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FBSM Solution of Optimal Control Problems using Hybrid Runge-Kutta based Methods

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Abstract. solving optimal control problems (OCP) with analytical methods has usually been difficult or not cost-effective. Therefore, solving these problems requires numerical methods. There are, of course, many ways to solve these problems. One of the methods available to solve OCP is a forward-backward sweep method (FBSM). In this method, the state variable is solved in a forward and co-state variable by a backward method where an explicit Runge–Kutta method (ERK) is often used to solve differential equations arising from OCP.In this paper, instead of the ERK method, two hybrid methods based on ERK method of order 3 and 4 are proposed for the numerical approximation of the OCP. Truncation errors and stability analysis of the presented methods are illustrated. Finally, numerical results of the five optimal control problems obtained by new methods, which shows that

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new methods give us more accurate results, are compared with those of ERK methods of orders 3 and 4 for solving OCP.

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1 Introduction

The goal of this work is to illustrate details of a new single step explicit Rung-Kutta (ERK) type method based on off-step points for the numerical solution of optimal control problems (OCP) of the form:

$$\max_u \int_{t_0}^{t_1} f\left(t,\, \boldsymbol{x}\left(t\right),\, \boldsymbol{u}\left(t\right)\right) dt, \tag{1}$$

subject to the state equation

$$\boldsymbol{x}'\left(t\right) = g\left(t,\, x\left(t\right),\, u\left(t\right)\right), \quad \boldsymbol{x}\left(t_0\right) = t_0, \tag{2}$$

where it is assumed that x and u are vector-valued functions on $[t_0, t_1]$ with values in R^n and R^m, respectively. Some of the ways to generalize this problem are as follows:
1) the terminal value of the $x(t)$ at $t = t_1$ may be fixed;
2) the value of end time $t_1$could be considered as a variable;
3) a scrap function $\phi(t_1)$ could be included in addition to the objective function (1) [9, 16, 17, 18] .
The generalized problem can be solved by using indirect methods which are numerical techniques to solve them. The forward backward sweep method (FBSM) is one of these methods. In [7], convergence analysis of the FBSM has been done. In fact, by using FBSM, the differential equations arising from the maximum principle are numerically solved. Euler, Trapezoidal and Runge–Kutta methods can be used for the numerical solution of OCP by using FBSM where we are faced with initial value problems(IVPS) arising from the state and adjoint equations. In 2015, D. P. Moualeu et al. used FBSM to solve derived optimality system for a tuberculosis model with undetected cases in Cameroon numerically. An iterative method used for solving obtained optimality system. The state system solved by forward in time and the adjoint system solved backward using the transversality condition [3]. Lhous et al. presented

a discrete mathematical modeling and optimal control of the marital status in 2017 and solved the control problem using FBSM in which an iterative method used for the numerical solution of ordinary differential equations with initial guess [2]. In [5], authors solved basic OCP using Runge-Kutta based FBSM and compared the numerical results with those of Euler and trapezoidal based FBSM. Note that, for non-stiff models of OCP a number of numerical schemes for solving ODEs in the literature can be used to solve forward as well as backward. On the other hand, for mildly stiff and stiff problems of ODEs which may be appeared in OCP, we need to use numerical methods with wide stability regions and domains as well as good accuracy. In this work, we illustrate three implicit hybrid methods of orders 3 and 4 and then convert them into explicit methods using explicit Runge-Kutta methods of orders 3 and 4 as a predictor of the scheme. The stability and order of truncation error of the methods discussed showing that new methods have wide stability regains by which more accurate results can be obtained compared to the FBSM based explicit Runge-Kutta methods of orders 3 and 4. The paper is organized as follows: In Section 2, hybrid methods of orders 3 and 4 is described and their orders of truncation errors discussed. In Section 3, stability of the presented methods is analyzed. Numerical results for solving some optimal control problems presented in Section 4. Finally, we conclude the paper in Section 5.

2 Hybrid methods and order of truncation errors

For the numerical solution of initial value problems (IVP) of the form

$$x' = f(t, x) \ , \quad x \in \mathbb{R}^n \ , x(t_0) = x_0 \ , \quad t_0 \ \leq t \leq t_1, \tag{3}$$

where $f : [t_0, t_1] \times \mathbb{R}^n \to \mathbb{R}^n$, one can use an explicit or implicit method. Methods based of off-step points, such as hybrid BDF, (HBDF), new class of HBDFs and class $2+1$ hybrid BDF-like methods have wide stability regions and higher order compared to some Runge-Kutta method and implicit BDF methods [1, 11, 13, 14, 15]. Let us consider the IVP of the form (3). Linear k-step methods of the

$$x_{n+1} = \alpha_1 x_n + \alpha_2 x_{n-1} + ... + \alpha_k x_{n-k+1} + h\{\beta_0 f_{n+1} + \beta_1 f_n + ... + \beta_k f_{n-k+1}\} \quad (4)$$

has $2k+1$ arbitrary parameter and can be written as

$$\rho(E)x_{n-k+1} - h\sigma(E)f_{n-k+1} = 0$$

where E is the shift operator as $E(x(t)) = x(t+h)$, with the step length h and ρ and σ are first and second characteristic polynomials defined by

$$\rho(\xi) = \xi^k - \alpha_1\xi^{k-1} - \alpha_2\xi^{k-2} - ... - \alpha_k, \quad (5)$$

$$\sigma(\xi) = \beta_0\xi^k + \beta_1\xi^{k-1} + ... + \beta_k \quad , \quad (6)$$

