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Existence Results for a Class of p-Hamiltonian Systems

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Abstract. The existence and multiplicity of periodic solutions for a class of p-Hamiltonian systems are established using the variational methods in critical point theory. In fact, using two fundamental theorems attributed to Bonanno, some important results would be achieved.

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1 Introduction

Consider the following p-Hamiltonian system

$$\begin{cases} -(|u'|^{p-2}u')' - q(t)|u'|^{p-2}u' + A(t)|u|^{p-2}u = \\ \lambda \nabla F(t,u) \qquad a.e.\ \ t \in [0,T], \\ u(0) - u(T) = u'(0) - e^{Q(T)}u'(T) = 0, \end{cases} \tag{1}$$

where $T > 0$, $p > 1$, $q \in L^1(0,T;\mathbb{R})$, $Q(t) = \int_0^t q(s)ds$, $A : [0,T] \to \mathbb{R}^{N\times N}$ is a continuous map from the interval $[0,T]$ to the set of N-order symmetric matrices, $\lambda > 0$ and $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ is measurable in

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t for all $x \in \mathbb{R}^N$ and continuously differentiable in x for almost every $t \in [0,T]$, and there exist $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $b \in L^1(0,T;\mathbb{R}^+)$ such that

$$|F(t,x)| \leq a(|x|)b(t), \quad |\nabla F(t,x)| \leq a(|x|)b(t) \tag{2}$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$.

Hamiltonian systems are a special case of dynamical systems. These types of equations play an important role in fluid mechanics and gas dynamics. There are various Hamiltonian systems are shown in [12, 14]. When $p = 2$, problem (1) is the second order Hamiltonian systems. In recent years, the existence of periodic solutions for the second order Hamiltonian systems have been studied in many papers; see [3,5–10,16–18] and the references contained therein. For example in [10], the authors proved the existence of periodic solutions by the variational methods in the critical point theory for the following second-order Hamiltonian system

$$\begin{cases} -\ddot{u}(t) - q(t)\dot{u}(t) + A(t)u(t) = \\ \lambda \nabla F(t,u(t)) + \mu \nabla G(t,u(t)) \qquad a.e. \ t \in [0,T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \end{cases}$$

where $\mu \geq 0$ and $G : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ is measurable in t for all $x \in \mathbb{R}^N$ and continuously differentiable in x for almost every $t \in [0,T]$. For the general case $p > 1$, some authors presented interesting results (see [11, 13, 19, 20]).

For instance Xu and Tang in [19] using minimax methods in the critical point theory established the existence of periodic solutions for the problem

$$\begin{cases} -(|u'(t)|^{p-2}u'(t))' = \lambda \nabla F(t,u(t)) \qquad a.e. \ t \in [0,T], \\ u(0) - u(T) = u'(0) - u'(T) = 0. \end{cases}$$

Also, in [20] some authors have considered the following problem

$$\begin{cases} -(|u'|^{p-2}u')' + A(t)|u|^{p-2}u = \\ \lambda \nabla F(t,u) + \mu \nabla G(t,u) \qquad a.e. \ t \in [0,T], \\ u(0) - u(T) = u'(0) - u'(T) = 0. \end{cases} \tag{3}$$

They studied the existence of at least three periodic solutions for problem (3) using two theorems due respectively to Ricceri (see reference [17] in [20]) and Averna-Bonanno(see reference [13] in [20]).
In this paper, using two kinds of critical points theorems obtained in [1] and [2], we ensure the existence of periodic solutions for problem (1).
The paper is organized as follows. In § 2 we establish all preliminary results that we need and in § 3 we present our main results.

