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Differential Transform Method for Solving the Linear and Nonlinear Westervelt Equation

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Abstract. In this paper, a differential transform method (DTM) is used to find the numerical solution of the linear and nonlinear Westervelt equation. Exact solution can also be achieved by the known forms of the series solution. In this paper, we present the definition and operation of the three-dimensional differential transform and investigate the particular exact solution of system of partial differential equations that usually arise in applied mechanic by a three-dimensional differential transform method. The numerical result of the present method is presented and compared with the exact solution that is calculated by the Laplace transform method.

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1. Introduction

Nonlinear wave phenomena appear in various scientific and engineering fields, such as fluid mechanics, plasma physics, optical fibers, biology, solid state physics, chemical kinematics, chemical physics and geochemistry. A variety of powerful methods has been presented, such as the inverse scattering transform ([1]), $(\frac{G'}{G})$-expansion method ([2]), Laplace

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Adomian decomposition method ([3]), homotopy analysis method ([4,5]) variational iteration method ([6]), Adomian decomposition method ([6]), homotopy perturbation method ([7]), Exp-function method ([8]) and so on. The concept of differential transform was first introduced by Zhou ([9]), who solved linear and nonlinear initial value problems in electric circuit analysis. Chen and Ho ([10]) developed this method for PDEs and obtained closed form series solutions for linear and nonlinear initial value problems. Jang ([11]) states that the differential transform is an iterative procedure for obtaining Taylor series solutions of differential equations. This method reduces the size of computational domain and applicable to many problems easily. The linear Westervelt equation describing the propagation of finite amplitude sound has the following form [12]

$$\nabla^2 \mathrm{p} - \frac{1}{\mathrm{c}_0^2}\frac{\partial^2 \mathrm{p}}{\partial \mathrm{t}^2} = 0, \qquad \nabla^2 = \frac{\partial^2}{\partial \mathrm{x}^2} + \frac{\partial^2}{\partial \mathrm{y}^2}, \tag{1}$$

where p is the acoustic pressure, ρ_0 and c_0 are the ambient density and sound speed respectively. The first two terms in Eq. (2), the D'Alembertian operator acting on the acoustic pressure, describe linear lossless wave propagation at the small-signal sound speed. The final term describes nonlinear distortion of the wave due to finite-amplitude effects. The Westervelt equation describing the propagation of finite amplitude sound has the following form [12]

$$\nabla^2 \mathrm{p} - \frac{1}{\mathrm{c}_0^2}\frac{\partial^2 \mathrm{p}}{\partial \mathrm{t}^2} + \frac{\beta}{\rho_0 \mathrm{c}_0^4}\frac{\partial^2 \mathrm{p}^2}{\partial \mathrm{t}^2} = 0, \tag{2}$$

where p is the acoustic pressure, ρ_0 and c_0 are the ambient density and sound speed, respectively. $\beta = 1 + (B/2A)$ is the nonlinearity coefficient for the fluid and B/A is the nonlinearity parameter. The first two terms in Eq. (2), theD'Alembertian operator acting on the acoustic pressure, describe linear lossless wave propagation at the small-signal sound speed. The final term describes nonlinear distortion of the wave due to finite-amplitude effects. If the medium is assumed to be a thermoviscous fluid, the Westervelt equation (Eq. (2)) takes the following form [13]

$$\nabla^2 \mathrm{p} - \frac{1}{\mathrm{c}_0^2}\frac{\partial^2 \mathrm{p}}{\partial \mathrm{t}^2} + \frac{\delta}{\mathrm{c}_0^4}\frac{\partial^3 \mathrm{p}}{\partial \mathrm{t}^3} + \frac{\beta}{\rho_0 \mathrm{c}_0^4}\frac{\partial^2 \mathrm{p}^2}{\partial \mathrm{t}^2} = 0. \tag{3}$$

