability region of rk4 and Method 1 where $p_1 = 4$.

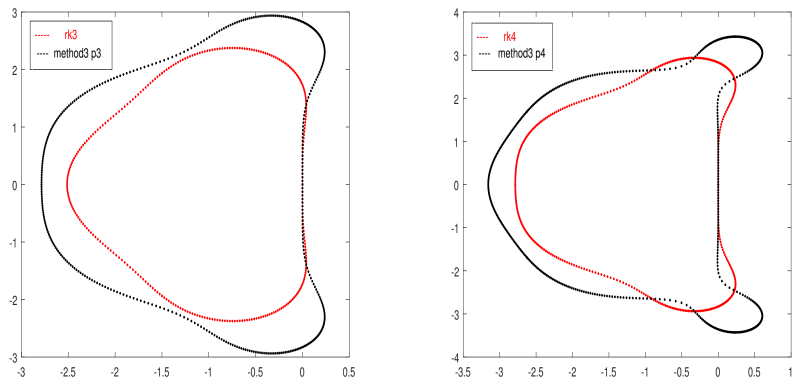
We show the stability of the Method1, where $p_1 = 3$ in Figure 1 and compared Runge- Kutta of Rank 3 method. It can be seen that the stability zone of the proposed scheme is larger than the stability region of the rk3 scheme. Furthermore, this proves the efficiency of the propsed method. We show the stability of the Method1, where $p_1 = 4$. in Figure 1 and We compared Runge- Kutta of Rank 4 methods. It can be seen that the stability zone of the proposed scheme is larger than the stability region of the rk4 scheme. Furthermore, this proves the efficiency of the proposed method. We show the stability of the



(a) (b)

Figure 2: (a) Stability region of rk3 and Method2 where $p_1 = 3$. (b) Stability region of rk4 and Method2 where $p_1 = 4$.

Method2, where $p_1 = 3$ in Figure 2 and compared Runge- Kutta of rank 3 methods. One can observe the stability region in the current scheme is larger than the stability region of the rk4 and rk3 methods. Furthermore, this proves the efficiency of the propsed method, and we show the stability of the Method2 where $p_1 = 4$ in Figure1 and compared Runge- Kutta of rank 4 method. One can observe that the stability region of the propsed methodology is larger than the stability region of the rk4 and rk3 methods.In aditional, this proves the efficiency of the propsed method. We show the stability of the Method3, where $p_1 = 3$ in Figure 3 and compared Runge-Kutta of rank 3 method. One can observe that the stability region of the new scheme is larger, and this proves the efficiency of the new method, and we show the stability of



(a) (b)

Figure 3: (a) Stability region of rk3 and Method3 where $p_1 = 3$. (b) Stability region of rk4 and Method3 where $p_1 = 4$.

the Method3 where $p_1 = 4$ in Figure 3 and compared Runge- Kutta of rank 4 method. One can observe that the stability region of the new scheme is larger.

6 Convergence

6.1 Convergence of FBSMs

In simpler terms, solving the OC problem with the FBSM method is actually finding $(x(t), \lambda(t), u(t))$ as

$$\begin{aligned} &x'(t) = g(t, x(t), u(t)), \quad x(t_0) = x_0, \\ &\lambda'(t) = p_1(t, x(t), u(t)) + \lambda(t)p_2(t, x(t), u(t)), \ \lambda(t_N) = 0, \\ &u(t) = p_3(t, x(t), u(t)). \end{aligned}$$

Here $x_0 \in R^n$ and $t_0 < t_N$ are given real numbers. To prove the convergence of the FBSM method, it is necessary to have the following conditions.
(B) The functions g, p_1, p_2 and p_3 are Lipschitz-continuous based on

their 2nd and 3th arguments, with Lipschitz-constants L_g, L_{p_1},

$$|g(t, x_1, u_1) - g(t, x_2, u_2)| \leq L_g(|x_1 - x_2| + |u_1 - u_2|).$$

Moreover, $\Lambda = \|\lambda\|_\infty$ and $H = \|p_2\|_\infty < \infty$.