2 Preliminaries

We assume that the matrix A satisfies the following conditions:

(i) $A(t) = (a_{ij}(t))$, $i = 1, \dots, N$, $j = 1, \dots, N$, is a symmetric matrix with $a_{ij} \in L^\infty[0, T]$ for any $t \in [0, T]$,

(ii) there is a positive constant $\underline{\delta}$ such that $\langle A(t)|x|^{p-2}x, x\rangle \geq \underline{\delta}\,|x|^p$ for all $x \in \mathbb{R}^N$ and $t \in [0, T]$, where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^N$ and in the other hand we know that $\langle A(t)|x|^{p-2}x, x\rangle \leq \bar{\delta}\,|x|^p$ (see [20]) for any $x \in \mathbb{R}^N$ and for every $t \in [0, T]$ where

$$\bar{\delta} \leq \sum_{i,j=1}^{N} \|a_{ij}\|. \tag{4}$$

Let us recall some notions and results that are needed later. Here and in the sequel E denotes the Sobolev space

$$E = \{u : [0, T] \to \mathbb{R}^N,\ u \text{ is absolutely continuous},\\ u(0) = u(T),\ \ u' \in L^p([0, T], \mathbb{R}^N)\}$$

endowed with the norm

$$\|u\|_E = \left(\int_0^T (|u'(t)|^p + |u(t)|^p)dt\right)^{\frac{1}{p}},\ \forall\ u \in E.$$

Also , we consider E with the norm

$$\|u\| = \left(\int_0^T e^{Q(t)}[|u'(t)|^p + \langle A(t)|u(t)|^{p-2}u(t), u(t)\rangle]dt\right)^{\frac{1}{p}}.$$

E is separable and reflexive. It follows from [12], E is also an uniformly convex Banach space.

Due to the inequality

$$K_1 \min\{1, \underline{\delta}\}\|u\|_E^p \leq \|u\|^p \leq K_2 \max\{1, \bar{\delta}\}\|u\|_E^p,$$

where $K_1 = \min_{t\in[0,T]} e^{Q(t)}$, and $K_2 = \max_{t\in[0,T]} e^{Q(t)}$, the norm $\|\cdot\|$ is equivalent to the norm $\|\cdot\|_E$.

Since $(E, \|\cdot\|)$ is compactly embedded in $C([0,T], \mathbb{R}^N)$ (see [12]), there is a positive constant

$$c \leq c_0 = \sqrt[q]{2} \max\{T^{\frac{1}{q}}, T^{\frac{-1}{p}}\}(K_1 \min\{1, \underline{\delta}\})^{\frac{-1}{p}} \tag{5}$$

such that

$$\|u\|_\infty \leq c \parallel u \parallel, \tag{6}$$

where $q = \dfrac{p}{p-1}$ and $\|u\|_\infty = \max_{t\in[0,T]} |u(t)|$. The proof is similar to the corresponding parts in [20].

Let $\Phi,\ \Psi : E \to \mathbb{R}$ be defined by

$$\Phi(u) = \frac{1}{p} \parallel u \parallel^p = \frac{1}{p}\int_0^T e^{Q(t)}[|u'(t)|^p + \langle A(t)|u(t)|^{p-2}u(t), u(t)\rangle]dt \tag{7}$$

and

$$\Psi(u) = \int_0^T e^{Q(t)} F(t, u(t))dt \tag{8}$$

for every $u \in E$. It is well known that Ψ is a continuously Gâteaux differentiable functional whose Gâteaux dervative is compact (see [18]) and for each $u, v \in E$

$$\Psi'(u)(v) = \int_0^T e^{Q(t)}\langle \nabla F(t, u(t)), v(t)\rangle dt,$$

and, Φ is continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional. Moreover , the Gâteaux dervative of Φ admits a continuous inverse on E^* (see [20]). In particular, we have

$$\Phi'(u)(v) = \int_0^T e^{Q(t)}[\langle |u'(t)|^{p-2}u'(t), v'(t)\rangle + \langle A(t)|u(t)|^{p-2}u(t), v(t)\rangle]dt$$

for each $u, v \in E$.

Definition 2.1. Let Φ and Ψ be defined as above. Put $I_\lambda = \Phi - \lambda\Psi$. We say that $u \in E$ is a critical point of I_λ when $I'_\lambda(u) = 0_{\{E^*\}}$, that is, $I'_\lambda(u)(v) = 0$ for all $v \in E$.

