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Dominions and Zigzag theorem for Γ-semigroups

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Abstract. Dominions have been studied from different perspectives however their major application lies to study the closure property for monoids. The most useful characterization of semigroup domonions is provided by the famous Isbells Zigzag Theorem. In this paper, we introduce the dominion of a Γ-semigroups and give the analogue of Isbell's zigzag theorem in Γ-semigroups.

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1 Introduction

Dominions and zigzags were first studied in 1965 by Isbell [4] in connection with epimorphisms. Γ-semigroups was introduced by Sen [8]

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in 1981. In this paper, we introduce Γ-acts and Γ-tensor products on Γ-semigroups. Let S and Γ be any nonempty sets. Then S is called a Γ-semigroup if there exists a mapping from $S \times \Gamma \times S$ to S which maps (a, α, b) to $a\alpha b$ satisfying the condition $(a\alpha b)\gamma c = a\alpha(b\gamma c)$ for all $a, b, c \in S$ and for all $\alpha, \gamma \in \Gamma$. Let S_1 and S_2 be a Γ_1-semigroup and a Γ_2-semigroup respectively. A pair of mappings $f_1 : S_1 \to S_2$ and $f_2 : \Gamma_1 \to \Gamma_2$ is said to be homomorphism from (S_1, Γ_1) to (S_2, Γ_2) if $(s_1\gamma s_2)f_1 = (s_1)f_1(\gamma)f_2(s_2)f_1$ for all $s_1 \in S_1, s_2 \in S_2$ and $\gamma \in \Gamma_1$. Let U be a nonempty subset of a Γ-semigroup S. Then U is called a Γ-subsemigroup of S if $U\Gamma U \subset U$. A Γ-semigroup is called a Γ-monoid if there exists $1 \in S$ such that $1\gamma s = s = s\gamma 1$ for all $s \in S$ and $\gamma \in \Gamma_1$. Similarily, we can define a Γ-submonoid. Let U be a Γ-subsemigroup of S. Then we say that U dominates an element d of S if for every Γ-semigroup T and for all homomorphisms $\alpha, \beta : S \to T$ and $\alpha^{'}, \beta^{'} : \Gamma \to \Gamma^{'}$, $u\alpha = u\beta$ and $\gamma\alpha^{'} = \gamma\beta^{'}$ for all $u \in U$ implies $d\alpha = d\beta$. The set of all elements of S dominated by U is called the *dominion* of U in S, and we denote it by Γ-$Dom(U, S)$. It may easily be seen that Γ-$Dom(U, S)$ is a Γ-subsemigroup of S containing U.

In this paper, dominion of a Γ-semigroup has been characterized, and throughout this paper, Γ-semigroups are the member of special class of Γ-semigroups defined by

$$\mathcal{C} = \{S \text{ is } \Gamma\text{-semigroup} : a\gamma_1 b = c\gamma_2 d \implies \gamma_1 = \gamma_2 \text{ for all } a, b, c, d \in S, \forall \gamma_1, \gamma_2 \in \Gamma\}.$$

2 Γ-systems and Γ-tensor products

We generalize the concepts of systems and tensor product of semigroups [2] to Γ-semigroups by introducing Γ-systems and Γ-tensor products of Γ-semigroups.

Definition 2.1. Let S be a Γ-monoid and X be a any nonempty set. Then X is said to be a right Γ_S-system if there exists a map $X \times \Gamma \times S \to X$ defined by $(x, \gamma, s) \longmapsto x\gamma s$ such that following two conditions are satisfied:

(i) $x\gamma_1(s\gamma_2 t) = (x\gamma_1 s)\gamma_2 t, \quad \forall x \in X, \forall \gamma_1, \gamma_2 \in \Gamma, \forall s, t \in S$ and

(ii) $x\gamma_1 1 = x$.

Dually, we can define a left Γ_S-system.

Definition 2.2. Let S be a Γ'-semigroup, T be a Γ''-semigroup and X be a non-empty set. Then X is said to be (Γ'_S, Γ''_T)-bisystem if X is left Γ'_S-system and right Γ''_T-system. Moreover,

$$s\gamma'(x\gamma''t) = (s\gamma'x)\gamma''t, \quad \forall x \in X, \gamma' \in \Gamma', \gamma'' \in \Gamma'', \forall s \in S, t \in T.$$

Note 1. If X is (Γ'_S, Γ''_T)-bisystem, then we write $X \in \Gamma'_S$-ENS-Γ''_T.

