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Fixed Point Theorems in C^*-Algebra-Valued $b_v(s)$-Metric Spaces with Application and Numerical Methods

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Abstract. We first introduce a novel notion named C^*-algebra-valued $b_v(s)$-metric spaces. Then, we give proofs of the Banach contraction principle, the expansion mapping theorem, and Jungck's theorem in C^*-algebra-valued $b_v(s)$-metric spaces. As an application of our results, we establish a result for an integral equation in a C^*-algebra-valued $b_v(s)$-metric space. Finally, a numerical method is presented to solve the proposed integral equation, and the convergence of this method is also studied. Moreover, a numerical example is given to show applicability and accuracy of the numerical method and guarantee the theoretical results.

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1 Introduction and Preliminaries

Banach contraction principle [4] was introduced by Banach in 1922, and later it is called the fixed point theorem. The fixed point theorem is a strong tool for solving existence problems in many branches of mathematics and physics.
Bakhtin [3] introduced b-metric spaces as a generalization of metric spaces and proved analogue of the Banach contraction principle in b-metric spaces. In the paper [6], Branciari introduced the concept of v-generalized metric spaces. Radenović and Mitrović [12] introduced the concept $b_v(s)$-metric spaces as a generalization of metric spaces, b-metric spaces, and v-generalized metric spaces. On the other hand, Ma et al. [11] presented the concept of C^*-algebra-valued metric spaces. Later, this line of research was continued in [1, 5, 8, 9, 10, 14, 15, 16], where several other fixed point results were obtained in the framework of C^*-algebra-valued metric, as well as (more general) C^*-algebra-valued b-metric spaces. Now, in this paper, we introduce a new notion C^*-algebra-valued $b_v(s)$-metric spaces. Then, we prove the Banach contraction principle, expansion mapping theorem, and Jungck's theorem [2] for C^*-algebra-valued $b_v(s)$-metric spaces. Also, we state a result for an integral equation in a C^*-algebra-valued $b_v(s)$-metric space, which demonstrates an application of our main theorem. Finally, we propose a numerical method for solving the integral equation and investigate the convergence of this method. Moreover, to illustrate an application and accuracy, we present a numerical example, which guarantees the theoretical results.

We provide some auxiliary facts which will be used in the rest of the paper. Throughout this paper, $\mathbb{A}$ always denotes a unital C^*-algebra with a unit $1_{\mathbb{A}}$. We call an element $a \in \mathbb{A}$ a *positive element*, denoted $a \succeq 0_{\mathbb{A}}$, if $a \in \mathbb{A}_h$ and $\sigma(a) \subseteq \mathbb{R}_+ = [0, +\infty)$, where $\mathbb{A}_h = \{a \in \mathbb{A} : a^* = a\}$. The set $\mathbb{A}_+$ indicates the positive elements of $\mathbb{A}$. Also, $\mathbb{A}' = \{a \in \mathbb{A} : xa = ax, \text{for all } x \in \mathbb{A}\}$.

Lemma 1.1. *[13] Let $\mathbb{A}$ be a unital C^*-algebra with unit $1_{\mathbb{A}}$.*

(1) If $a, b \in \mathbb{A}_h$ with $a \preceq b$ and $c \in \mathbb{A}$, then $c^*ac \preceq c^*bc$.

(2) For all $a, b \in \mathbb{A}_h$, if $0_{\mathbb{A}} \preceq a \preceq b$, then $\|a\| \preceq \|b\|$.

Lemma 1.2. *[7, 13] Suppose that $\mathbb{A}$ is a unital C^*-algebra with unit $1_{\mathbb{A}}$.*

(1) For any $x \in \mathbb{A}_+$, it follows that $x \preceq 1_{\mathbb{A}}$ if and only if $\|x\| \preceq 1$.

(2) If $a \in \mathbb{A}_+$ with $\|a\| \prec \frac{1}{2}$, then $1_{\mathbb{A}} - a$ is invertible and $\|a(1_{\mathbb{A}} - a)^{-1}\| \prec 1$.

(3) Suppose that $a, b \in \mathbb{A}_+$ with $ab = ba$; then $ab \succeq 0_{\mathbb{A}}$.

(4) For $a \in \mathbb{A}'$, if $b \succeq c \succeq 0_{\mathbb{A}}$ and $1_{\mathbb{A}} - a \in \mathbb{A}'_+$ is an invertible element, then $(1_{\mathbb{A}} - a)^{-1}b \succeq (1_{\mathbb{A}} - a)^{-1}c$.

2 Main results

Definition 2.1. Let X be a nonempty set and let $\mathbb{A}$ be a C^*-algebra. The mapping $d : X \times X \to \mathbb{A}_+$ is called *C^*-algebra-valued $b_v(s)$-metric*, if there exists $s \in \mathbb{A}'_+$ with $\|s\| \succeq 1$ such that d satisfies

(1) $d(x, y) = 0_{\mathbb{A}}$ if and only if $x = y$ for all $x, y \in X$.

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$.