Definition 2.2. A function $u \in E$ is a weak solution to problem (1), if

$$\int_0^T e^{Q(t)}[\langle |u'(t)|^{p-2}u'(t), v'(t)\rangle + \langle A(t)|u(t)|^{p-2}u(t), v(t)\rangle$$

$$-\lambda\langle \nabla F(t, u(t)), v(t)\rangle]dt = 0$$

for every $v \in E$.

Remark 2.3. We clearly observe that the weak solutions of the problem (1) are exactly the solutions of the equation $I'_\lambda(u)(v) = \Phi'(u)(v) - \lambda\Psi'(u)(v) = 0$.

Definition 2.4. A Gâtuax differentiable function I satisfies the Palais-Smale condition (in short (PS) -condition) if any sequence $\{u_n\}$ such that
(a) $\{I(u_n)\}$ is bounded,
(b) $\lim_{n\to+\infty} \|I'(u_n)\|_{X^*} = 0, \quad \forall n \in \mathbb{N}$,
has a convergent subsequence.

A non-standard state of the Palais-Smale condition is introduced in [1] as follows.

Definition 2.5. Fix $r \in]-\infty, +\infty]$. A Gâtuax differentiable function I satisfies the Palais-Smale condition cut off upper at r (in short $(PS)^{[r]}$ -condition) if any sequence $\{u_n\}$ such that
(a) $\{I(u_n)\}$ is bounded,
(b) $\lim_{n\to+\infty} \|I'(u_n)\|_{X^*} = 0$,
(c) $\Phi(u_n) < r \quad \forall n \in \mathbb{N}$,
has a convergent subsequence.

Two propositions will be needed to prove the main theorems of this paper are discussed.

Proposition 2.6. [*[1], Proposition 2.1*] *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous, coercive and continuously Gâteaux differentiable function whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable function whose Gâteaux derivative is compact. Then, for all $r \in \mathbb{R}$, the function $\Phi - \Psi$ satisfies the $(PS)^{[r]}$-condition.*

Remark 2.7. Fix $\lambda > 0$. According to proposition 2.6, the functional $I_\lambda = \Phi - \lambda\Psi$ satisfies the $(PS)^{[r]}$-condition for any $r > 0$.

In the next proposition, using Ambrosetti-Rabinowitz conditions obtained in [15], ensured that functional I_λ is unbounded from below.

Proposition 2.8. *Assume that there are $M > 0$ and $\theta > p$ such that*

$$0 < \theta F(t, x) \leq \langle \nabla F(t, x), x\rangle$$

for all $x \in \mathbb{R}^N$ with $|x| \geq M$ and a.e. $t \in [0, T]$. Then $I_\lambda = \Phi - \lambda\Psi$ satisfies the (PS)-condition and it is unbounded from below.

Proof. First we prove that I_λ satisfies (PS)-condition for every $\lambda > 0$. For this purpose we will prove that for an arbitrary sequence $\{u_n\} \subset E$ satisfying

$$|I_\lambda(u_n)| \leq D \;\; \text{for some } D > 0 \text{ and for all } n \in \mathbb{N}, \tag{9}$$

$$\lim_{n\to+\infty} \|I'(u_n)\|_{E^*} = 0, \quad \forall n \in \mathbb{N} \tag{10}$$

contains a convergent subsequence. For n large enough, from (9) we have

$$D \geq I_\lambda(u_n) = \frac{1}{p}\|u_n\|^p - \lambda \int_0^T F(t, u_n)dt \geq$$

$$\frac{1}{p}\|u_n\|^p - \frac{\lambda}{\theta}\int_0^T \langle \nabla F(t, u_n), u_n\rangle dt =$$

$$\Big(\frac{1}{p} - \frac{1}{\theta}\Big)\|u_n\|^p + \frac{1}{\theta} I'_\lambda(u_n)(u_n). \tag{11}$$