Definition 2.3. Let X and Y be left Γ_S-systems. Then a map $\phi : X \longrightarrow Y$ satisfying $(s\gamma x)\phi = s\gamma(x\phi)$, $\forall x \in X, s \in S, \gamma \in \Gamma$ is called Γ_S-map from X to Y.

Definition 2.4. A relation ρ on a left Γ_S-system X is called a congruence if ρ is an equivalence relation on X such that for all $x, y \in X, \gamma \in \Gamma$ and $s \in S$, $x\rho y \Rightarrow (s\gamma x)\rho(s\gamma y)$.

Let $X/\rho = \{x\rho \mid x \in X\}$. Then it can be easily verified that X/ρ is a left Γ_S-system, where action is defined by $s\gamma(x\rho) = (s\gamma x)\rho$.

Definition 2.5. Let X be left Γ'_S-system and Y be right Γ''_T-system. We see that $X \times Y$ is a (Γ'_S, Γ''_T)-bisystem. Define $s\gamma'(x, y) = (s\gamma'x, y)$ and $(x, y)\gamma''t = (x, y\gamma''t)$ where $(x, y) \in X \times Y, \gamma' \in \Gamma', \gamma'' \in \Gamma'', s \in S, t \in T$.

One can easily verify that $X \times Y$ is a (Γ'_S, Γ''_T)-bisystem with respective to the above action.

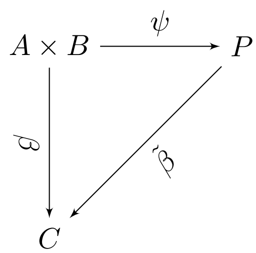
Definition 2.6. Let $A \in \Gamma'_T$-ENS-$\Gamma''_S, B \in \Gamma''_S$-ENS-$\Gamma'''_U$, and $C \in \Gamma'_T$-ENS-Γ'''_U. A map $\beta : A \times B \longrightarrow C$ is said to be (Γ'_T, Γ'''_U) map if for all $a \in A, b \in B, \gamma' \in \Gamma, \gamma''' \in \Gamma''', t \in T$ and $u \in U$

$$(t\gamma'(a, b))\beta = t\gamma'((a, b))\beta \text{ and } ((a, b)\gamma'''u)\beta = (a, b)\beta\gamma'''u.$$

Definition 2.7. A (Γ'_T, Γ'''_U) map $\beta : A \times B \longrightarrow C$ is called a bimap if

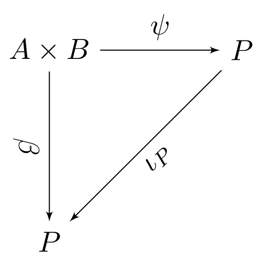
$$(a\gamma''s, b)\beta = (a, s\gamma''_1 b)\beta, \quad \forall a \in A, b \in B, \gamma'', \gamma''_1 \in \Gamma''.$$

Definition 2.8. A pair (ρ, ψ) consisting of a $((\Gamma'_T, \Gamma'''_U))$-bisystem P and a bimap $\psi : A \times B \longrightarrow P$ will be called a tensor product of A and B if for every (Γ'_T, Γ'''_U)-bisystem C and every bimap $\beta : A \times B \longrightarrow C$, there exists a unique (Γ'_T, Γ'''_U) map $\tilde{\beta} : P \longrightarrow C$ such that the diagram



commutes.

Moreover , when $C = P$ and $\beta = \psi$, the unique $\tilde{\beta}$ is ι_P (identity map)



Lemma 2.9. *If a tensor product of A and B over S exists then it is unique upto isomophism.*

Proof. It follows on the lines similar to the lines of the proof lemma 8.17 [2]. $\square$

Define $A \otimes B = A \times B/\tau$, where τ is the equivance relation on $A \times B$ generated by the relation

$$T = \{(a\gamma s, b), (a, s\gamma b) \mid a \in A, b \in B, s \in S, \gamma \in \Gamma\}.$$

We denote the τ class of (a, b) by $(a, b)\tau = a \otimes_{\Gamma_S} b$.
Note 2. $a\gamma s \otimes_{\Gamma_S} b = a \otimes_{\Gamma_S} s\gamma b$ for all $a \in A, b \in B, s \in S, \gamma \in \Gamma$.