To increase the order of k-step methods of the form (4), we use a linear combination of the slopes at several points between t_n and t_{n+1} where $t_{n+1} = t_n + h$ and h is the step length on $[t_0, t_1]$. Then, the modified form of (4) with m slops is given by

$$x_{n+1} = \sum_{j=1}^{k} \alpha_j x_{n-j+1} + h\sum_{j=0}^{k} \beta_j f_{n-j+1} + h\sum_{j=1}^{m} \gamma_j f_{n-\theta_j+1} \quad (7)$$

where α_j, β_j, γ_j and θ_j are $2k+2m+1$ arbitrary parameters [11]. Methods of the form (7) with m off-step points are called hybrid methods where $0 < \theta_j < 1, j = 1, 2, ..., m$. In this work, we set $\beta_0 = 0, k = 1$ and $m = 1$. Hence, we write (7) as

$$x_{n+1} = \alpha_1 x_n + h\{\beta_0 f_{n+1} + \beta_1 f_n\} + h\gamma_1 f_{n-\theta_1+1} \quad (8)$$

where $\alpha_1, \beta_0, \beta_1, \gamma_1$ and θ_1 are arbitrary parameters and $\theta_1 \neq 0$ or 1. Expanding terms y_{n+1}, $f_{n+1}, f_{n-\theta_1+1}$ in Taylor's series about t_n, we can obtain a family of third order methods if the equations

$$\begin{aligned}
&\alpha_1 = 1\\
&\beta_1 + \beta_0 + \gamma_1 = 1\\
&\beta_0 + (1-\theta_1)\gamma_1 = \tfrac{1}{2}\\
&\tfrac{1}{2}\beta_0 + \tfrac{1}{2}(1-\theta_1)^2\gamma_1 = \tfrac{1}{6}
\end{aligned}$$

are satisfied where the principal term of the truncation error is

$$\frac{1}{4!}c_4 h^4 x^{(4)}(t_n) + o(h^5), \qquad c_4 \; = 1 - 4\beta_2 - 4\gamma_1(1-\theta_1)^3.$$

For more details, we refer the reader to [11]. Considering the following three cases:

1. $\beta_1 = 0, \alpha_1 = 1, \beta_0 = \frac{1}{4}, \gamma_1 = \frac{3}{4}, \theta_1 = \frac{2}{3}, c_4 = -\frac{1}{9}$,
2. $\beta_1 = \frac{1}{4}, \alpha_1 = 1, \beta_0 = 0, \gamma_1 = \frac{3}{4}, \theta_1 = \frac{1}{3}, c_4 = \frac{1}{9}$,
3. $\beta_1 = \frac{1}{6}, \alpha_1 = 1, \beta_0 = \frac{1}{6}, \gamma_1 = \frac{2}{3}, \theta_1 = \frac{1}{2}, c_4 = 0$.

gives us the following methods of orders 3, 3, and 4 respectively (say New 3_1, New 3_2 and New 4 in this work) [11]:

$$x_{n+1} = x_n + \frac{h}{4}\{f_{n+1} + 3f_{n+\frac{1}{3}}\}, \tag{9}$$

$$x_{n+1} = x_n + \frac{h}{4}\{f_n + 3f_{n+\frac{2}{3}}\}, \tag{10}$$

$$x_{n+1} = x_n + \frac{h}{4}\{f_{n+1} + 4f_{n+\frac{1}{2}} + f_n\}, \tag{11}$$

where $f_{n+1} = f(t_n, x_{n+1}), f_{n+m} = f(t_n + mh, x_{n+m})$ and $f_n = f(t_n, x_n)$ for $m = \frac{1}{3}, \frac{2}{3}$ and $\frac{1}{2}$. Note that, x_{n+1}, x_{n+m} and x_n are numerical approximations according to the exact values of the solution $x(t)$ at $t_{n+1} = t_n + h, t_{n+m} = t_n + mh,\ t_n = t_n$ for $m = \frac{1}{3}, \frac{2}{3}$ and $\frac{1}{2}$ respectively. In order to convert methods (9)–(11) into explicit methods at each step, we predict the values of x_{n+1} and x_{n+m} used on the right hand side of the new methods using fourth or third order explicit Runge-Kutta method as follows, respectively:

$$\begin{aligned} & x_{n+1} = x_n + \tfrac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4), \\ & k_1 = f(t_n, x_n), \\ & k_2 = f(t_n + \tfrac{1}{2}h, x_n + \tfrac{1}{2}k_1 h), \\ & k_3 = f(t_n + \tfrac{1}{2}h, x_n + \tfrac{1}{2}k_2 h), \\ & k_4 = f(t_n + h, x_n + k_3 h). \end{aligned} \tag{12}$$

or

$$\begin{aligned} &x_{n+1} = x_n + \tfrac{h}{6}(k_1 + 4k_2 + k_3),\\ &k_1 = f(t_n, x_n),\\ &k_2 = f(t_n + \tfrac{1}{2}h, x_n + \tfrac{1}{2}k_1 h),\\ &k_3 = f(t_n + h, x_n - k_1 h + 2k_2 h). \end{aligned} \tag{13}$$

In general, we rewrite methods (9) –(11) using RK4 method as a predictor as follows:

$$\bar{x}_{n+1} = x_n + \tfrac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4), \tag{14}$$

$$\bar{x}_{n+m} = x_n + \tfrac{mh}{6}(k_1 + 2k_{2m} + 2k_{3m} + k_{4m}), m = \tfrac{1}{3}, \tfrac{2}{3} \text{ or } \tfrac{1}{2}, \tag{15}$$

$$x_{n+1} = x_n + h\{\beta_0 \bar{f}_{n+1} + \gamma_1 \bar{f}_{n+m} + \beta_1 f_n\}, \tag{16}$$

where

$$\begin{aligned} &k_1 = f(t_n, x_n),\\ &k_{2m} = f(t_n + mh, x_n + mk_1 h),\\ &k_{3m} = f(t_n + mh, x_n + mk_2 h),\\ &k_{4m} = f(t_n + mh, x_n + mk_3 h), \end{aligned}$$