From $\lim_{n\to+\infty} \|I'_\lambda(u_n)\|_{E^*} = 0$, there is a sequence $\{\varepsilon_n\}$, with $\varepsilon_n \to 0^+$, such that

$$|I'_\lambda(u_n)(v_n)| \leq \varepsilon_n \tag{12}$$

for all $n \in \mathbb{N}$ and for all $v \in E$ with $\| v \| \leq 1$. Taking into account $v(x) = \frac{u_n(x)}{\|u_n\|}$, from (12) one has

$$|I'_\lambda(u_n)(u_n)| \leq \varepsilon_n \| u_n \| \tag{13}$$

for all $n \in \mathbb{N}$. Hence from (11) and (13) we have

$$D + \frac{\varepsilon_n}{\theta} \| u_n \| \geq \Big(\frac{1}{p} - \frac{1}{\theta}\Big)\|u_n\|^p. \tag{14}$$

Thus, (14) ensures that $\{u_n\}$ is bounded in E and hence, passing to a subsequence if necessary we can assume that there is $u_0 \in E$ such that $u_n \rightharpoonup u_0$ ([4]-Theorem 3.18). Now since Ψ' is compact then $\Psi'(u_n) \to \Psi'(u_0)$. But from (10) we have $I'_\lambda(u_n) = \Phi'(u_n) - \lambda\Psi'(u_n) \to 0$. This implies that $u_n \to \Phi'^{-1}(\lambda\Psi'(u_0))$ (because Φ admits a continuous inverse on E^*) and finally according to the uniqueness of the weak limit, $u_n \to u_0$ in E and so I_λ satisfies $(PS)^{[r]}$-condition.
Take $h(t) := \min_{|\xi|=M} F(t,\xi)$. From (2.8), by standard computations, we have

$$F(t,x) \geq \, h(t)\frac{|x|^\theta}{M^\theta} - \big(\max_{s\in[0,M]} a(s)\big)b(t) \tag{15}$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$. Fixed $u_0 \in E - \{0\}$. For each $s > 1$, we have

$$I_\lambda(su_0) = \frac{1}{p} \| su_0 \|^p - \lambda \int_0^T F(t, su_0)dx.$$

Taking into account (15), one has

$$I_\lambda(su_0) \leq \frac{s^p}{p} \| u_0 \|^p - \lambda \int_0^T \left(h(t)\frac{s^\theta\,|u_0|^\theta}{M^\theta} - \big(\max_{z\in[0,M]} a(z)\big)b(t)\right)dt$$

and since $\theta > p$, this condition guarantees that I_λ is unbounded from below. $\square$
Our main tools are the following critical points theorems.

Theorem 2.9 ([2], Theorem 2.3)**.** *X is a real Banach space, and $\Phi, \Psi : X \longrightarrow \mathbb{R}$ are two continuously Gâteaux differentiable functionals such that $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$.*
Assume that there are $r \in \mathbb{R}$ and $\tilde{u} \in X$, with $0 < \Phi(\tilde{u}) < r$, such that

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}{r} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})}, \tag{16}$$

and, for each $\lambda \in \left]\frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}\right[$ the functional $\Phi - \lambda\Psi$ satisfies the $(PS)^{[r]}$-condition.
Then, for each $\lambda \in \left]\frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}\right[$ there is $u_\lambda \in \Phi^{-1}(]0, r[)$ (hence $u_\lambda \neq 0$) such that $I_\lambda(u_\lambda) < I_\lambda(u)$ for all $u \in \Phi^{-1}(]0, r[)$ and $I'_\lambda(u_\lambda) = 0$.

Theorem 2.10 ([1], Theorem 3.2)**.** *X is a real Banach space, and let $\Phi, \Psi : X \longrightarrow \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that Φ is bounded from below and*
$\Phi(0) = \Psi(0) = 0$. Fix $r > 0$ such that $\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u) < +\infty$ and assume that for each $\lambda \in \left]0, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}\right[$ the functional $\Phi - \lambda\Psi$ satisfies the (PS)-condition and it is unbounded from below. Then, for each $\lambda \in \left]0, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}\right[$ the functional I_λ admits two distinct critical points.