(3) $d(x, y) \preceq s[d(x, u_1) + d(u_1, u_2) + \cdots + d(u_{v-1}, u_v) + d(u_v, y)]$, for all $x, y \in X$ and for all distinct elements $u_1, u_2, \ldots, u_v \in X - \{x, y\}$ in which $v \in \mathbb{N}$.

Definition 2.2. Suppose that $(X, \mathbb{A}, d)$ is a C^*-algebra-valued $b_v(s)$-metric space. Then $T : X \to X$ is called a *C^*-algebra-valued contractive mapping*, if there exists $B \in \mathbb{A}$ with $\|B\| \prec 1$ such that

$$d(Tx, Ty) \preceq B^* d(x, y) B \quad \text{for all } x, y \in X. \tag{1}$$

Example 2.3. Let $X = \ell^p = \{x = \{x_n\} \subseteq \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^p \prec +\infty\}$, $p \in (0, 1)$, and let $\mathbb{A} = \mathbb{M}_{2\times 2}(\mathbb{R})$.
Define $d(x, y) = \mathrm{diag}\Big((\sum_{n=1}^{\infty} |x_n - y_n|^p)^{\frac{1}{p}}, (\sum_{n=1}^{\infty} |x_n - y_n|^p)^{\frac{1}{p}}\Big)$ in which "diag" denotes a diagonal matrix and $x, y \in X$. It is easy to verify that $d(.,.)$ is a C^*-algebra-valued $b_v(s)$-metric. For proving (3) of Definition 2.1 with $v = 2$, we only need to use the following inequality:

$$\Big(\sum_{n=1}^{\infty}|x_n-y_n|^p\Big)^{\frac{1}{p}}$$

$$\preceq 2^{(\frac{2}{p})}\Big[\Big(\sum_{n=1}^{+\infty}|x_n-u_n|^p\Big)^{\frac{1}{p}}+\Big(\sum_{n=1}^{+\infty}|u_n-z_n|^p\Big)^{\frac{1}{p}}+\Big(\sum_{n=1}^{\infty}|z_n-y_n|^p\Big)^{\frac{1}{p}}\Big],$$

which implies that $d(x,y)\preceq s\Big[d(x,u)+d(u,z)+d(z,y)\Big]$, where $s=2^{(\frac{2}{p})}I\in\mathbb{A}'_+$, for all $x,y\in X$ and for all distinct elements $u,z\in X-\{x,y\}$.

Lemma 2.4. *Let $(X,\mathbb{A},d)$ be a C^*-algebra-valued $b_v(s)$-metric space and let $s\in\mathbb{A}'_+$. Then $(X,\mathbb{A},d)$ is a C^*-algebra-valued $b_{2v}(s^2)$-metric space.*

Proof. Let $(X,\mathbb{A},d)$ be a C^*-algebra-valued $b_v(s)$-metric space. Let

$$d(x,y)\preceq s\Big[d(x,u_1)+d(u_1,u_2)+\cdots+d(u_v,y)\Big]$$

for all $x,y\in X$ and for all distinct elements $u_1,u_2,\dots,u_v\in X-\{x,y\}$. Then, for different $s_1,s_2,\dots,s_v\in X-\{x,y,u_1,u_2,\dots,u_v\}$, we have

$$d(u_v,y)\preceq s\Big[d(u_v,s_1)+d(s_1,s_2)+\cdots+d(s_v,y)\Big].$$

On the other hand, for every C^*-algebra, if a and b are positive elements with $a\preceq b$, and in addition $s\in\mathbb{A}'_+$, then $sa\preceq sb$.
Now, by the above inequality, we have

$$sd(u_v,y)\preceq s\Big[s[d(u_v,s_1)+\cdots+d(s_v,y)]\Big].$$

Furthermore, if $a,b\succeq 0_{\mathbb{A}}$, then $a+b\succeq 0_{\mathbb{A}}$. Hence we can write

$$d(x,y)$$

$$\preceq s\Big[d(x,u_1)+d(u_1,u_2)+\cdots+d(u_{v-1},u_v)+s[d(u_v,s_1)+\cdots+d(s_v,y)]\Big].$$

Since $I\preceq s$ and $s\in\mathbb{A}'_+$, so $s\preceq s^2$ and $sb\preceq s^2b$ for all positive element $b\in\mathbb{A}$. Hence, for all positive elements $a,b,c\in\mathbb{A}$, if $a\preceq sb+s^2c$, then

$a \preceq s^2 b + s^2 c$. Thus, we get

$d(x,y)$

$$\preceq s^2\Big[d(x,u_1)+d(u_1,u_2)+\cdots+d(u_v,s_1)+d(s_1,s_2)+\cdots+d(s_v,y)\Big],$$

which implies that $(X,\mathbb{A},d)$ is a C^*-algebra-valued $b_{2v}(s^2)$-metric space.
□