and $f_{n+1} = f(t_n, x_{n+1}), f_{n+m} = f(t_n + mh, x_{n+m}), f_n = f(t_n, x_n)$. Now, suppose that the order of stage equation (13) is $p_1, p_1 = 4$, as like as (14). Thus, the difference between exact and numerical solution at $t = t_{n+m} = t_n + mh, m = \frac{1}{3}, \frac{2}{3}, \frac{1}{2}$ and 1 is

$$y(t_{n+m}) - y_{n+m} = C_m h^{p_1} y^{(p_1)}(t_n) + O(h^{p_1+1}) \tag{17}$$

where C_m is the error constant of the method (14) or (16) with corresponding m which can take only one of the values $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{2}$, together with the value 1 related to methods (13) and (14) respectively. The difference operator associated to method (15), of order $p, p = 3$ or 4, can be written as

$$y(t_{n+1}) - y_{n+1} = Ch^p y^{(p)}(t_n) + O(h^{p+1}) \tag{18}$$

where C is the error constant of the method (16). Therefore, we have the following theorem:

Theorem 2.1. *Given that*

1. *formula (14) is of order p_1,*

2. *formula (15) is of order p_1 too,*

3. *formula (16) is of order p,*

then, the order of (14) –(15) is p.

Proof. Suppose that m can only take one of the values $\frac{1}{3}, \frac{2}{3}$ or $\frac{1}{2}$ and y_n is exact. From (18) and (16) one can write

$$y(t_{n+1}) - y_{n+1} = h\beta_m[f(t_{n+m}, y(t_{n+m})) - f(t_{n+m}, \overline{y}_{n+m})] \tag{19}$$

$$+h\beta_1[f(t_{n+1}, y(t_{n+1})) - f(t_{n+1}, \overline{y}_{n+1})] + Ch^p y^{(p)}(t_n) + O(h^{p+1}).$$

Considering properties of the IVPs of the form (3), for some values such as η_m and η_1 belong to intervals $(\overline{y}_{n+m}, y(t_{n+m}))$ and $(\overline{y}_{n+1}, y(t_{n+1}))$ respectively, we can write

$$f(t_{n+m}, y(t_{n+m})) - f(t_{n+m}, \overline{y}_{n+m}) = \frac{\partial f}{\partial y}(t_{n+m}, \eta_{n+m})(y(t_{n+m}) - \overline{y}_{n+m}),$$

$$f(t_{n+1}, y(t_{n+1})) - f(t_{n+1}, \overline{y}_{n+1}) = \frac{\partial f}{\partial y}(t_{n+1}, \eta_{n+1})(y(t_{n+1}) - \overline{y}_{n+1}).$$

Therefore, by using (19), we have

$$y(t_{n+1}) - y_{n+1} = h\beta_m \left[\frac{\partial f}{\partial y}(t_{n+m}, \eta_{n+m})(y(t_{n+m}) - \overline{y}_{n+m})\right]$$

$$+h\beta_1 \left[\frac{\partial f}{\partial y}(t_{n+1}, \eta_{n+1})(y(t_{n+1}) - \overline{y}_{n+1})\right] + Ch^p y^{(p)}(t_n) + O(h^{p+1}).$$

Applying equation (16) to this gives us

$$y(t_{n+1}) - y_{n+1} =$$

$$h\beta_m \left[\frac{\partial f}{\partial y}(t_{n+m}, \eta_{n+m})C_m h^{p_1} y^{(p_1)}(t_n) + O(h^{p_1+1})\right]$$

$$+h\beta_1 \left[\frac{\partial f}{\partial y}(t_{n+1}, \eta_{n+1})C_1 h^{p_1} y^{(p_1)}(t_n) + O(h^{p_1+1})\right] + Ch^p y^{(p)}(t_n) + O(h^{p+1})$$

$$= h^p \left\{\beta_m \left[\frac{\partial f}{\partial y}(t_{n+m}, \eta_{n+m})C_m h^{p_1-p+1} y^{(p_1)}(t_n)\right] + \beta_1 \left[\frac{\partial f}{\partial y}(t_{n+1}, \eta_{n+1})C_1 h^{p_1-p+1} y^{(p_1)}(t_n)\right]\right.$$

$$\left. +Cy^{(p)}(t_n)\right\} + O(h^{p+1})$$

where $p_1 \geq p$. Thus, it can be concluded that the method (14) –(16) is of order p and so the proof is completed. □

By following the same way as presented above, it can be proved that the methods (9) –(11) using RK3 method as a predictor (Runge-kutta of order 3) of the form

$$\bar{x}_{n+1} = x_n + \tfrac{h}{6}(k_1 + 4k_2 + k_3), \tag{20}$$

$$\bar{x}_{n+m} = x_n + \tfrac{mh}{6}(k_1 + 2k_{2m} + 2k_{3m}), m = \tfrac{1}{3}, \tfrac{2}{3} \text{ or } \tfrac{1}{2}, \tag{21}$$

$$x_{n+1} = x_n + h\{\beta_1 \bar{f}_{n+1} + \gamma_1 \bar{f}_{n+m} + \beta_0 f_n\}, \tag{22}$$

where

$$\begin{aligned} &k_1 = f(t_n, x_n),\\ &k_{2m} = f(t_n + mh, x_n + mk_1 h),\\ &k_{3m} = f(t_n + mh, x_n + mk_2 h), \end{aligned}$$

and $f_{n+1} = f(t_n, x_{n+1}), f_{n+m} = f(t_n + mh, x_{n+m}), f_n = f(t_n, x_n)$.