The additional term is a loss term, which is due to the thermal conduction and the viscosity of the fluid. Here δ is the diffusivity of sound; in a thermoviscous fluid, the absorption coefficient α is related to δ and $\varpi = 2\pi f$ by

$$\delta = \frac{2c_0^3\alpha}{\varpi^2}. \qquad (4)$$

The absorption coefficient is a constant, specific to a single frequency.
It is interesting to point out that Eq. (3) has attracted a considerable amount of research work such as in [12,13,20-24]. The usual fourth-order finite difference representation of the second partial derivative would lead t an unconditionally unstable scheme, a technique given by Cohen ([22]) based on the "modified equation approach" to obtain fourth- order accuracy in time was used.This technique, while improving the accuracy in time, preserves the simplicity of the second- order accurate time-step scheme. This was performed in order that all derivatives have the same order of accuracy as the boundary conditions which are also fourth order. It was found ([21]) that lower order boundary conditions did not suppress spurious reflections at the computational boundary that could contaminate low amplitude signals. A technique known as the complementary operator method(COM) was utilized ([23]) in the development of the absorbing boundary conditions(ABC). This technique was first utilized in electromagnetics where it was shown to yield excellent results ([23]). The COM method is a differential equation-based ABC method, and differs from the other common approach of terminating the grid with the use of an absorbing material. An example of this type of boundary condition is the perfectly matched layer(PML) method originally proposed by Berenger ([24]). The COM is based on one-way wave equations such as Higdon's boundary operators ([24]). Also, In [20] Norton et. al investigated he Westervelt equation with viscous attenuation versus a causal propagation operator and also have done a numerical comparison.
The article is organized as follows: In Section 2, first we briefly give the steps of the method and apply the method to solve the nonlinear partial differential equations. In Section 3 linear Westervelt equation will be introduced briefly and obtained exact solutions for related equations. In Section 4 we Summarize our results. Finally some references are given at the end of this paper.

2. Basic Idea of Differential Transform Method

The basic definitions and fundamental operations of the three-dimensional differential transform are given in [10,11,14,16-19] as follows: Consider a function of two variable $w(x, y, t)$, be analytic in the domain K and let $(x, y, t) = (x_0, y_0, t_0)$ in this domain. The function $w(x, y, t)$ is then represented by one series whose center is located at (x_0, y_0, t_0). The differential transform of the function w(x, y, t) has the form

$$W(k,h,m)=\frac{1}{k!h!m!}\left[\frac{\partial^{k+h+m}w(x,y,t)}{\partial^{k}x\partial^{h}y\partial^{m}t}\right], \tag{5}$$

where w(x, y, t) is the original function and W(k, h, m) is the transformed function. The differential inverse transform of W(k, h, m) is defined as

$$w(x,y,t)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\sum_{p=0}^{\infty}W(k,h,m)(x-x_0)^{k}(y-y_0)^{h}(t-t_0)^{m}. \tag{6}$$

In a real application, and when (x_0, y_0, t_0) are taken as (0, 0, 0), then the function w(x, y, t) is expressed by a finite series and Eq. (6) can be written as

$$w(x,y,t)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\sum_{p=0}^{\infty}\frac{1}{k!h!m!}\left[\frac{\partial^{k+h+m}w(x,t)}{\partial^{k}x\partial^{h}y\partial^{m}t}\right]x^{k}y^{h}t^{m}. \tag{7}$$

We recall the following properties from [16]. Then collect all these 3 theorems in one theorem, as 3 different parts (i), (ii), and (iii).

Theorem 2.1.
(i) If $w(x,y,t)=u(x,y,t)\pm v(x,y,t)$, *then*

$$W(k,h,m)=U(k,h,m)\pm V(k,h,m).$$

(ii) If $w(x,y,t)=cu(x,y,t)$, *(where* c *is a constant) then*

$$W(k,h,m)=cU(k,h,m).$$

(iii) If $w(x,y,t)=\frac{\partial u(x,y,t)}{\partial x}$, *then*

$$W(k,h,m)=(k+1)U(k+1,h,m).$$

Theorem 2.2.
(i) If $w(x,y,t)=\frac{\partial^2 u(x,y,t)}{\partial x^2}$, *then*

$$W(k,h,m)=(k+1)(k+2)U(k+2,h,m).$$

Proof. By Definition 5, we write

$$W(k,h,m)=\frac{1}{k!h!m!}\left[\frac{\partial^{k+h+m}}{\partial x^{k}\partial y^{h}\partial t^{m}}\left[\frac{\partial^{2}u(x,y,t)}{\partial x^{2}}\right]\right],$$