Lemma 2.5. *Let $(X,\mathbb{A},d)$ be a C^*-algebra-valued $b_v(s)$-metric space, let $T: X \to X$, and let $\{x_n\}$ be a sequence in X defined by $x_0 \in X$ and $x_{n+1} = Tx_n$ such that $x_n \neq x_{n+1}$ $(n \succeq 0)$. If T is a C^*-algebra-valued contractive mapping, then $x_n \neq x_m$ for all distinct numbers $m,n \in \mathbb{N}$.*

Proof. Suppose, to the contrary, that $x_n = x_{n+p}$ for some $n \succeq 0$ and $p \succeq 1$.
Since T is a C^*-algebra-valued contractive mapping, there exists $B \in \mathbb{A}$ with $\|B\| \prec 1$ such that

$$d(Tx,Ty) \preceq B^* d(x,y) B, \quad \text{for all } x,y \in X.$$

On the other hand, we have $Tx_n = Tx_{n+p}$, and the assumptions imply $x_{n+1} = x_{n+p+1}$.
Now, we get

$$d(x_{n+1},x_n) = d(x_{n+p+1},x_{n+p}) \preceq B^* d(x_{n+p},x_{n+p-1}) B.$$

Similarly,

$$d(x_{n+p},x_{n+p-1}) \preceq B^* d(x_{n+p-1},x_{n+p-2}) B.$$

Now, using Lemma 1.1, we conclude

$$\begin{aligned} 0_{\mathbb{A}} \preceq d(x_{n+1},x_n) = d(x_{n+p+1},x_{n+p}) &\preceq B^* d(x_{n+p},x_{n+p-1}) B \\ &\preceq (B^*)^2 d(x_{n+p-1},x_{n+p-2}) B^2 \\ &\vdots \\ &\preceq (B^*)^p d(x_{n+1},x_n) B^p. \end{aligned}$$

Finally, by applying Lemma 1.1 again, we obtain

$$\begin{aligned}
\|d(x_{n+1},x_n)\| &\preceq \|(B^*)^p d(x_{n+1},x_n)B^p\| \\
&\preceq \|(B^*)^p\|\|d(x_{n+1},x_n\|\|B^p\| \\
&\preceq \|B^*\|^p\|d(x_{n+1},x_n)\|\|B\|^p \\
&= \|B\|^{2p}\|d(x_{n+1},x_n)\| \\
&\prec \|d(x_{n+1},x_n)\|,
\end{aligned}$$

which is a contradiction. □

Definition 2.6. Let $(X,\mathbb{A},d)$ be a C^*-algebra-valued $b_v(s)$-metric space. Suppose that $\{x_n\}\subset X$ and $x\in X$. If, for any $\varepsilon\succ 0$, there is a natural number N such that $\|d(x_n,x)\|\preceq\varepsilon$ for all $n\succ N$, then $\{x_n\}$ is said to be *convergent with respect to* $\mathbb{A}$, also $\{x_n\}$ *converges to* x, or x *is the limit of* $\{x_n\}$. We denote it by $\lim_{n\to+\infty} x_n = x$.
For any $\varepsilon\succ 0$, if there is a natural number N such that $\|d(x_n,x_m)\|\preceq\varepsilon$ for all $n,m\succ N$, then $\{x_n\}$ is called a *Cauchy sequence with respect to* $\mathbb{A}$.
We say $(X,\mathbb{A},d)$ is a *complete C^*-algebra-valued $b_v(s)$-metric space* if every Cauchy sequence with respect to $\mathbb{A}$ is convergent.

Theorem 2.7. *Suppose that $(X,\mathbb{A},d)$ is a complete C^*-algebra-valued $b_v(s)$-metric space with coefficient s. Let $T: X\to X$ be a C^*-algebra-valued contractive mapping with constant B. If there exists a natural number n_0 such that $s(B^*)^{n_0}B^{n_0}\prec 1_{\mathbb{A}}$ and $B^{n_0}\in\mathbb{A}'$, then T has a unique fixed point in X.*

Proof. It is clear that if $B=0_{\mathbb{A}}$, then T maps X into a single point. Thus without loss of generality, one can suppose that $B\neq 0_{\mathbb{A}}$.
Choose $x_0\in X$, and set $\{x_n\}$ by $x_{n+1}=Tx_n=T^{n+1}x_0, n=0,1,2,\ldots$. If $x_n=x_{n+1}$ for some $n\succeq 0$, then T has a unique fixed point in X. Otherwise, we consider $x_n\neq x_{n+1}$ $(n\succeq 0)$. Using Lemma 2.5 implies that $x_n\neq x_m$ for all distinct numbers $n,m\in\mathbb{N}$. On the other hand, notice that $s[d(x,u_1)+d(u_1,u_2)+\cdots+d(u_{v-1},u_v)+d(u_v,y)]$, $s\in\mathbb{A}'_+$, is also a positive element. Now, by Lemma 1.1 and the condition (1) on

T, it follows that

$$\begin{aligned} d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) &\preceq B^* d(x_n, x_{n-1}) B \\ &\preceq (B^*)^2 d(x_{n-1}, x_{n-2}) B^2 \\ &\vdots \\ &\preceq (B^*)^n d(x_1, x_0) B^n. \end{aligned}$$

We consider the following two cases:

(1) $v \succeq 2$

(2) $v = 1$.