3 Stability analysis of the new methods

Now we want to examine the stability analysis of new methods. We consider Dahlquist test problem $x' = \lambda x$ to investigate the stability region of the methods presented in this study. Using the Dahlquist test problem to the methods (14)-(16) inserting $p_1 = 4$, the following equations can be obtained:

$$\bar{x}_{n+1} = \left(1 + \bar{h} + \frac{(\bar{h})^2}{2!} + \frac{(\bar{h})^3}{3!} + \frac{(\bar{h})^4}{4!}\right) x_n, \tag{23}$$

$$\bar{x}_{n+m} = \left(1 + \frac{m\bar{h}}{1!} + \frac{(m\bar{h})^2}{2!} + \frac{(m\bar{h})^3}{3!} + \frac{(m\bar{h})^4}{4!}\right) x_n, m = \tfrac{1}{3}, \tfrac{2}{3} \text{ or } \tfrac{1}{2}, \tag{24}$$

$$x_{n+1} = x_n + h\{\beta_0 \bar{f}_{n+1} + \gamma_1 \bar{f}_{n+m} + \beta_1 f_n\}, \tag{25}$$

where $\bar{h} = h\lambda$. By substituting (23) and (24) into (25), the following equation is obtained:

$$x_{n+1} = x_n + h\left\{\beta_0 \left(1 + \bar{h} + \frac{(\bar{h})^2}{2!} + \frac{(\bar{h})^3}{3!}\frac{(\bar{h})^4}{4!}\right) x_n\right\}$$

$$+h\left\{\gamma_1\left(1+\frac{m\bar{h}}{1!}+\frac{(m\bar{h})^2}{2!}+\frac{(m\bar{h})^3}{3!}+\frac{(m\bar{h})^4}{4!}\right)x_n+\beta_1 x_n\right\}. \tag{26}$$

By inserting $x_n = r^n$ into (26) and dividing by r^n we can obtain:

$$r^{n+1} = r^n\left\{1+\bar{h}(\beta_0+\beta_1+\gamma_1)+(\beta_0+\gamma_1 m)\bar{h}^2+\frac{(\beta_0+\gamma_1 m^2)\bar{h}^3}{2}\right\}$$

$$+r^n\left\{\frac{(\beta_0+\gamma_1 m^3)\bar{h}^4}{6}+\frac{(\beta_0+\gamma_1 m^4)\bar{h}^5}{24}\right\} \tag{27}$$

$$\Rightarrow r = 1+\bar{h}(\beta_0+\beta_1+\gamma_1)+(\beta_0+\gamma_1 m)\bar{h}^2+\frac{(\beta_0+\gamma_1 m^2)\bar{h}^3}{2}+\frac{(\beta_0+\gamma_1 m^3)\bar{h}^4}{6}$$
$$+\frac{(\beta_0+\gamma_1 m^4)\bar{h}^5}{24}.$$

which is the stability polynomial of the methods (14) – (16) for $m = \frac{1}{3}, \frac{2}{3}$ or $\frac{1}{2}$ where $p_1 = 4$. By following the same way for $p_1 = 3$, we can obtain:

$$\bar{x}_{n+1} = \left(1+\bar{h}+\frac{(\bar{h})^2}{2!}+\frac{(\bar{h})^3}{3!}\right)x_n, \tag{28}$$

$$\bar{x}_{n+m} = \left(1+\frac{m\bar{h}}{1!}+\frac{(m\bar{h})^2}{2!}+\frac{(m\bar{h})^3}{3!}\right)x_n, m=\tfrac{1}{3}, \tfrac{2}{3} \text{ or } \tfrac{1}{2}, \tag{29}$$

$$x_{n+1} = x_n + h\{\beta_0\bar{f}_{n+1}+\gamma_1\bar{f}_{n+m}+\beta_1 f_n\}, \tag{30}$$

where $\bar{h} = h\lambda$. By substituting (28) and (29) into (30), the following equation is obtained:

$$x_{n+1} = x_n + h\left\{\beta_0\left(1+\bar{h}+\frac{(\bar{h})^2}{2!}+\frac{(\bar{h})^3}{3!}\frac{(\bar{h})^4}{4!}\right)x_n\right\}$$

$$+h\left\{\gamma_1\left(1+\frac{m\bar{h}}{1!}+\frac{(m\bar{h})^2}{2!}+\frac{(m\bar{h})^3}{3!}+\right)x_n+\beta_1 x_n\right\}. \tag{31}$$

By inserting $x_n = r^n$ into (31) and dividing by r^n we can obtain:

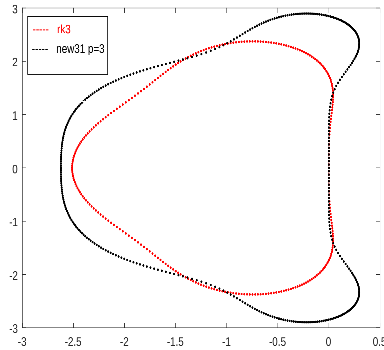
$$r^{n+1} = r^n \left\{1 + \bar{h}(\beta_0 + \beta_1 + \gamma_1) + (\beta_0 + \gamma_1 m)\bar{h}^2\right\}$$

$$+r^n \left\{\frac{(\beta_0 + \gamma_1 m^2)\bar{h}^3}{2} + \frac{(\beta_0 + \gamma_1 m^3)\bar{h}^4}{6}\right\}$$

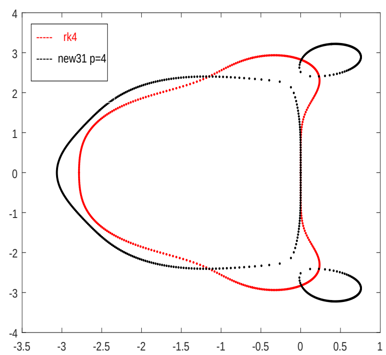
$$\Rightarrow r = 1 + \bar{h}(\beta_0 + \beta_1 + \gamma_1) + (\beta_0 + \gamma_1 m)\bar{h}^2$$

$$+\frac{(\beta_0 + \gamma_1 m^2)\bar{h}^3}{2} + \frac{(\beta_0 + \gamma_1 m^3)\bar{h}^4}{6}. \tag{32}$$