3 Main results

In this section, the following notation is used:

$$F^\theta := \int_0^T e^{Q(t)} \sup_{|x|\leq\theta} F(t,x)dt, \quad t \in [0,T], \quad \forall\theta > 0. \tag{17}$$

Also, in the sequel, $\bar{\delta}$ and c are the constants defined in (4) and (5), respectively.

Now, we formulate our main result.

Theorem 3.1. *Let* $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ *satisfy assumption* (2) *and* $\int_0^T e^{Q(t)} F(t,0) = 0$ *for a.e.* $t \in [0,T]$. *Assume that the following condition hold:*
(*A*) *there exists positive constant* θ *and a point* $0 \neq x_0 \in \mathbb{R}^N$ *with* $|x_0| c(\bar{\delta}\ \int_0^T e^{Q(t)}dt)^{\frac{1}{p}} < \theta$, *such that*

$$\frac{F^\theta}{\theta^p} < \frac{\displaystyle\int_0^T e^{Q(t)} F(t,x_0)dt}{c^p \bar{\delta}\, |x_0|^p \int_0^T e^{Q(t)}dt}. \tag{18}$$

Then for every

$$\lambda \in \Lambda := \left] \frac{\bar{\delta}\, |x_0|^p \int_0^T e^{Q(t)}dt}{p \displaystyle\int_0^T e^{Q(t)} F(t,x_0)dt}, \frac{\theta^p}{p\, c^p F^\theta} \right[, \tag{19}$$

the problem (1) *admits at least one non-trivial weak solution* $u_\lambda \in E$ *such that* $\|u_\lambda\|_\infty < \theta$.

Proof. Our aim is to apply Theorem 2.9, to problem (1). Fix $\lambda \in \Lambda$. Take $X = E$ and Φ and Ψ as in the previous section. Notice first that by the definition of the functional Φ and the condition $\int_0^T e^{Q(t)} F(t,0) = 0$ for a.e. $t \in [0,T]$ we will have

$$\inf_X \Phi = \Phi(0) = \Psi(0) = 0.$$

Also, we observe that the another regularity assumptions of Theorem 2.9 on Φ and Ψ are satisfied and according to Remark 2.7, the functional $I_\lambda = \Phi - \lambda\Psi$ satisfies the $(PS)^{[r]}$-condition for all $r > 0$.

Put $r = \frac{1}{p}(\frac{\theta}{c})^p$ and $\tilde{u}(t) = x_0$ for all $t \in [0,T]$. We clearly observe that $\tilde{u} \in E$ and according to assumption $|x_0| c(\bar{\delta}\ \int_0^T e^{Q(t)}dt)^{\frac{1}{p}} < \theta$, one has $0 < \Phi(\tilde{u}) < r$.

For each $u \in E$ bearing in mind (6), we see that

$$\begin{aligned}
\Phi^{-1}(]-\infty, r[) &= \{u \in E;\ \Phi(u) < r\} \\
&= \left\{u \in E;\ \frac{\|u\|^p}{p} < r\right\} \\
&\subseteq \{u \in E;\ |u(t)| \leq \theta \text{ for each } t \in [0,T]\}.
\end{aligned}$$

Now we have

$$\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u) = \sup_{\Phi(u)<r} \int_0^T e^{Q(t)} F(t,u(t)dt \leq \int_0^T e^{Q(t)} \sup_{|x|\leq\theta} F(t,x)dt = F^\theta.$$

Therefore, we have

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}{r} \leq \frac{F^\theta}{\frac{1}{p}\left(\frac{\theta}{c}\right)^p} = \frac{p\, c^p\, F^\theta}{\theta^p} < \frac{1}{\lambda}. \tag{20}$$

On the other hand

$$\frac{\Psi(\tilde{u})}{\Phi(\tilde{u})} = \frac{\int_0^T e^{Q(t)} F(t,x_0)dt}{\frac{1}{p}\|x_0\|^p} \geq \frac{\int_0^T e^{Q(t)} F(t,x_0)dt}{\frac{1}{p}\bar{\delta}|x_0|^p \int_0^T e^{Q(t)}dt} > \frac{1}{\lambda}. \tag{21}$$

Now from (20) and (21) we have,

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}{r} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})}$$

and (16) is proved.