Theorem 6.1. *Under the assumptions* (B)*, if*

$$c_0 \equiv L_{p_3} \{[\exp(L_g(t_N - t_0)) - 1]\} + L_{p_3} \Big\{(L_{p_1} + \Lambda L_{p_2}) \frac{1}{H}[\exp(H(t_N - t_0)) - 1][\exp(L_g(t_N - t_0)) + 1]\Big\} < 1,$$

then one has convergence: as $k \to \infty$,

$$\max_{t_0 \leq t \leq t_N} |x(t) - x^{(k)}(t)| + \max_{t_0 \leq t \leq t_N} |\lambda(t) - \lambda^{(k)}(t)| + \max_{t_0 \leq t \leq t_N} |u(t) - u^{(k)}(t)| \to 0.$$

Proof. We refer the reader to [22]. □

6.2 Convergence of the proposed method

One important property of numerical methods related to truncation errors is convergence. In this section, after mentioning a few definitions, we prove that our proposed method is convergent.

Definition 6.2. linear Mulistep Methods are generally described as follows:

$$\sum_{j=1}^{k} \alpha_j y_{n+j} = h \sum_{j=1}^{k} \beta_j f_{n+j} \qquad LMM.$$

Definition 6.3. The first characteristic polynomial of LMM methods is defined as follows:

$$\rho(\xi) = \alpha_1 \xi^1 + \alpha_2 \xi^2 + ... + \alpha_k \xi^k.$$

Definition 6.4. The second characteristic polynomial of LMM methods is defined as follows:

$$\sigma(\xi) = \beta_1 \xi^1 + \beta_2 \xi^2 + ... + \alpha_k \xi^k.$$

Definition 6.5. If the method at least rank one, then it is compatible, or in other word:

$$\rho(1) = 0, \quad \rho'(1) = \sigma(1).$$

Definition 6.6. A polynomial applies to the root condition, when if any roots satisfies $|r| \leq 1$ and then satisfies $|r| = 1$ are simple.

Definition 6.7. Every LMM is called zero-stable ,when the polynomial of its first characteristic $\rho(r)$ applies to the root condition.

Theorem 6.8 (Dalquist)**.** *Every LMM is convergent, if and only if, at the same time, it be consistent and zero-stable.*

Proof. We refer the reader to [14]. □

Remark 6.9. Because the methods presented in this article are consistent and zero stable. therefor, according to Theorem 6.8 the proposed method is covergent.

7 Numerical results using FBSM and new methods

One of The important cases of OC is the problem of linear regulators. In this part of the article we state its general state and solve an example in this field with new methods and compare its numerical results with FBSM method. Let E,P(t) and S(t) be nonnegative and symmetric definite matrices. A linear regulator problem with linear state space is called and its cost function is as follows:

$$F(u) = \frac{1}{2}x^T(t_f)Ex(t_1) + \frac{1}{2}\int_{t_0}^{t_f}[x^tP(t)x(t) + u^T(t)S(t)u(t)]dt.$$

Example 7.2 is related to the effect of Medicine on Cancerous Cells, Example 7.3. the general form is a Linear-Qudratic Models and Example 7.4 is a nonlinear Model and Related to the OC of rubella.

Example 7.1. (linear regulator problem)
Let we have the following OC problem [22]:

$$\begin{array}{l} \min_u \frac{1}{2}\int_0^1 x(t)^2 + u(t)^2 dt \\ st. \quad x'(t) = -x(t) + u(t), \quad x(0) = 1. \end{array}$$

The Pontryagin's Maximum-Principle can be utilized to construction of any analytic answer

$$H(t,x,u,\lambda)=\frac{1}{2}(x(t)^2+u(t)^2)+\lambda(-x(t)+u(t)),$$

$$\frac{\partial H}{\partial u}=0 \quad at \quad u^* \quad \Rightarrow u^*+\lambda=0 \Rightarrow u^*=-\lambda,$$

$$\lambda'=-\frac{\partial H}{\partial x}=-x+\lambda, \quad \lambda(1)=0.$$

And by the state equation, one gets a linear differential-algebric system as follows.