Let $v \succeq 2$. Also, suppose that $m \succ n$; then the triangle inequality for the $b_v(s)$-metric d implies that

$$\begin{aligned} d(x_n, x_m) &\preceq s\Big[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+v-3}, x_{n+v-2}) \\ &\quad + d(x_{n+v-2}, x_{n+n_0}) + d(x_{n+n_0}, x_{m+n_o}) + d(x_{m+n_o}, x_m)\Big] \\ &\preceq s\Big[(B^*)^n d(x_0, x_1) B^n + (B^*)^{n+1} d(x_0, x_1) B^{n+1} + \cdots \\ &\quad + (B^*)^{n+v-3} d(x_0, x_1) B^{n+v-3} + (B^*)^n d(x_{v-2}, x_{n_0}) B^n \\ &\quad + (B^*)^{n_0} d(x_n, x_m) B^{n_0} + (B^*)^m d(x_{n_0}, x_0) B^m\Big]. \end{aligned}$$

So,
$d(x_n, x_m) - s(B^*)^{n_0} d(x_n, x_m) B^{n_0}$

$$\begin{aligned} &\preceq s(B^*)^n d(x_0, x_1) B^n + s(B^*)^{n+1} d(x_0, x_1) B^{n+1} \\ &\quad + \cdots + s(B^*)^{n+v-3} d(x_0, x_1) B^{n+v-3} \\ &\quad + s(B^*)^n d(x_{v-2}, x_{n_0}) B^n \\ &\quad + s(B^*)^m d(x_{n_0}, x_0) B^m. \end{aligned}$$

On the other hand, by Lemma 1.1, we have
$d(x_n, x_m)(1_{\mathbb{A}} - s(B^*)^{n_0} B^{n_0})$

$$
\begin{aligned}
&\preceq \|d(x_n,x_m)(1_{\mathbb{A}}-s(B^*)^{n_0}B^{n_0})\|1_{\mathbb{A}}\\
&\preceq \|s(B^*)^n d(x_0,x_1)B^n+s(B^*)^{n+1}d(x_0,x_1)B^{n+1}+\cdots\\
&\quad +s(B^*)^{n+v-3}d(x_0,x_1)B^{n+v-3}+s(B^*)^n d(x_{v-2},x_{n_0})B^n\\
&\quad +s(B^*)^m d(x_{n_0},x_0)B^m\|1_{\mathbb{A}}\\
&\preceq \|s(B^*)^n d(x_0,x_1)B^n\|1_{\mathbb{A}}+\|s(B^*)^{n+1}d(x_0,x_1)B^{n+1}\|1_{\mathbb{A}}+\cdots\\
&\quad +\|s(B^*)^{n+v-3}d(x_0,x_1)B^{n+v-3}\|1_{\mathbb{A}}+\|s(B^*)^n d(x_{v-2},x_{n_0})B^n\|1_{\mathbb{A}}\\
&\quad +\|s(B^*)^m d(x_{n_0},x_0)B^m\|1_{\mathbb{A}}\\
&\preceq \|s\|\|(B^*)^n\|\|d(x_0,x_1)\|\|B^n\|1_{\mathbb{A}}+\|s\|\|(B^*)^{n+1}\|\|d(x_0,x_1)\|\|B^{n+1}\|1_{\mathbb{A}}\\
&\quad +\|s\|\|(B^*)^{n+v-3}\|\|d(x_0,x_1)\|\|B^{n+v-3}\|1_{\mathbb{A}}\\
&\quad +\|s\|\|(B^*)^n\|\|d(x_{v-2},x_{n_0})\|\|B^n\|1_{\mathbb{A}}\\
&\quad +\|s\|\|(B^*)^m\|\|d(x_{n_0},x_0)\|\|B^m\|1_{\mathbb{A}}\\
&\preceq \|s\|\|B^*\|^n\|d(x_0,x_1)\|\|B\|^n 1_{\mathbb{A}}+\|s\|\|B^*\|^{n+1}\|d(x_0,x_1)\|\|B\|^{n+1}1_{\mathbb{A}}\\
&\quad +\|s\|\|B^*\|^{n+v-3}\|d(x_0,x_1)\|\|B\|^{n+v-3}1_{\mathbb{A}}+\|s\|\|B^*\|^n\|d(x_{v-2},x_{n_0})\|\|B\|^n 1_{\mathbb{A}}\\
&\quad +\|s\|\|B^*\|^m\|d(x_{n_0},x_0)\|\|B\|^m 1_{\mathbb{A}}.
\end{aligned}
$$