Finally, for each $\lambda \in \Lambda \subseteq \left]\frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}\right[$, Theorem 2.9 guarantees the existence of at least one non-trivial critical point for the functional $I_\lambda = \Phi - \lambda\Psi$, and the conclusion is obtained. □

A corollary of Theorem 3.1, is as follows.

Corollary 3.2. *$F : \mathbb{R}^N \to \mathbb{R}$ is a non-negative function such that $F(0) = 0$ and ∇F is continuous in $\mathbb{R}^N$. Moreover, suppose that*

$$\limsup_{|\xi|\to+0} \frac{F(\xi)}{|\xi|^p} = +\infty. \tag{22}$$

Then, for each $\theta > 0$ and $\lambda \in \left]0, \frac{\theta^p}{p\,c^p\left(\sup_{|\xi|\le\theta} F(\xi)\right)\int_0^T e^{Q(t)}dt}\right[$ the problem

$$\begin{cases} -(|u'|^{p-2}u')' - q(t)|u'|^{p-2}u' + A(t)|u|^{p-2}u = \\ \lambda\nabla F(u) \quad a.e.\ t\in[0,T], \\ u(0)-u(T) = u'(0) - e^{Q(T)}u'(T) = 0, \end{cases} \tag{23}$$

admits at least one non-trivial classical solution $u_\lambda \in E$ such that $\|u_\lambda\|_\infty < \theta$.

Proof. Fix $\theta > 0$, $\lambda \in \left]0, \frac{\theta^p}{p\,c^p\left(\sup_{|\xi|\le\theta} F(\xi)\right)\int_0^T e^{Q(t)}dt}\right[$.

By (22), there exists $x_0 \in \mathbb{R}^N$ with $|x_0|c(\bar\delta\ \int_0^T e^{Q(t)}dt)^{\frac{1}{p}} < \theta$, such that $\frac{F(x_0)}{|x_0|^p} > \frac{\bar\delta}{p\,\lambda}$.

Taking into account that $\lambda \in \left]0, \frac{\theta^p}{p\,c^p\left(\sup_{|\xi|\le\theta} F(\xi)\right)\int_0^T e^{Q(t)}dt}\right[$, one has

$$\frac{\left(\sup_{|\xi|\le\theta} F(\xi)\right)\int_0^T e^{Q(t)}dt}{\theta^p} < \frac{1}{\lambda\,p\,c^p} < \frac{F(x_0)}{c^p\,\bar\delta|x_0|^p}$$

and so condition (18) of Theorem 3.1 is verified. Now the desired result can be obtained from Theorem 3.1. □

Now, we will present an example for Corollary 3.2.

Example 3.3. Let $T = 1, p = 3$ and $A(t) = I$, where I is an identity matrix of order $N\times N, q(t) = 1$ and therefore $Q(t) = t$ for all $t \in [0,1]$. Due to the (5), we can consider $c = \sqrt[3]{4}$.

Also let $F(x) = \sinh(|x|^2)$ for all $x \in \mathbb{R}^N$ and hence $\sup_{|\xi|\le\theta} F(\xi) = \sinh(\theta^2)$ for all $\theta > 0$. Then for every $\lambda \in \left]0, \frac{\theta^3}{12(e-1)\sinh(\theta^2)}\right[$ all the hypotheses of Corollary 3.2 are satisfied and therefore the problem

$$\begin{cases} -(|u'|u')' - |u'|u' + |u|u = 2\lambda u\cos(|u|^2) \qquad a.e.\ t\in[0,1], \\ u(0)-u(1) = u'(0) - e\,u'(1) = 0, \end{cases}$$

admits at least one non-trivial classical solution $u_\lambda \in E$ such that $\|u_\lambda\|_\infty < \theta$.