$$W(k,h,m) = \frac{(k+1)(k+2)}{(k+2)!h!m!}\left[\frac{\partial^{k+h+m}}{\partial^{k+2}x\partial^{h}y\partial^{m}t}u(x,y,t)\right],$$

then

$$W(k,h,m) = (k+1)(k+2)U(k+2,h,m). \quad \square$$

(ii) If $w(x,y,t) = \frac{\partial^2 u(x,y,t)}{\partial y^2}$, then

$$W(k,h,m) = (h+1)(h+2)U(k,h+2,m).$$

Proof. In the same manner, proof can be concluded by using Theorem 2.2 section (i). $\square$

(iii) If $w(x,y,t) = \frac{\partial^2 u(x,y,t)}{\partial t^2}$, then

$$W(k,h,m) = (m+1)(m+2)U(k,h,m+2).$$

Proof. Also, in the same manner, proof can be concluded by using Theorem 2.2 section (i). $\square$

Theorem 2.3. *If* $w(x,y,t) = \frac{\partial^{r+s+p}u(x,y,t)}{\partial x^r \partial y^s \partial t^p}$, *then*

$$W(k,h,m) = (k+1)(k+2)...(k+r)(h+1)(h+2)...(h+s)$$
$$(m+1)(m+2)...(m+p)U(k+r,h+s,m+p).$$

Proof. By Definition 5, we write

$$W(k,h,m) = \frac{1}{k!h!m!}\left[\frac{\partial^{k+h+m}}{\partial x^k \partial y^h \partial t^m}\left[\frac{\partial^{r+s+p}u(x,y,t)}{\partial x^r \partial y^s \partial t^p}\right]\right],$$

$$= \frac{(k+1)...(k+r)(h+1)...(h+s)(m+1)...(m+p)}{(k+r)!(h+s)!(m+p)!}$$

$$\left[\frac{\partial^{k+r+h+s+m+p}}{\partial x^{k+r}\partial y^{h+s}\partial t^{m+p}}u(x,y,t)\right],$$

then

$$W(k,h,m) = (k+1)(k+2)...(k+r)(h+1)(h+2)...(h+s)$$
$$(m+1)(m+2)...(m+p)U(k+r,h+s,m+p). \quad \square$$

Theorem 2.4. *If* $w(x,y,t) = u(x,y,t)\frac{\partial^2 u(x,y,t)}{\partial t^2}$, *then*

$$W(k,h,m) = \sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{q=0}^{m}(m-q+1)(m-q+2)$$

$$P(k-r,h-s,m-q)P(r,s,m-q+2).$$

Theorem 2.5. *If* $w(x,y,t) = \frac{\partial u(x,y,t)}{\partial t}\frac{\partial u(x,y,t)}{\partial t}$, *then*

$$W(k,h,m) = \sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{q=0}^{m}(q+1)(m-q+1)$$

$$P(k-r,h-s,m-q+1)P(r,s,m-q).$$

3. Application

To illustrate the effectiveness of the present method, two examples are considered in this section.

Example 1. We first consider the linear Westervelt equation

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} - \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} + \frac{\delta}{c_0^4}\frac{\partial^3 p}{\partial t^3} = 0, \tag{8}$$

subject to the initial condition of

$$p(x,y,0) = \exp[\lambda(x+y)], \qquad \lambda = \frac{(2c_0^2-1)c_0^2}{\delta}, \tag{9}$$