Now, since $(1_{\mathbb{A}}-s(B^*)^{n_0}B^{n_0})\in\mathbb{A}'_+$ and it is invertible. Hence, by Lemma 1.2, we have

$$
\begin{aligned}
&d(x_n,x_m)\\
&\preceq \Big(\|s\|\|B^*\|^n\|d(x_0,x_1)\|\|B\|^n+\|s\|\|B^*\|^{n+1}\|d(x_0,x_1)\|\|B\|^{n+1}\\
&\quad +\|s\|\|B^*\|^{n+v-3}\|d(x_0,x_1)\|\|B\|^{n+v-3}+\|s\|\|B^*\|^n\|d(x_{v-2},x_{n_0})\|\|B\|^n\\
&\quad +\|s\|\|B^*\|^m\|d(x_{n_0},x_0)\|\|B\|^m\Big)(1_{\mathbb{A}}-s(B^*)^{n_0}B^{n_0})^{-1}\\
&\longrightarrow 0_{\mathbb{A}}\ \ (as\ \ m,n\to+\infty).
\end{aligned}
$$

Therefore $\{x_n\}$ is a Cauchy sequence with respect to $\mathbb{A}$. If $v=1$, then the proof follows from Lemma 2.4. By completeness of $(X,\mathbb{A},d)$, there exists $x^*\in X$ such that $\lim_{n\to+\infty}x_n=\lim_{n\to+\infty}Tx_{n-1}=x^*$. Since

$$
\begin{aligned}
&d(Tx^*,x^*)\\
&\preceq s\Big[d(Tx^*,x_{n+1})+d(x_{n+1},x_{n+2})+\cdots+d(x_{n+v},x^*)\Big]\\
&= s\Big[d(Tx^*,Tx_n)+d(Tx_n,Tx_{n+1})+\cdots+d(x_{n+v},x^*)\Big]\\
&\preceq s\Big[B^*d(x^*,x_n)B+(B^*)^n d(x_0,x_1)B^n+\cdots+d(Tx_{n+v-1},x^*)\Big],
\end{aligned}
$$

it follows that

$\|d(Tx^*, x^*)\|$

$$\begin{aligned}
&\preceq \|s[B^*d(x^*, x_n)B + (B^*)^n d(x_0, x_1)B^n + \cdots + d(Tx_{n+v-1}, x^*)\| \\
&\preceq \|s\|\|B^*\|\|d(x^*, x_n)\|\|B\| + \|s\|\|(B^*)^n\|\|d(x_0, x_1)\|\|B^n\| + \cdots \\
&\quad +\|s\|\|(B^*)^{n+v-2}\|\|d(x_0, x_1)\|\|B^{n+v-2}\| + \|d(x_{n+v}, x^*)\| \\
&\preceq \|s\|\|B^*\|\|d(x^*, x_n)\|\|B\| + \|s\|\|B^*\|^n\|d(x_0, x_1)\|\|B\|^n + \cdots \\
&\quad +\|s\|\|B^*\|^{n+v-2}\|d(x_0, x_1)\|\|B\|^{n+v-2} + \|d(x_{n+v}, x^*)\| \\
&\longrightarrow 0 \quad (as \;\; n \to +\infty),
\end{aligned}$$

which shows that $Tx^* = x^*$.
To prove that x^* is the unique fixed point, we suppose that $y^*(\neq x^*)$ is another fixed point of T. Then by applying condition (1), we have

$$0_{\mathbb{A}} \preceq d(x^*, y^*) = d(Tx^*, Ty^*) \preceq B^*d(x^*, y^*)B.$$

Using the norm of $\mathbb{A}$, we have

$$\begin{aligned}
0 &\preceq \|d(x^*, y^*)\| = \|d(Tx^*, Ty^*)\| \\
&\preceq \|B^*\|\|d(x^*, y^*)\|\|B\| \\
&= \|B\|^2\|d(x^*, y^*)\| \\
&\prec \|d(x^*, y^*)\|,
\end{aligned}$$

which is impossible. So $d(x^*, y^*) = 0_{\mathbb{A}}$ and $x^* = y^*$, which implies that the fixed point is unique. □

Definition 2.8. [11] let X be a nonempty set. We call a mapping T is a C^**-algebra-valued expansion mapping on* X, if $T : X \to X$ satisfies

(1) $T(X) = X$;

(2) $d(Tx, Ty) \succeq B^*d(x, y)B, \quad$ for all $x, y \in X$,

where $B \in \mathbb{A}$ is an invertible element and $\|B^{-1}\| \prec 1$.

Theorem 2.9. *Consider a complete* C^**-algebra-valued* $b_v(s)$*-metric space* $(X, \mathbb{A}, d)$ *with coefficient* s*. Let* $T : X \to X$ *be a* C^**-algebra-valued expansion mapping with constant* B*. If there exists a natural number* n_0 *such that* $(B^{-1})^{n_0} \in \mathbb{A}'$ *and* $s((B^{-1})^*)^{n_0}(B^{-1})^{n_0} \prec 1_{\mathbb{A}}$*, then* T *has a unique fixed point in* X*.*

Proof. First, we show that T is invertible. Since by condition (1) of Definition 2.8, T is surjective, it is enough to show that T is injective. Indeed, for any $x, y \in X$ with $x \neq y$, if $T(x) = T(y)$, we have

$$0_{\mathbb{A}} = d(Tx, Ty) \succeq B^* d(x, y) B.$$

Since $B^* d(x, y)B \in \mathbb{A}_+$, therefore $B^* d(x, y)B = 0_{\mathbb{A}}$. On the other hand, B is invertible, then $d(x, y) = 0_{\mathbb{A}}$, which is impossible. Thus T is injective.