$$x'(t)=-x(t)+u(t), \quad x(0)=1$$

$$\lambda'(t)=\lambda(t)-x(t), \quad \lambda(1)=0$$

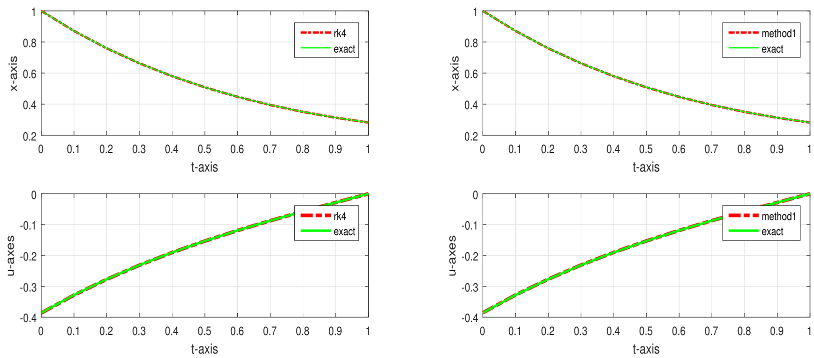
$$u(t)=-\lambda(t).$$

The solution is

$$x^*(t)=\frac{\sqrt{2}\cosh(\sqrt{2}(t-1))-\sinh(\sqrt{2}(t-1))}{\sqrt{2}\cosh(\sqrt{2})+\sinh(\sqrt{2})} \quad and$$

$$u^*(t)=\frac{\sinh(\sqrt{2}(t-1))}{\sqrt{2}\cosh(\sqrt{2})+\sinh(\sqrt{2})}.$$

The optimal condition of the performance index is $J=.1929092981$ and The final condition of the state is $x(1)=.2819695346$. The last condition of the control is 0 and the initial condition of the co state is $\lambda(0)=.3858185962$. Matlab implementation of the new methods of Example 7.1 was determined as follows. The results are shown in Figure4 with $h=\frac{1}{10}$ and in Tables 1- 4. These results show that the new methods are more accurate than the FBSM method under the 4th and 3th order Runge-Kutta. To avoid increasing the number of pages of the article, the Figure of method 1 has been selected and drawn as an example. The Figure of the other methods is similar to method 1.



(a) (b)

Figure 4: (a) The optimal state and control in Example 7.1 (Rk4). (b) The optimal state and control values in Example 7.1 (Method1, where $p_1 = 4$).

Table 1: End Error of control values in Example 7.1 where $p_1 = 4, \quad t = .9$.

N	FBSM_rk4	Method1	Method2	Method3
10	3.5468e-5	1.5003e-5	1.1891e-5	9.1624e-6
30	1.4072e-5	8.5749e-6	1.1443e-5	9.1645e-6
100	1.9391e-5	1.8900e-5	1.9155e-5	1.8955e-5
500	3.3824e-8	1.4209e-8	2.4338e-8	1.6390e-8

Table 2: End Error of control values in Example 7.1 where $p_1 = 3, \quad t = .9$.

N	FBSM_rk3	Method1	Method2	Method3
10	3.3172e-5	9.2263e-6	3.2725e-5	9.1004e-6
30	1.3993e-5	9.1904e-6	6.1536e-6	9.1953e-6
100	1.9381e-5	1.8955e-5	1.8683e-5	1.8955e-5
500	3.3821e-8	1.6389e-8	5.4958e-9	1.6390e-8

Table 3: The Error of estimate of the value of the objective J_{approx} in Example 7.1 where $p_1 = 4$.

N	eta	FBSM_rk4	Method1	Method2	Method3
10	0.001	1.4263e-3	1.3302e-3	1.4070e-3	1.3407e-3
30	0.001	1.5403e-4	1.4361e-4	1.5190e-4	1.4477e-4
100	0.001	6.0107e-5	5.9178e-5	5.9915e-5	5.9281e-5
500	0.000001	5.6024-7	5.2301e-7	5.5246e-7	5.2715e-7

Table 4: The Error of estimate of the value of the objective J_{approx} in Example 7.1 where $p_1 = 3$.