Now, we point out the following existence results, as consequences of Theorem 2.10.

Theorem 3.4. *Let $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ satisfy assumption (2) and $\int_0^T e^{Q(t)} F(t,0) = 0$ for a.e. $t \in [0,T]$. Moreover Suppose that there are $M > 0$ and $\theta > p$ such that*

$$0 < \theta F(t,x) \leq \langle \nabla F(t,x), x \rangle \tag{24}$$

for all $x \in \mathbb{R}^N$ with $|x| \geq M$ and a.e. $t \in [0,T]$. Then for each $\lambda \in \left]0, \frac{1}{F^{cp^{\frac{1}{p}}}}\right[$ where $F^{cp^{\frac{1}{p}}}$ is defined by (17), problem (1) admits at least two distinct weak solutions.

Proof. Our aim is to apply Theorem 2.10, to problem (1). Put $r = 1$ and fixed $\lambda \in \left]0, \frac{1}{F^{cp^{\frac{1}{p}}}}\right[$. Let E, Φ and Ψ be as given in the proof of Theorem 3.1. We observe that the regularity assumptions of Theorem 2.10 on Φ and Ψ are satisfied and also according to proposition 2.8, the functional I_λ satisfies the (PS)-condition and it is unbounded from below. If $u \in \Phi^{-1}(]-\infty, 1[)$ then $\Phi(u) < 1$ and so $\|u\| < p^{\frac{1}{p}}$. Hence according to (6) we get

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}{r} = \sup_{u\in\Phi^{-1}(]-\infty,1[)} \Psi(u) =$$

$$\sup_{u\in\Phi^{-1}(]-\infty,1[)} \int_0^T e^{Q(t)} \, F(t,u)dt \leq \int_0^T e^{Q(t)} \sup_{|x|\leq cp^{\frac{1}{p}}} F(t,x)dt =$$

$$F^{cp^{\frac{1}{p}}} < \frac{1}{\lambda}. \tag{25}$$

From (25) we have

$$\lambda \in \left]0, \frac{1}{F^{cp^{\frac{1}{p}}}}\right[\subseteq \left]0, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}\right[.$$

So all the hypotheses of Theorem 2.10 are verified. Therefore, for each $\lambda \in \left]0, \frac{1}{F^{cp^{\frac{1}{p}}}}\right[$, the functional I_λ admits at least two distinct critical points and therefore the proof is completed. □

Finally, we present the following example to illustrate Theorem 3.4.

Example 3.5. Let $T = 2\pi$, $p = 3$ and $\theta = 4$. Again we can consider $c = \sqrt[3]{4}$ (see example 3.3) .

Now if we consider $F(t,x) = e^{-Q(t)}(\sin t + |x|^6)$, one has

$$0 < \theta\, e^{-Q(t)}(\sin t + |x|^6) \leq 6\, e^{-Q(t)}|x|^6$$

for all $x \in \mathbb{R}^N$ with $|x| \geq \sqrt[6]{2}$ and a.e. $t \in [0,2\pi]$ and so (24) is verified. Therefore according to Theorem 3.4 for each $\lambda \in \left]0, \frac{1}{F^{c\,p^{\frac{1}{p}}}}\right[$,

where $F^{c\,p^{\frac{1}{p}}} = F^{\sqrt[3]{12}} = \int_0^{2\pi} e^{Q(t)} \sup_{|x| \leq \sqrt[3]{12}} F(t,x)dt =$

$\int_0^{2\pi} (\sin t + 144)dt = 288\pi$
the problem

$$\begin{cases} -(|u'|u')' - q(t)|u'|u' + A(t)|u|u = \lambda e^{-Q(t)}|u|^4\, u & a.e.\ \ t \in [0,2\pi], \\ u(0) - u(2\pi) = u'(0) - e^{Q(2\pi)}u'(2\pi) = 0, \end{cases}$$

admits at least two classical solutions.

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