Next, we will show that T has a unique fixed point in X. In fact, since T is invertible, for any $x, y \in X$, it follows that

$$d(Tx, Ty) \succeq B^* d(x, y)B.$$

In the above formula, we replace x and y by $T^{-1}(x)$ and $T^{-1}(y)$, respectively, and we get

$$d(x, y) \succeq B^* d(T^{-1}x, T^{-1}y)B.$$

Now by part (1) of Lemma 1.1, we have

$$\begin{aligned} (B^{-1})^* d(x, y) B^{-1} &\succeq (B^{-1})^* B^* d(T^{-1}x, T^{-1}y) B B^{-1} \\ &= (B^*)^{-1} B^* d(T^{-1}x, T^{-1}y) B B^{-1} \\ &= d(T^{-1}x, T^{-1}y). \end{aligned}$$

Using Theorem 2.7, there exists unique $x^* \in X$ such that $T^{-1}x^* = x^*$, which means that there is a unique fixed point $x^* \in X$ such that $Tx^* = x^*$. □ In the following theorem, we prove Jungcks theorem in C^*-algebra-valued $b_v(s)$-metric spaces.

Theorem 2.10. *Consider $(X, \mathbb{A}, d)$ is a complete C^*-algebra-valued $b_v(s)$-metric space with coefficient s. Let T and I be commuting mappings of X into itself such that the range of I contains the range of T and I is continuous and satisfies the inequality*

$$d(Tx, Ty) \preceq B^* d(Ix, Iy) B \quad \text{for all } x, y \in X, \tag{2}$$

where $B \in \mathbb{A}$ with $\|B\| \prec 1$. If there exists a natural number n_0 such that $s(B^*)^{n_0} B^{n_0} \prec 1_{\mathbb{A}}$ and $B^{n_0} \in \mathbb{A}'$. Then T and I have a unique common fixed point.

Proof. Let $x_0 \in X$ be arbitrary. Then Tx_0 and Ix_0 are well-defined. Since $Tx_0 \in I(X)$, there is $x_1 \in X$ such that $Ix_1 = Tx_0$. In general, if x_n is chosen, then we choose a point x_{n+1} in X such that $Ix_{n+1} = Tx_n$. Now, we show that $\{Ix_n\}$ is Cauchy. From (2), for all $m, n \in \mathbb{N}$, we have

$$d(Ix_m, Ix_n) = d(Tx_{m-1}, Tx_{n-1}) \preceq B^* d(Ix_{m-1}, Ix_{n-1})B. \tag{3}$$

Now, we have the following two cases.
Case 1 If $Ix_n = Ix_{n+1}$ for some $n \succeq 0$, then $Ix_n = Ix_{n+1} = Tx_n = \omega$. We show that ω is a unique common fixed point of T and I. Since T and I commute, thus $I\omega = I(Tx_n) = T(Ix_n) = T\omega$.
Now, let $d(T\omega, \omega) \succ 0_{\mathbb{A}}$. Hence

$$d(T\omega, \omega) = d(T\omega, Tx_n) \preceq B^* d(I\omega, Ix_n)B = B^* d(T\omega, \omega)B.$$

Using the norm of $\mathbb{A}$, we have

$$\|d(T\omega, \omega)\| \prec \|d(T\omega, \omega)\|.$$

This is a contradiction. Thus $\|d(T\omega, \omega)\| = 0$, $d(T\omega, \omega) = 0_{\mathbb{A}}$, and $T\omega = \omega = I\omega$. By condition (2), ω is a unique common fixed point of T and I.
Case 2 Now suppose that $Ix_n \neq Ix_{n+1}$ for all $n \succeq 0$. From Lemma 2.5 and inequality (3), we have $Ix_n \neq Ix_{n+p}$ for all $n \succeq 0$ and $p \succeq 1$. With a similar argument used in the proof of Theorem 2.7, we can prove that the sequence $\{Ix_n\}$ is Cauchy. Since the C^*-algebra-valued $b_v(s)$-metric space $(X, \mathbb{A}, d)$ is complete, so $\{Ix_n\}$ converges to $u \in X$ such that

$$\lim_{n\to+\infty} Ix_n = \lim_{n\to+\infty} Tx_{n-1} = u.$$

Since I is continuous, inequality (2) implies that both I and T are continuous. Since T and I commute, we obtain

$$\begin{aligned} Iu &= I(\lim_{n\to+\infty} Tx_{n-1}) = I(\lim_{n\to+\infty} Tx_n) = \lim_{n\to+\infty} ITx_n \\ &= \lim_{n\to+\infty} TIx_n = T(\lim_{n\to+\infty} Ix_n) = Tu. \end{aligned}$$