N	eta	FBSM_rk3	Method1	Method2	Method3
10	0.001	1.4158e-3	1.3414e-3	1.2875e-3	1.3410e-3
30	0.001	1.5366e-4	1.4478e-4	1.3897e-4	1.4477e-4
100	0.001	6.0098e-5	5.9281e-5	5.8764e-5	5.9281e-5
500	0.000001	5.6017e-7	5.2715e-7	5.0646e-7	5.2715e-7

7.1 Application of the proposed method in Effect of drug on cancer cells

Example 7.2. In general, suppose the following set-up,

$$\max_u \left[\phi(x(t_1)) + \int_{t_0}^{t_1} f(t, x(t), u(t))dt\right]$$

$$\text{subject to} \quad x' = g(t, x(t), u(t)), \quad x(t_0) = x_0.$$

Suppose $x(t)$ is the number of tumor cells at moment t (with exponential growth rate α), and $u(t)$ the drug concentration. We want to minimize the number of tumor cells at the end of the treatment period and minimize the harmful influence of drug accumulation in the body. Therefore, the problem is as follows, which is solved by another method in reference

[19].

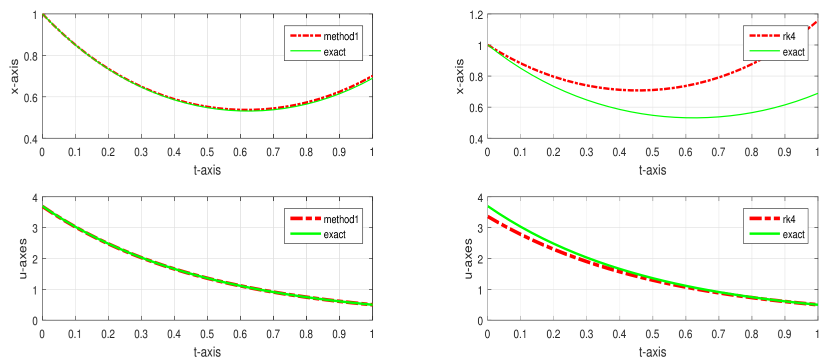
$$\min_u \, x(T) + \int_0^T u(t)^2 dt$$
$$\text{subject to} \quad x'(t) = \alpha x(t) - u(t),\ x(0) = x_0 > 0$$

Notice that $\phi(s) = s$ here, so that $\phi'(s) = 1$. First, one constructs the Hamiltonian and then calculate the necessary conditions:

$$H = u^2 + \lambda(\alpha x - u), \quad \frac{\partial H}{\partial u} = 2u - \lambda = 0 \quad at \quad u^* \Rightarrow u^* = \frac{\lambda}{2},$$
$$\lambda' = -\frac{\partial H}{\partial x} = -\alpha\lambda \Rightarrow \lambda = ce^{-\alpha t} \quad \lambda(T) = 1 \Rightarrow \lambda(t) = e^{\alpha(T-t)}$$
$$\Rightarrow u^*(t) = \frac{e^{\alpha(T-t)}}{2},$$
$$x' = \alpha x - u = \alpha x - u = \alpha x - \frac{e^{\alpha(T-t)}}{2}, \quad x(0) = x_0.$$
$$\Rightarrow x^*(t) = x_0 e^{\alpha t} + e^{\alpha T}\frac{e^{-\alpha t} - e^{\alpha t}}{4\alpha}.$$

The optimal value of the objective functional is $J = 4.0391717218591$. The final value of the state is $x(1) = 6.8928734478762e - 1$. The final value of the control is 0, the final value of the controller is 0 and the estimate of value of control with new methods and FBSM methods is $3.7252e - 9$.
The numerical answer of this OCP related to x(t), and u(t) are obtained and their results have been plotted in Figure 5 with $\alpha = 2, \quad h = \frac{1}{30}$ and the results are indicated in Tables 5-6. These results show that the new methods are more accurate than the Runge-Kutta of rank 4 method.