We show the stability of the method New 3_1, where $p_1 = 3$. in Figure. 1 and compared Runge- Kutta of Rank 3 method. It can be seen that the stability zone of the new method is larger, and this proves the efficiency of the new method. We show the stability of the method New 3_1, where $p_1 = 4$. in Figure. 1 and We compared Runge- Kutta of Rank 4 methods. It can be seen that the stability zone of the new method is larger, and this proves the efficiency of the new method. We show the stability of the method New 3_2, where $p_1 = 3$ in Figure. 2 and compared Runge- Kutta of rank 3 methods. It can be seen that the stability region of the new method is larger, and this proves the efficiency of the new method, and we show the stability of the method New 3_2 where $p_1 = 4$ in Figure 2 and compared Runge- Kutta of rank 4 method. It can be seen that the stability region of the new method is larger. We show the stability of the method New4, where $p_1 = 3$ in Figure. 3 and compared Runge- Kutta of rank 3 method. It can be seen that the stability region of the new method is larger, and this proves the efficiency of the new method, and we show the stability of the method New4 where $p_1 = 4$ in Figure 3 and compared Runge- Kutta of rank 4 method. It can be seen that the stability region of the new method is larger.



(a)



(b)

Figure 1: (a) Stability region of rk3 and new 3_1 methods where $p_1 = 3$. (b) Stability region of rk4 and new3_1 methods where $p_1 = 4$.

4 Numerical results using FBSM and new methods:

Example 4.1. Consider the following optimal control problem:

$$\begin{aligned} &\min_u \int_0^1 3x(t)^2 + u(t)^2 dt \\ &st. \quad x'(t) = x(t) + u(t), \quad x(0) = 1. \end{aligned}$$

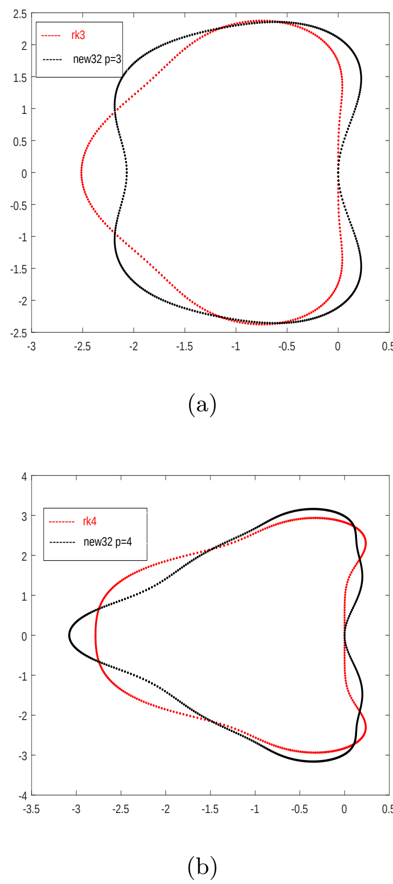
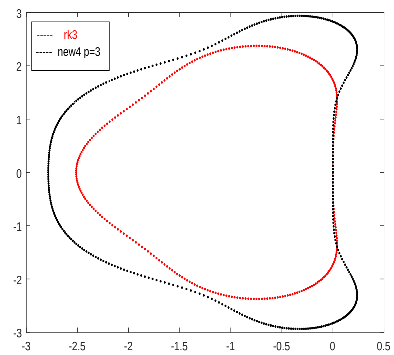


Figure 2: (a) Stability region of rk3 and new3_2 methods where $p_1 = 3$. (b) Stability region of rk4 and new3_2 methods where $p_1 = 4$.

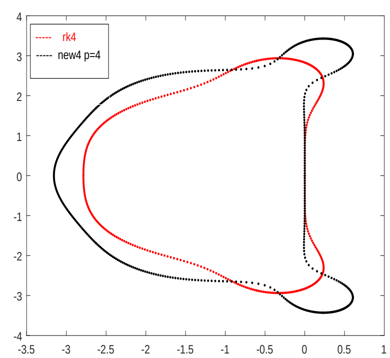
Analytical solutions which are as follows: [9]

$u^*(t) = \frac{3e^{-4}}{3e^{-4}+1}e^{2t} - \frac{3}{3e^{-4}+1}e^{-2t}$, $x^*(t) = \frac{3e^{-4}}{3e^{-4}+1}e^{2t} + \frac{1}{3e^{-4}+1}e^{-2t}$

The state variable at the end point is $x = 0.51314537669$. And variable control endpoint is equal to zero. Matlab implementation of the three methods of Example 4.1 was determined as follows. The results are shown in Figure 4 with $h = \frac{1}{10}$ and in Tables $1, 2$.These results show that the new methods are more accurate than the Runge-Kutta of rank



(a)



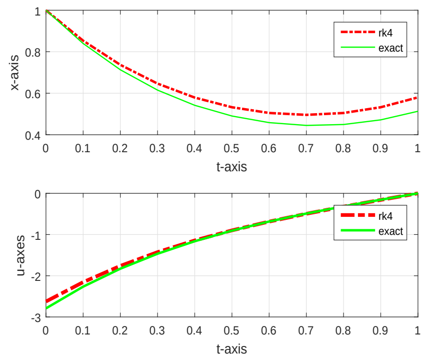
(b)

Figure 3: (a) Stability region of rk3 and new34 methods where $p_1 = 3$. (b) Stability region of rk4 and new4 methods where $p_1 = 4$.