Proposition 2.10. *Let $a \otimes_{\Gamma_S} b, c \otimes_{\Gamma_S} d \in A \otimes_{\Gamma_S} B$. Then $a \otimes b = c \otimes d$ iff there exist $a_1, a_2, \cdots a_{n-1} \in A$, $b_1, b_2, \cdots b_{n-1}, b_n \in B$, $s_1, s_2, \cdots s_{n-1}, s_n, t_1, t_2, \cdots t_{n-1} \in$*

S such that

$$
\begin{aligned}
a &= a_1\gamma s_1, & s_1\gamma b &= t_1\gamma b_1,\\
a_1\gamma t_1 &= a_2\gamma s_2, & s_2\gamma b_1 &= t_2\gamma b_2\\
a_2\gamma t_2 &= a_3\gamma s_3, & s_3\gamma b_2 &= t_2\gamma b_3\\
&. & &.\\
&. & &.\\
&. & &.\\
a_{n-1}\gamma t_{n-1} &= c\gamma s_n, & s_n\gamma b_{n-1} &= d.
\end{aligned}
$$

Proof.

$$
\begin{aligned}
a\otimes_{\Gamma_S} b &= a_1\gamma s_1\otimes b\\
&= a_1\otimes s_1\gamma b\\
&= a_1\otimes t_1\gamma b_1\\
&= a_1\gamma t_1\otimes b_1\\
&= a_2\gamma s_2\otimes b_1\\
&= a_2\otimes s_2\gamma b_1\\
&= a_2\otimes t_2\gamma b_2\\
&.\\
&.\\
&.\\
&= a_{n-1}\otimes t_{n-1}\gamma b_{n-1}\\
&= a_{n-1}\gamma t_{n-1}\otimes b_{n-1}\\
&= c\gamma s_n\otimes b_{n-1}\\
&= c\otimes s_n\gamma b_{n-1}\\
&= c\otimes_{\Gamma_S} d.
\end{aligned}
$$

Conversely suppose that $a\otimes b = c\otimes d$, then by Theorem 1.4.10 [2],

$$(a,b) = (p_1,q_1)\to(p_2,q_2)\to\cdots\to(p_{n-1},q_{n-1})\to(p_n,q_n) = (c,d),$$

Where $((p_{i-1},q_{i-1}),(p_{i+1},q_{i+1}))\in T\cup T^{-1}$. We can assume that the sequence begins and end with right move $(a,b) \to (a\gamma b)$.

$$\begin{aligned}
(a,b)=(p_1,q_1) &= (a_1\gamma s_1, b) \to (a_1, s_1\gamma b)\\
&= (a_1, t_1\gamma b_1) \to (a_1\gamma t_1, b_1)\\
&= (a_2\gamma s_2, b_1) \to (a_2, s_2\gamma b_1)\\
&= (a_2, t_2\gamma b_2) \to (a_2\gamma t_2, b_2)\\
&= (a_3\gamma s_3, b_2) \to (a_2, s_3\gamma b_2)\\
&= (a_3, t_3\gamma b_3) \to (a_3\gamma t_3, b_3)\\
&.\\
&.\\
&.\\
&= (a_{n-1}, t_{n-1}\gamma b_{n-1}) \to (a_{n-1}\gamma t_{n-1}, b_{n-1})\\
&= (c\gamma s_n, b_{n-1}) \to (c, s_n\gamma b_{n-1}) = (c,d).
\end{aligned}$$

This gives

$$\begin{aligned}
a &= a_1\gamma s_1, & s_1\gamma b &= t_1\gamma b_1,\\
a_1\gamma t_1 &= a_2\gamma s_2, & s_2\gamma b_1 &= t_2\gamma b_2\\
a_2\gamma t_2 &= a_3\gamma s_3, & s_3\gamma b_2 &= t_2\gamma b_3\\
&. & &.\\
&. & &.\\
&. & &.\\
a_{n-1}\gamma t_{n-1} &= c\gamma s_n, & s_n\gamma b_{n-1} &= d.
\end{aligned}$$