As clearly seen in Section b of Figure 5. The curves of the control variables and the state of the FBSM method under the Runge-Kutta order of 4 times N equals 30 are very different from the real answer. While the control curves and the state of our proposed method for each N are exactly the same as the real answer. To avoid increasing the number of pages of the article, the Figure of method 1 has been selected and drawn as an example. The Figure of the other methods is similar to method 1.



(a) (b)

Figure 5: (a) The optimal state and control values of Example 7.2 (Method1, where $p_1 = 4$). (b) The optimal state and control values of Example 7.2 (rk4).

Table 5: End Error of control values in Example 7.2 where $(Alpha = 2, \quad eta = .000001, \quad t = .5, \quad p_1 = 4)$.

N	FBSM_rk4	Method1	Method2	Method3
10	1.0071e-2	3.3690e-5	3.3693e-5	3.3692e-5
30	4.0578e-3	5.4275e-6	5.4278e-6	5.4276e-6
100	2.0338e-3	1.3668e-6	1.3669e-6	1.3668e-6
500	1.0181e-3	3.4961e-7	3.4961e-7	3.4961e-7

Table 6: The Error of estimate of the value of the performance index J_{approx} in Example 7.2 where $p_1 = 4, \quad eta = 0.000001$.

N	FBSM_rk4	Method1	Method2	Method3
200	5.1952e-4	1.4077e-4	1.1992e-4	1.3230e-4
500	8.7354e-5	2.5344e-5	2.1874e-5	2.3861e-5
1000	2.2999e-5	7.3634e-6	6.4948e-6	6.9919e-6
2000	6.0393e-6	2.1111e-6	1.8945e-6	2.0189e-6

Example 7.3. (linear-quadratic prblem) Suppose the following OC problem [29]:

$$\begin{array}{l} \min_u \int_0^1 \frac{5}{8}x(t)^2 + \frac{1}{2}x(t)u(t) + \frac{1}{2}u(t)^2 dt \\ \text{subject to} \quad x'(t) = \frac{1}{2}x(t) + u(t), \quad x(0) = 1. \end{array}$$

To solve the above Example, using the FBSM and new methods, we should use the Pontryagin's Theorem as below

$$H(t,x,u,\lambda) = \frac{5}{8}x(t)^2 + \frac{1}{2}x(t)u(t) + \frac{1}{2}u(t)^2 + \lambda(\frac{1}{2}x(t) + u(t)),$$

$$\frac{\partial H}{\partial u} = 0 \quad at \quad u^* \quad \Rightarrow u^* = -\lambda - \frac{1}{2}x,$$

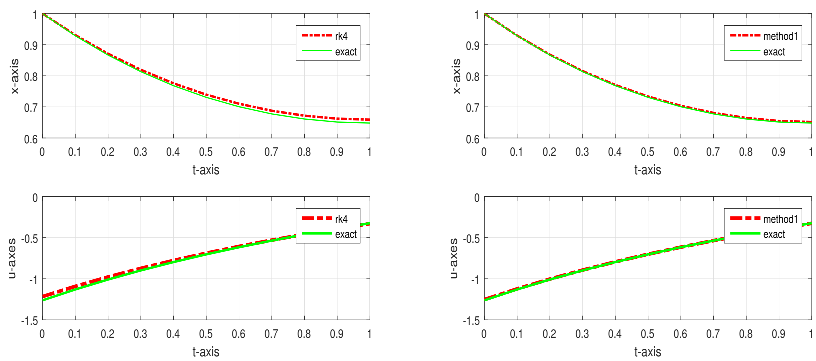
$$\lambda' = -\frac{\partial H}{\partial x} = -\frac{10}{8}x - \frac{1}{2}u - \frac{1}{2}\lambda, \quad \lambda(1) = 0.$$