3 and 4

Example 4.2. Consider the following optimal control problem with a payoff term:

$$\begin{aligned} &\min_u \; x(T) + \int_0^T u(t)^2 dt \\ &\text{st} \quad x'(t) = \alpha x(t) - u(t), \; x(0) = x_0 > 0 \end{aligned}$$



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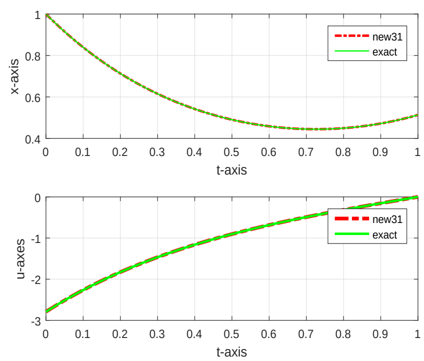


(b)

Figure 4: (a) The optimal state and control of Example 4.1(Rk4). (b) The optimal state and control values of Example 4.1 (new3_1, where $p_1 = 4$)

The analytical solution of this problem is as follows [9]:

$$H = u^2 + \lambda(\alpha x - u) \ , \quad \frac{\partial H}{\partial u} = 2u - \lambda = 0 \quad at \quad u^* \quad \Rightarrow u^* = \frac{\lambda}{2} \ ,$$
$$\lambda' = -\frac{\partial H}{\partial x} = -\partial\lambda \Rightarrow \lambda = ce^{-\alpha t} \ , \quad \lambda(T) = 1 \Rightarrow \lambda(t) = e^{\alpha(T-t)} \ \Rightarrow u^*(t) = \frac{e^{\alpha(T-t)}}{2},$$

$$x' = \alpha x - u = \alpha x - u = \alpha x - \frac{e^{\alpha(T-t)}}{2}, x(0) = x_0. \Rightarrow x^*(t) = x_0 e^{\alpha t} + e^{\alpha T}\frac{e^{-\alpha t} - e^{\alpha t}}{4\alpha}.$$

The numerical solution of this problem related to x (t) , and u(t) are obtained and their results have been plotted in figure 5 with $\alpha = 2, h = \frac{1}{30}$:

Table 1: End Error of state values in Example 4.1 where $p_1 = 4$

New3_2	New4	New3_1	FBSM_rk4	FBSM_Rk3	h
1.2332e-6	1.1930e-5	1.4443e-4	7.1135e-3	7.1134e-3	$\frac{1}{100}$
3.6697e-7	2.9247e-6	6.9423e-5	3.5690e-3	3.5960e-3	$\frac{1}{200}$
1.1662e-7	1.2755e-6	4.5676e-5	2.3820e-3	2.3820e-3	$\frac{1}{300}$
7.4674e-8	2.9113e-7	2.2526e-5	1.1924e-3	1.1924e-3	$\frac{1}{600}$

Table 2: End Error of state values in Example 4.1 where $p_1 = 3$

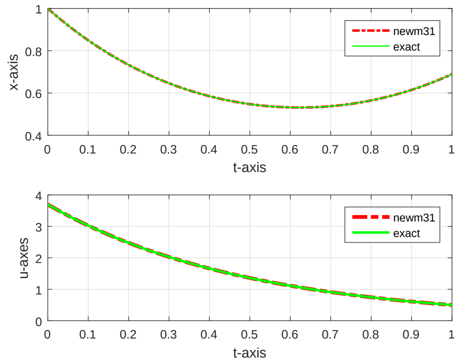
New3_2	New4	New3_1	FBSM_rk4	FBSM_Rk3	h
1.4565e-5	4.6274e-6	4.6161e-6	7.1135e-3	7.1134e-3	$\frac{1}{100}$
3.7039e-6	1.1471e-6	1.1457e-6	3.5690e-3	3.5960e-3	$\frac{1}{200}$
1.6712e-6	5.6625e-7	5.6583e-7	2.3820e-3	2.3820e-3	$\frac{1}{300}$
3.7482e-7	1.6944e-7	1.6939e-7	1.1924e-3	1.1924e-3	$\frac{1}{600}$

These results show that the new methods are more accurate than the Runge-Kutta of rank 4 method.

Example 4.3. Consider the example 4.2.
The numerical solution of Example 2 related to x (t) , and u(t) are obtained and their results have been plotted in figure 6 with $\alpha = -30, h = \frac{1}{40}$:

Table 3: End Error for state values of Example 4.2 (Alpha=2,eta=.000001, $p_1 = 4$)

New4	New3_2	New3_1	Rk4	h
2.7526-4	2.6289e-4	2.8447e-4	7.6088e-2	$\frac{1}{200}$
4.5617e-5	4.3631e-5	4.7100e-5	3.0713e-2	$\frac{1}{500}$
1.2621e-5	1.2122e-5	1.2993e-5	1.5404e-2	$\frac{1}{1000}$
1.7077e-6	1.7027e-6	1.7112e-6	1.5460e-3	$\frac{1}{2000}$



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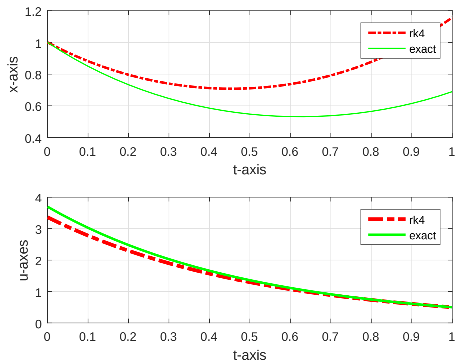


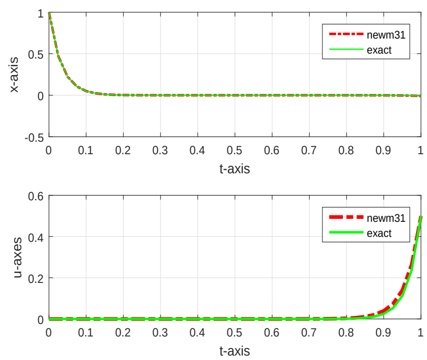
(b)

Figure 5: (a) The optimal state and control values of Example 4.2 (new3_1, where $p_1 = 4$). (b) The optimal state and control values of Example 4.2 (rk4).