The transformed version of Eq. (8) is

$$(k+1)(k+2)P(k+2,h,m) + (h+1)(h+2)P(k,h+2,m) \tag{10}$$

$$-\frac{1}{c_0^2}(m+1)(m+2)P(k,h,m+2)$$

$$+\frac{\delta}{c_0^4}(m+1)(m+2)(m+3)P(k,h,m+3) = 0.$$

The transformed version of Eq. (9) is

$$P(k,h,0)=\frac{\lambda^k h^m}{k!h!,} \qquad k,h=0,1,2,... \tag{11}$$

By substituting (11) in (10), we obtain by the closed form series solution as

$$p(x,y,t)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\sum_{p=0}^{\infty}P(k,h,m)x^k y^h t^m=\left(1+\frac{(\lambda x)^1}{1!}+\frac{(\lambda x)^2}{2!}+...\right) \tag{12}$$

$$\left(1+\frac{(\lambda y)^1}{1!}+\frac{(\lambda y)^2}{2!}+...\right)\left(1-\frac{(\lambda t)^1}{1!}+\frac{(\lambda t)^2}{2!}-...\right)=\exp\left[\lambda(x+y-t)\right]$$

which is the exact solution.

Example 2. We first consider the nonlinear Westervelt equation

$$\nabla^2 p-\frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2}+\frac{\delta}{c_0^4}\frac{\partial^3 p}{\partial t^3}+\frac{\beta}{\rho_0 c_0^4}\frac{\partial^2 p^2}{\partial t^2}=0. \tag{13}$$

subject to the initial condition of

$$p(x,y,0)=\frac{\rho_0}{2\beta}(c_0^2-2c_0^4)+\frac{\rho_0\delta}{\beta}\tanh\left[(x+y)\right], \tag{14}$$

The transformed version of Eq. (13) by using of Theorem 2.5 is

$$\begin{aligned}&(k+1)(k+2)P(k+2,h,m)+(h+1)(h+2)P(k,h+2,m)\\&-\frac{1}{c_0^2}(m+1)(m+2)P(k,h,m+2)\\&+\frac{\delta}{c_0^4}(m+1)(m+2)(m+3)P(k,h,m+3)+\\&\frac{2\beta}{\rho_0 c_0^4}\left(\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{q=0}^{m}(q+1)(m-q+1)P(k-r,h-s,m-q+1)P(r,s,m-q)+\right.\\&\left.\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{q=0}^{m}(m-q+1)(m-q+2)P(k-r,h-s,m-q)P(r,s,m-q+2)\right)=0.\end{aligned} \tag{15}$$

The transformed version of Eq. (14) is

$$P(k,h,0)=\frac{\rho_0}{2\beta}(c_0^2-2c_0^4)+\frac{\rho_0\delta}{\beta}\left(1-\frac{2}{3!}+\frac{16}{5!}+\frac{16}{7!}-\frac{3584}{9!}+...\right) \tag{16}$$

By substituting (16) in (15), we obtain the closed form series solution as

$$p(x,y,t)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\sum_{p=0}^{\infty}P(k,h,m)x^{k}y^{h}t^{m}=\frac{\rho_0}{2\beta}(c_0^2-2c_0^4)+\frac{\rho_0\delta}{\beta}$$

$$\left((x+y+t)-\frac{2(x+y+t)^3}{3!}+\right). \tag{17}$$

$$\left(\frac{16(x+y+t)^5}{5!}+\frac{16(x+y+t)^7}{7!}-\frac{3584(x+y+t)^9}{9!}+...\right)$$

$$=\frac{\rho_0}{2\beta}(c_0^2-2c_0^4)+\frac{\rho_0\delta}{\beta}\tanh\left[(x+y+t)\right]$$

which is the exact solution.

4. Conclusion

Three-dimensional differential transform has been applied to linear Westervelt equation. We obtained the exact solution for aforementioned equations. Using the differential transform method, the solution of the equation of partial differential equation can be obtained in Taylor's series form. All the calculations in the method are very easy. The present study has confirmed that the differential transform method offers significant advantages in terms of its straightforward applicability, its computational effectiveness and its accuracy. The calculated results are quite reliable. Therefore, this method can be applied to many complicated linear and nonlinear PDEs.

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