Analytical solutions which are as follows [29]:
$u^*(t) = -\frac{(\tanh(1-t)+.5)\cosh(1-t)}{\cosh(1)}, \qquad x^*(t) = \frac{\cosh(1-t)}{\cosh(1)}$
The state variable at the end point is $x(1) = 6.4805427366388e-1$. The control variable at the end point is $u(1) = -3.24027136831e-1$ and the optimal value of the objective functional is $J = 0.3807970779$. Matlab implementation of the three methods of Example 7.3 was determined as follows. The results are shown in Figure 7 with $h = \frac{1}{10}$ and in Tables 7-11.These results show that the new methods are more accurate than the Runge-Kutta of rank 3 and 4.

Table 7: End Error of state values in Example 7.3 where $p_1 = 4$.

h	FBSM_rk4	Method1	Method2	Method3
$\frac{1}{10}$	1.0965e-2	3.4143e-3	3.4818e-4	4.0739e-4
$\frac{1}{100}$	1.2195e-3	3.5529e-4	4.2119e-5	4.2737e-5
$\frac{1}{200}$	6.1325e-4	1.7793e-4	2.1162e-5	2.1317e-5
$\frac{1}{300}$	4.0959e-4	1.1866e-4	1.4106e-5	1.4175e-5
$\frac{1}{600}$	2.0514e-4	5.9314e-5	7.0160e-6	7.0333e-6
$\frac{1}{1000}$	1.2313e-4	3.5554e-5	4.1705e-6	4.1767e-6

The numerical results of Table 7 show our proposed methodology is two digits more precise than the fbsm technique under the 4th order



(a) (b)

Figure 6: (a) The optimal state and control values of Example 7.3(Rk4). (b) The optimal state and control of Example 7.3 (Method1, where $p_1 = 4$).

Table 8: End Error of state values in Example 7.3 where $p_1 = 3$.

h	FBSM_Rk3	Method1	Method2	Method3
$\frac{1}{10}$	1.0936e-2	5.2716e-4	2.3723e-4	7.5223e-5
$\frac{1}{100}$	1.2194e-4	8.0464e-5	2.4718e-6	8.9277e-7
$\frac{1}{200}$	6.1325e-4	4.0935e-5	5.3996e-7	1.4577e-7
$\frac{1}{300}$	4.0959e-4	2.7422e-5	1.8050e-7	5.3951e-9
$\frac{1}{600}$	2.0514e-4	1.3741e-5	3.5639e-8	7.9314e-8
$\frac{1}{1000}$	1.2313e-4	8.2219e-6	8.1824e-8	9.7573e-8

Runge-Kutta. It can be seen from Table 7 that the values of the state variable at the end point, our proposed method, for the state $p = 3$, in most cases 4 digits are more accurate than the FBSM method under the 3th order Runge-Kutta. Also, the numerical results of Table 9 show that the values of the control variable of our proposed method are two digits more accurate than the fbsm method under the 4th order Runge-Kutta. And our proposed method for $p = 3$, according to Table 7, is three digits more accurate than fbsm method under the 3th order Runge-Kutta. The numerical results of Table 11 indicate that the approximate value of the

Table 9: End Error of control values in Example 7.3 where $p_1 = 4$.

h	FBSM_rk4	Method1	Method2	Method3
$\frac{1}{10}$	5.4830e-3	1.7074e-3	1.7442e-4	2.0403e-4
$\frac{1}{100}$	6.1008e-4	1.7796e-4	2.1387e-5	2.1696e-5
$\frac{1}{200}$	3.0695e-4	8.9291e-5	1.0909e-5	1.0986e-5
$\frac{1}{300}$	2.0512e-4	5.9656e-5	7.3809e-6	7.4154e-6
$\frac{1}{600}$	1.0289e-4	2.9983e-5	3.8358e-6	3.8444e-6
$\frac{1}{1000}$	6.1896e-5	1.8104e-5	2.4130e-6	2.4161e-6

Table 10: End Error of control values in Example 7.3 where $p_1 = 3$.