These results show that the new methods are more accurate than the Runge-Kutta of rank 4 method.

Example 4.4. Consider the following optimal control problem:

$$\begin{array}{l} \min_u \int_0^1 \frac{5}{8}x(t)^2 + \frac{1}{2}x(t)u(t) + \frac{1}{2}u(t)^2 dt \\ st. \quad x'(t) = \frac{1}{2}x(t) + u(t), \quad x(0) = 1. \end{array}$$



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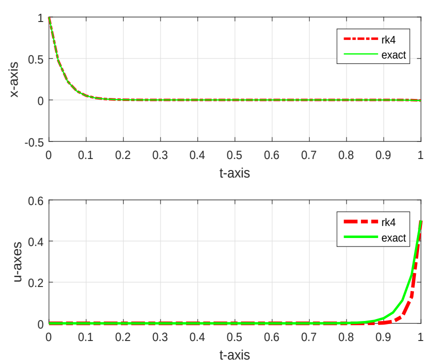


(b)

Figure 6: (a) The optimal state and control values for Example 4.3(new3_1,$p_1 = 4$). (b) The optimal state and control values of Example 4.3(rk4, $p_1 = 4$).

Analytical solutions which are as follows [12]:

$u^*(t) = -\frac{(tanh(1-t)+.5)cosh(1-t)}{cosh(1)}$, $x^*(t) = \frac{cosh(1-t)}{cosh(1)}$

The state variable at the end point is $x(1) = 6.4805427366388e-1$. And variable control endpoint is $u(1) = -3.24027136831e-1$. Matlab implementation of the three methods of Example 4.5 was determined as follows. The results are shown in Figure 7 with $h = \frac{1}{10}$ and in Tables $6-9$.These results show that the new methods are more accurate than

Table 4: End error for state values of Example 4.3 (Alpha=-30,eta=.000001, where $p_1 = 4$

New4	New3_2	New3_1	Rk4	h
1.0885e-5	1.3666e-5	7.8909e-6	3.0501e-4	$\frac{1}{200}$
1.6421e-6	2.0708e-6	1.2004e-6	1.2132e-4	$\frac{1}{500}$
4.0217e-7	5.0787e-7	2.9488e-7	6.0541e-5	$\frac{1}{1000}$
9.8872e-8	1.2510e-7	7.2440e-8	3.0240e-5	$\frac{1}{2000}$

Table 5: End Error of state values in Example 4.4 where $p_1 = 4$

New3_2	New4	New3_1	FBSM_rk4	FBSM_Rk3	h
3.4818e-4	4.0739e-4	3.4143e-3	1.0965e-2	1.0936e-2	$\frac{1}{10}$
4.2119e-5	4.2737e-5	3.5529e-4	1.2195e-3	1.2194e-4	$\frac{1}{100}$
2.1162e-5	2.1317e-5	1.7793e-4	6.1325e-4	6.1325e-4	$\frac{1}{200}$
1.4106e-5	1.4175e-5	1.1866e-4	4.0959e-4	4.0959e-4	$\frac{1}{300}$
7.0160e-6	7.0333e-6	5.9314e-5	2.0514e-4	2.0514e-4	$\frac{1}{600}$
4.1705e-6	4.1767e-6	3.5554e-5	1.2313e-4	1.2313e-4	$\frac{1}{1000}$

the Runge-Kutta of rank 3 and 4.

Example 4.5. .Consider the following optimal control problem[6]

$$\min_u \int_0^3 (Ax_3 + u^2)^2 dt$$
$$st. \quad \begin{aligned} {x_1}' &= b - b(px_2 + qx_2) - bx_1 - \beta x_1 x_3 - ux_1,, \\ {x_2}' &= bpx_2 + \beta x_1 x_3 - (e+b)x_2, \\ {x_3}' &= ex_2 - (g+b)x_3, \\ {x_4}' &= b - bx_4. \end{aligned}$$

With initial conditions $x1(1) = 0.0555$, $x2(1) = 0.0003$, $x3(1) = 0.00041$, $x4(1) = 1$ and the parameters $b = 0.012$, $p = 0.65$, $q = 0.65$, $\beta = 527.59$, $e = 36.5$, $g = 30.417$ and $A = 100$.

It is not easy to solve this, analytically and it is necessary to use numerical method.

The results in the table 9 and figures 8, 9 shows that the new methods

Table 6: End Error of state values in Example 4.4 where $p_1 = 3$

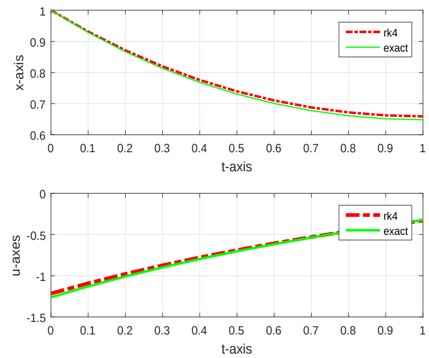
New3_2	New4	New3_1	FBSM_rk4	FBSM_Rk3	h
2.3723e-4	7.5223e-5	5.2716e-4	1.0965e-2	1.0936e-2	$\frac{1}{10}$
2.4718e-6	8.9277e-7	8.0464e-5	1.2195e-3	1.2194e-4	$\frac{1}{100}$
5.3996e-7	1.4577e-7	4.0935e-5	6.1325e-4	6.1325e-4	$\frac{1}{200}$
1.8050e-7	5.3951e-9	2.7422e-5	4.0959e-4	4.0959e-4	$\frac{1}{300}$
3.5639e-8	7.9314e-8	1.3741e-5	2.0514e-4	2.0514e-4	$\frac{1}{600}$
8.1824e-8	9.7573e-8	8.2219e-6	1.2313e-4	1.2313e-4	$\frac{1}{1000}$