Let $Tu = Iu = \nu$. Thus $T\nu = TIu = ITu = I\nu$.
If $Tu \neq T\nu$, then from (2), we get

$$\begin{aligned} \|d(Tu, T\nu)\| &\preceq \|B^* d(Iu, I\nu)B\| = \|B^* d(Tu, T\nu)B\| \\ &\preceq \|B^*\| \|d(Tu, T\nu)\| \|B\| \\ &\prec \|d(Tu, T\nu)\|. \end{aligned}$$

This is a contradiction. So $\|d(Tu,T\nu)\| = 0$, $d(Tu,T\nu) = 0_{\mathbb{A}}$, and $Tu = T\nu$. Thus, we obtain $T\nu = I\nu = \nu$.
Now, we claim ν is the unique common fixed point for T and I.
Let $\nu^*(\neq \nu)$ be another fixed point for T and I. By inequality (2), we have

$$d(\nu,\nu^*) = d(T\nu,T\nu^*) \preceq B^*d(I\nu,I\nu^*)B.$$

Now, by using the norm of $\mathbb{A}$, we have

$$\begin{aligned}\|d(\nu,\nu^*)\| = \|d(T\nu,T\nu^*)\| &\preceq \|B^*d(I\nu,I\nu^*)B\| \\ &\preceq \|B^*\|\ d(I\nu,I\nu^*)\|\|B\| \\ &\prec \|d(I\nu,I\nu^*)\| = \|d(\nu,\nu^*)\|.\end{aligned}$$

This is a contradiction, which implies that $\nu = \nu^*$. □

Remark 2.11. In Theorem 2.10, if I is the identity map on X, then, Theorem 2.7 holds.

3 Application

In this section, we give an existence theorem for a solution of the following integral equation.

$$x(t) = \int_E K(t,s,x(s))ds + g(t), \quad t \in E, \tag{4}$$

where $K : E \times E \times \mathbb{R} \to \mathbb{R}$ and $g \in C_{\mathbb{R}}(E)$.
Let $X = C_{\mathbb{R}}(E)$ be the set of all real valued continuous functions on E, where E is a nonempty Lebesgue measurable compact set in $\mathbb{R}_+$. Also, $\mathbb{A} = L(H)$ is the set of all bounded linear operators on $H = L^2(E)$ with usual operator norm. We define $d' : X \times X \to \mathbb{R}_+$ by $d'(x,y) = \sup\limits_{t\in E}(x(t) - y(t))^2$ for all $x, y \in X$. Then, (X, d') is a complete $b_2(3)$-metric space. Moreover, $\Pi_\gamma : H \to H$ is defined by $\Pi_\gamma(h) = \gamma.h$ for all $\gamma \in \mathbb{C}$ and $h \in H$. Now, define $d : X \times X \to \mathbb{A}_+$ by $d(x,y) = \Pi_{d'(x,y)}$. It is clear that $(X, \mathbb{A}, d)$ is a complete C^*-algebra-valued $b_v(s)$-metric space with $v = 2$ and $s = 3I$. We assume that the following conditions

are satisfied:
(i) There exists a continuous function $f : E \times E \to \mathbb{R}$ such that

$$|K(t,s,u) - K(t,s,v)| \preceq \alpha |f(t,s)(u-v)|,$$

for $t, s \in E$, $\alpha \in (0,1)$ and $u, v \in \mathbb{R}$.
(ii) It follows that $\sup_{t\in E} \int_E |f(t,s)| ds \preceq 1$ for any $t, s \in E$.

Theorem 3.1. *Under the assumptions* (i) *and* (ii) *equation (4) has a unique solution in* X

Proof. Let $T : X \to X$ be defined by $Tx(t) = \int_E K(t,s,x(s))ds + g(t)$, $t \in E$. Then

$$\begin{aligned}
&\|d(Tx,Ty)\| \\
&= \|\Pi_{d'(Tx,Ty)}\| = \sup_{\|h\|=1} \langle \Pi_{d'(Tx,Ty)}(h), h\rangle; h \in H \\
&= \sup_{\|h\|=1} \int_E d'(Tx,Ty)h(u)\overline{h}(u)d(u); u \in E \\
&= \sup_{\|h\|=1} \int_E \sup_{t\in E} \Big[Tx(t) - Ty(t)\Big]^2 h(u)\overline{h}(u)d(u); u \in E \\
&= \sup_{\|h\|=1} \int_E \sup_{t\in E} \Big[\int_E \big[K\big(t,s,x(s)\big) - K\big(t,s,y(s)\big)\big]ds\Big]^2 |h(u)|^2 du; u \in E \\
&\preceq \sup_{\|h\|=1} \int_E \sup_{t\in E} \Big[\int_E \alpha |f(t,s)| \big(x(s) - y(s)\big) ds\Big]^2 |h(u)|^2 du; u \in E \\
&= \alpha^2 d'(x,y) \sup_{\|h\|=1} \int_E \sup_{t\in E} \Big[\int_E |f(t,s)| ds\Big]^2 |h(u)|^2 du; u \in E \\
&\preceq \alpha^2 d'(x,y) \sup_{\|h\|=1} \int_E |h(u)|^2 du; u \in E \\
&= \alpha^2 \sup_{\|h\|=1} \int_E d'(x,y)|h(u)|^2 du; u \in E \\
&= \alpha^2 \|d(x,y)\|.
\end{aligned}$$