h	FBSM_Rk3	Method1	Method2	Method3
$\frac{1}{10}$	5.4687e-3	2.6391e-4	1.1895e-4	3.7945e-5
$\frac{1}{100}$	6.1007e-4	4.0560e-5	1.5637e-6	7.7423e-7
$\frac{1}{200}$	3.0695e-4	2.0795e-5	5.9778e-7	4.0069e-7
$\frac{1}{300}$	2.0512e-4	1.4039e-5	4.1804e-7	3.3049e-7
$\frac{1}{600}$	1.0289e-4	7.1985e-6	3.0997e-7	2.8809e-7
$\frac{1}{1000}$	6.1896e-5	4.4387e-6	2.8687e-7	2.7900e-7

Table 11: The Error of estimate of the value of the objective J_{approx} in Example 7.3 where $p_1 = 4, \quad eta = 0.000001$.

N	FBSM_rk4	Method1	Method2	Method3
10	2.7744e-3	1.2701e-3	1.2787e-3	1.3057e-3
100	4.1164e-4	1.2104e-5	1.1972e-5	1.2255e-5
200	2.0993e-4	2.7470e-6	2.7112e-6	2.7821e-6
300	1.4107e-4	8.9264e-7	8.7624e-7	9.0777e-7
600	7.0732e-5	3.4405e-7	3.4002e-7	3.4791e-7
1000	4.2459e-5	2.0901e-7	2.0761e-7	2.1045e-7

performance index in our proposed method, three digits, is calculated better than FBSM method under the 4th order Runge-Kutta. To avoid increasing the number of pages of the article, the Figure of Method1 has been selected and drawn as an example. The Figure of the other methods is similar to Method1.

7.2 Application of the proposed method in controlling Rubella

Rubella is a viral disease also called german measles or three-day measles because the patients fever goes away after 3 days. Rubella can occure at any age, but it is a relatively common disease in children and if it occurs during pregnancy, it can have irreversible consequences for the baby and the mother. The rubella virus is transmitted through contact with throat and masal secretions. or through the secretions of infected person's respiratory system. The incubation period of this disease is 14 to 27 days. The initial symptoms of the disease begin about ten days to two weeks after the virus enters the body. some of these symptoms include: Mild fever up to 9 degrees celsius, nasal stasis and runy nose, headache, conjunctivitis and joint pain, especially in young women and so on. So we have provided an OC problem for the study and control of rubella and we consider vaccination process as a control measure.

Example 7.4. Suppose using a vaccination process (u) as a value of control the disease. Let x_1 represent the talented people, x_2 the ratio of the population that is in the reproductive period, x_3 the number of people with the disease, and x_4 has the role of keeping people constant. Control problem can be defined as [30].

$$\min_u \int_0^3 (Ax_3 + u^2)^2 dt$$

$$\text{subject to} \quad \begin{aligned} x_1' &= a - a(px_2 + qx_2) - ax_1 - \alpha x_1 x_3 - ux_{1,}, \\ x_2' &= apx_2 + \alpha x_1 x_3 - (e+a)x_2, \\ x_3' &= ex_2 - (g+a)x_3, \\ x_4' &= a - ax_4. \end{aligned}$$

That its initial conditions $x_1(0) = 5.55e^{-2}$, $x_2(0) = 0.3e^{-3}$, $x_3(0) = 0.00041$, $x_4(0) = 1$ and the parameters $a = 1.2e^{-2}$, $p = 0.65$, $q = 0.65$, $\alpha = 527.59$, $e = 36.5$, $g = 30.417$ and $A = 100$.
Let $\overrightarrow{x}(t) = (x_1(t), x_2(t), x_3(t), x_4(t))$
and $\overrightarrow{\lambda}(t) = (\lambda_1(t), \lambda_2(t), \lambda_3(t), \lambda_4(t))$, the Hamiltonian of this problem

can be written as

$$H(t, \overrightarrow{x}(t), u(t), \overrightarrow{\lambda}(t)) = Ax_3 + u^2 + \lambda_1(a - a(px_2 + qx_2) - ax_1 - \alpha x_1 x_3 - ux_1)$$
$$+\lambda_2(apx_2 + \alpha x_1 x_3 - (e+a)x_2) + \lambda_3(ex_2 - (g+a)x_3) + \lambda_4(a - ax_4).$$