Table 7: End Error of control values in Example 4.4 where $p_1 = 4$

New3_2	New4	New3_1	FBSM_rk4	FBSM_Rk3	h
1.7442e-4	2.0403e-4	1.7074e-3	5.4830e-3	5.4687e-3	$\frac{1}{10}$
2.1387e-5	2.1696e-5	1.7796e-4	6.1008e-4	6.1007e-4	$\frac{1}{100}$
1.0909e-5	1.0986e-5	8.9291e-5	3.0695e-4	3.0695e-4	$\frac{1}{200}$
7.3809e-6	7.4154e-6	5.9656e-5	2.0512e-4	2.0512e-4	$\frac{1}{300}$
3.8358e-6	3.8444e-6	2.9983e-5	1.0289e-4	1.0289e-4	$\frac{1}{600}$
2.4130e-6	2.4161e-6	1.8104e-5	6.1896e-5	6.1896e-5	$\frac{1}{1000}$

Table 8: End Error of control values in Example 4.4 where $p_1 = 3$

New3_2	New4	New3_1	FBSM_rk4	FBSM_Rk3	h
1.1895e-4	3.7945e-5	2.6391e-4	5.4830e-3	5.4687e-3	$\frac{1}{10}$
1.5637e-6	7.7423e-7	4.0560e-5	6.1008e-4	6.1007e-4	$\frac{1}{100}$
5.9778e-7	4.0069e-7	2.0795e-5	3.0695e-4	3.0695e-4	$\frac{1}{200}$
4.1804e-7	3.3049e-7	1.4039e-5	2.0512e-4	2.0512e-4	$\frac{1}{300}$
3.0997e-7	2.8809e-7	7.1985e-6	1.0289e-4	1.0289e-4	$\frac{1}{600}$
2.8687e-7	2.7900e-7	4.4387e-6	6.1896e-5	6.1896e-5	$\frac{1}{1000}$



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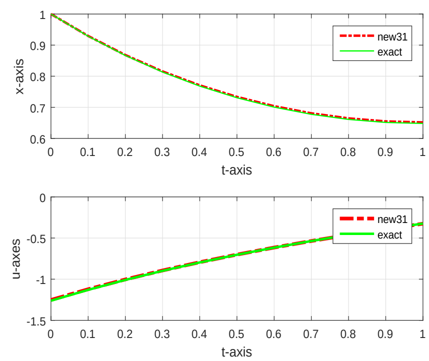

(b)

Figure 7: (a) The optimal state and control values of Example 4.4(Rk4). (b) The optimal state and control of Example 4.4 (new3_1, where $p_1 = 4$)

compete with Runge-Kutta methods in stiff and several variable problems.

5 Conclusion

In this work, three hybrid methods of orders 3 and 4 are presented and then, Runge - Kutta methods of orders 3 and 4 are used as predictor

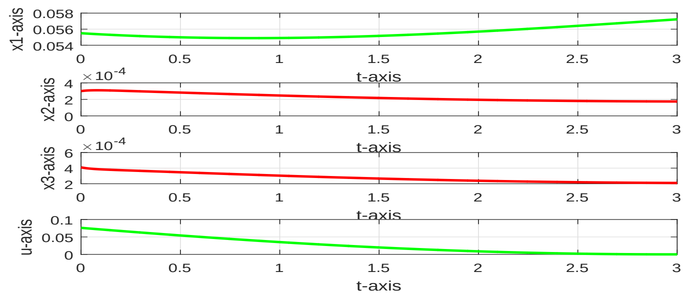

Figure 8: The optimal curves of the problem in Example 4.5(rk4)

Table 9: The optimal state values of methods in Example 4.5, $p_1 = 4$

Method	x1	x2	x3	x4
Rk4	0.057229233323	0.000174026727	0.000209796082	0
New3_1	0.057201314198	0.000175317514	0.000211408161	0
New3_2	0.057187036786	0.000176015581	0.000212278533	0
New4	0.057201267178	0.000175314688	0.000211404848	0

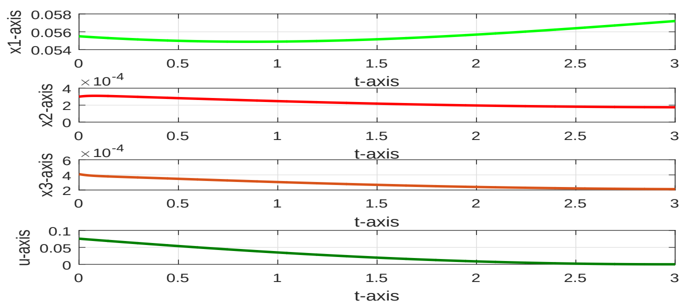

Figure 9: The optimal curves of the problem in Example 4.5(new3_1, $p_1 = 4$)

schemes to gain whole methods of the same orders. In section 2, order of truncation errors are investigated for the explicit hybrid based on Rune - Kutta methods. Then, stability analysis of the methods are discussed which shows that the stability domains of them are wider compared to explicit Rune - kutta methods of orders 3 and 4. Finally, five examples of optimal control problems are solved by using Matlab, FBSM scheme and presented methods. Numerical results to solve the examples presented by Tables $1 - 9$ and therefore one can conclude that hybrid methods, have a good performance in getting small end errors in solving optimal control problems numerically.

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