By take $B = \alpha 1_{\mathbb{A}}$, then $\|B\| \prec 1$. Using Theorem 2.7, the integral equation (4) has a unique solution in X. $\square$

Example 3.2. Consider the following functional integral equation:

$$x(t) = \int_0^1 \frac{4e^{-(t+1)s}}{3((t+1)^2+2)} \frac{|x(s)|}{1+|x(s)|} ds + t \tag{5}$$

for $t \in E = [0,1]$. Observe that this equation is a special case of (4) with

$K(t,s,x(s)) = \frac{4e^{-(t+1)s}}{3((t+1)^2+2)} \frac{|x(s)|}{1+|x(s)|}$,

$f(t,s) = \frac{4e^{-(t+1)s}}{(t+1)^2+2}$,

$g(t) = t$.

Notice that, for arbitrary fixed numbers $u, v \in \mathbb{R}$ and $t, s \in E = [0,1]$, we have

$$\begin{aligned} &|K(t,s,u) - K(t,s,v)| \\ &\quad = |\frac{4e^{-(t+1)s}}{3((t+1)^2+2)} \frac{|u|}{1+|u|} - \frac{4e^{-(t+1)s}}{3((t+1)^2+2)} \frac{|v|}{1+|v|}| \\ &\quad \preceq \frac{1}{3}|\frac{4e^{-(t+1)s}}{(t+1)^2+2}||u-v|. \end{aligned}$$

Thus the function K satisfies the assumption (i) with $\alpha = \frac{1}{3}$.
Also, we have
$\sup_{0 \preceq t \leq 1} \int_0^1 |f(t,s)|ds = \sup_{0 \preceq t \preceq 1} \int_0^1 |\frac{4e^{-(t+1)s}}{(t+1)^2+2}|ds = \sup_{0 \preceq t \preceq 1} \frac{4}{(t+1)^2+2} \int_0^1 e^{-(t+1)s} ds \prec 1.$ This shows that the assumption (ii) holds. Consequently, all the conditions of Theorem 3.1 are satisfied. Hence the integral equation (3.2) has a unique solution in $C_{\mathbb{R}}(E)$.

4 Iterative method for solving integral equation

Theorem 4.1. *Consider the integral equation (4). The following iteration process leads to the fixed point (function) solution of (4)*

$$x_{n+1}(t) = \int_E K(t,s,x_n(s))ds + g(t), \quad t \in E, \tag{6}$$

where the initial guess $x_0(t)$ *can be any arbitrary function such as* $0,\ 1,$ *or* t.

Proof. Assume that the exact solution of (4) is $\tilde{x}(t)$.
We have

$$\begin{aligned}|x_1(t)-\tilde{x}(t)| &= |\int_E \big(K(t,s,x_0(s)-K(t,s,\tilde{x}(s))\big)ds| \\ &\preceq \int_E \alpha|f(t,s)||x_0(s)-\tilde{x}(s)|ds \\ &\preceq \alpha M,\end{aligned}$$

where $M=\max|x_0(s)-\tilde{x}(t)|,\ t\in E$. One can show similarly that

$$\begin{aligned}|x_2(t)-\tilde{x}(t)| &\preceq \alpha\int_E |f(t,s)||x_1(s)-\tilde{x}(s)|ds \\ &\preceq \alpha^2 M\int_E |f(t,s)|ds \\ &\preceq \alpha^2 M.\end{aligned}$$

Finally,

$$|x_{n+1}(t)-\tilde{x}(t)|\preceq \alpha^{n+1}M.$$

It is clear that when n tends to infinity, $x_{n+1}(t)$ tends to the exact solution $\tilde{x}(t)$. □ Consider the integral equation (6), we set

$$H(x_n(t))=\int_E K\big(t,s,x_n(s)\big)ds+g(t),$$

so the integral equation (6) can be rewritten as follows:

$$x_{n+1}(t)=H(x_n(t)).$$

It is clear that the exact solution $\tilde{x}(t)$ satisfies

$$\tilde{x}(t)=H(\tilde{x}(t))$$

and $|\tilde{x}(t)-H(\tilde{x}(t))|=0$.
Now in order to start the iterations for Example 3.2, we consider $x_0(t)=$

0 and do four iterations according to relation (6) to obtain $x_4(t)$. we have used Maple 2018 to plot

$$|x_4(t) - x_3(t)| = |H(x_3(t)) - x_3(t)|$$

in Figure 1, which shows small errors between $x_3(t)$ and $x_4(t)$, and it can be considered as a good approximation for the exact solution $\tilde{x}(t)$. In Figure 2, we have plotted $x_4(t)$ in the interval $[0, 1]$.

Figure 1: graph of $|x_4(t) - x_3(t)|$

Figure 2: graph of $x_4(t)$

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