Using the Pontriagin maximum principle, the OCP can be studied with the following differential equation system.

$$\begin{cases} {x_1}' = a - a(px_2 + qx_2) - ax_1 - \alpha x_1 x_3 - ux_{1,}, \\ {x_2}' = apx_2 + \alpha x_1 x_3 - (e+a)x_2, \\ {x_3}' = ex_2 - (g+a)x_3, \\ {x_4}' = a - ax_4. \end{cases}$$

subject to initial values $x_1(0) = 5.55e^{-2}$, $x_2(0) = 0.3e^{-3}$, $x_3(0) = 0.41e^{-3}$, $x_4(0) = 1$, and the adjoint system

$$\begin{cases} {\lambda_1}' = \lambda_1(a + u + \alpha x_3) - \lambda_2 \alpha x_3, \\ {\lambda_2}' = \lambda_1 ap + \lambda_2(e + a + pa) - \lambda_3 e, \\ {\lambda_3}' = -A + \lambda_1(ap + \alpha x_1) - \lambda_2 \alpha x_1 + \lambda_3(g+a), \\ {\lambda_4}' = \lambda_4 a. \end{cases}$$

with transversality conditions $\lambda_i(3) = 0, i = 1, ..., 4$. The OC is presented by

$$\begin{cases} 0 & if \quad \frac{\partial H}{\partial u} < 0, \\ \frac{\lambda_1 x_1}{2} & if \quad \frac{\partial H}{\partial u} = 0, \\ 0.9 & if \quad \frac{\partial H}{\partial u} > 0. \end{cases}$$

This is a stiff problem and a nonlinear model for the OC problem. Also, it is difficult to solve this, analytically and it is necessary to utilize numerical method.This example is solved in [29] by the forward-backward sweep method. Our proposed method also solves this example well. The results in the table 12 and Figures 7, 8 shows that the new methods compete with Runge-Kutta methods in stiff and several variable problems.

8 Conclusion

In this research, three hybrid methods of orders three and four are presented and then, the third and fourth order of Runge-Kutta methodologies are utilized as predictor schemes to gain whole methods of the

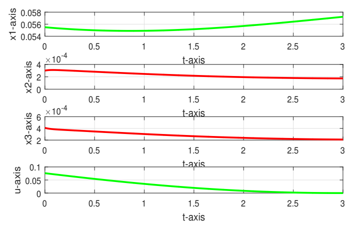


Figure 7: The optimal curves of the problem in Example 7.4.

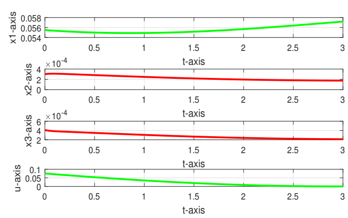


Figure 8: The optimal curves of the problem in Example 7.4(Method1, $p_1 = 4$).

Table 12: The optimal state values of methods in Example 7.4, $p_1 = 4$.

Method	x_1	x_2	x_3	x_4
FBSM_rk4	0.057229233323	0.000174026727	0.000209796082	0
Method1	0.057201314198	0.000175317514	0.000211408161	0
Method2	0.057187036786	0.000176015581	0.000212278533	0
Method3	0.057201267178	0.000175314688	0.000211404848	0

same orders. Then the order of truncation errors are investigated for the explicit hybrid based on Runge-Kutta methods.The stability analysis as well as covergence of the methods are consequently discussed. It shows that the stability domains of them are wider compared to the explicit the third and fourth order of Runge-Kutta methods. At last, several examples of OCPs are solved by the FBSM scheme and presented methodologies. The numerical results are presented with the aid of tables and figures. It is concluded that the hybrid schemes have pleasant performance index getting small end-point errors for solving optimal control problems numerically.

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