□

Proposition 2.11. *The equivalence τ defined on $A \times B$ is a (Γ_S, Γ_S)-congruence and $t\gamma(a \otimes b) = (t\gamma a) \otimes b$, $(a \otimes b)\gamma a = a \otimes (b\gamma a)$.*

Proof. Suppose $(a,b)\tau(c,d) \Rightarrow (a,b)\tau = (c,d)\tau \Rightarrow a \otimes b = c \otimes d$. Then, by Proposition 2.10, we have

$$t\gamma(a \otimes b) = (t\gamma a) \otimes b \text{ and } (a \otimes b)\gamma a = a \otimes (b\gamma a)$$

for all $s, t \in S, a, c \in A$ and $b, d \in B$. This implies

$$\begin{aligned} &(t\gamma a) \otimes b = (t\gamma c) \otimes d \\ \Rightarrow &((t\gamma a), b)\tau = ((t\gamma c), d)\tau \\ \Rightarrow &(t\gamma(a, b))\tau = (t\gamma(c, d))\tau. \end{aligned}$$

Similarly

$$((a, b)\gamma s)\tau = ((c, d)\gamma s)\tau.$$

So τ is a congruence. □

Proposition 2.12. *Let $A, B \in \Gamma_S$-ENS-Γ_S. Then $(A \otimes_{\Gamma_S} B, \tau^{\#})$ is a tensor product of A and B over S.*

Proof. We have $\tau^{\#} : A \times B \to (A \times B)/\tau = A \otimes_{\Gamma_S} B$ defined by $(a, b)\tau^{\#} = (a, b)\tau = a \otimes_{\Gamma_S} b$. We have

$$\begin{aligned} (a, b)\tau^{\#} &= (s\gamma a, b)\tau^{\#} \\ &= (s\gamma a) \otimes b \\ &= s\gamma(a \otimes b) \\ &= s\gamma(a, b)\tau^{\#} \end{aligned}$$

and

$$\begin{aligned} (a, b)\gamma s\tau^{\#} &= (a, b\gamma s)\tau^{\#} \\ &= a \otimes (b\gamma s) \\ &= (a \otimes b)\gamma s \\ &= (a, b)\tau^{\#}\gamma s. \end{aligned}$$

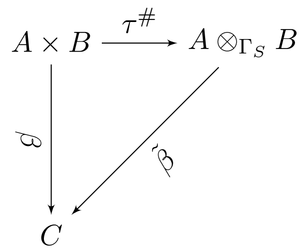
Therefore $\tau^{\#}$ is a (Γ_S, Γ_S) map. Again

$$\begin{aligned} (a\gamma s, b)\tau^{\#} &= a\gamma s \otimes b \\ &= a \otimes s\gamma b \\ &= (a, s\gamma b)\tau^{\#}. \end{aligned}$$

Therefore $\tau^{\#}$ is a bimap. Now let $C \in \Gamma_S$-ENS-Γ_S and let $\beta : A \times B \longrightarrow C$ be a bimap. Define $\tilde{\beta} : A \otimes_{\Gamma_S} B \longrightarrow C$ by

$$(a \otimes b)\tilde{\beta} = (a, b)\beta \tag{1}$$

By using Proposition 2.10, we can easily verify that $\tilde{\beta}$ is well defined and the diagram



commutes.
Since $((a, b)\tau^{\#}) = a \otimes b$, from (1), we have

$$((a, b)\tau^{\#})\tilde{\beta} = (a, b)\beta \Rightarrow \tilde{\beta} = \beta.$$

Thus $(A \otimes_{\Gamma_S} B, \tau^{\#})$ is a tensor product. $\square$

3 Isbell *zigzag* theorem for Γ-semigroups

In [4], Isbell gave the characterisation of semigroup dominion. In the next theorem, we generalize the Isbell *zigzag* theorem for Γ-semigroups and give the charaterization of Γ-semigroup dominion. Infact we prove the following:

Theorem 3.1. *Let U be a Γ-submonoid of a Γ-monoid S. Then $d \in$ Γ-Dom(U, S) iff $d \in U$ or there exists a series of factorization for d as follows:*

$$d = a_0\gamma t_1 = y_1\gamma a_1\gamma t_1 = y_1\gamma a_2\gamma t_2 = y_2\gamma a_3\gamma t_2 = \cdots = y_m\gamma a_{2m-1}\gamma t_2 m = y_m\gamma a_{2m}$$

Where $m \geq 1, a_i \in U$ $(i = 0, 1, 2, ..., 2m)$, $y_i, t_i \in S$ $(i = 1, 2, ..., m)$ and

$$a_0 = y_1\gamma a_1, \qquad a_{2m-1}\gamma t_m = a_{2m},$$
$$a_{2i-1}\gamma t_i = a_{2i}\gamma t_{i+1}, \qquad y_i\gamma a_{2i} = y_{i+1}\gamma a_{2i+1}$$

where $1 \leq i \leq m - 1$. Such a series of factorization is called a zigzag in S over U with value d, length m and spines $a_0, a_1, a_2, ..., a_{2m}$.

To prove theorem, we first prove the following lemma:

Lemma 3.2. *Let U be a Γ-submonoid of a Γ-monoid S. Then $d \in \Gamma$-$Dom(U,S)$ iff $d \otimes_\Gamma 1 = 1 \otimes_\Gamma d$ in $S \otimes_{\Gamma_U} S$.*

Proof. Suppose $d \otimes_\Gamma 1 = 1 \otimes_\Gamma d$. Let T be a Γ'-monoid and $\alpha, \beta : S \longrightarrow T$ and $\alpha', \beta' : \Gamma \longrightarrow \Gamma'$ such that $u\alpha = u\beta\ \forall u \in U$ and $\gamma(\alpha') = \gamma(\beta')\ \forall \gamma \in \Gamma$. We show that $d\alpha = d\beta$. To, show for this, first we show that T is (Γ_U, Γ_U)-bisystem. Define

$$u\gamma' t = (u\alpha)\gamma' t\ [= (u)\beta\gamma' t] \text{ and } t\gamma' u = t\gamma'(u\beta)\ [t\gamma'(u\alpha)].$$

Define $\psi : S \times S \longrightarrow T$ by $(s,s')\psi = (s\alpha)\gamma'(s'\beta)$. Then ψ is a (Γ_U, Γ_U) map and is also a bimap.

$$\begin{aligned}(u\gamma(s,s'))\psi &= (u\gamma s, s')\psi \\ &= (u\gamma s)\alpha\gamma'(s')\beta \\ &= (u)\alpha(\gamma)\alpha'(s)\alpha\gamma'(s')\beta \\ &= u\gamma_1'(s\alpha)\gamma'(s')\beta \\ &= u\gamma_1'(s,s')\psi.\end{aligned}$$

Similarly,

$$((s,s')\gamma a)\psi = (s,s')\psi\gamma' u \text{ where } \gamma'\alpha' = \gamma_1'.$$

$$\begin{aligned}(s\gamma u, s')\psi &= (s\gamma u)\alpha\gamma' s'\beta \\ &= (s\alpha)\gamma_1'(u\alpha)\gamma' s'\beta \\ &= (s\alpha)\gamma_1'(u\beta)\gamma' s'\beta \\ &= (s\alpha)\gamma_1'(u\gamma s')\beta \\ &= (s, u\gamma s')\psi.\end{aligned}$$

Therefore (T,ψ) is a tensor product. But $(S \otimes_\Gamma S, \tau^\psi)$ is also tensor product. Therefore by Proposition 2.12, there exists a map $\tilde{\psi} : S \otimes_\Gamma S \longrightarrow T$ such that

$$(s \otimes s')\tilde{\psi} = (s,s')\psi = (s\alpha)\gamma'(s'\beta)\ (\forall s \otimes s' \in S \otimes_\Gamma S). \qquad (2)$$

Now,

$$\begin{aligned} d\alpha &= (d\gamma 1)\alpha \\ &= (d\alpha)(\gamma\alpha')(1\alpha) \\ &= (d\alpha)(\gamma\alpha')(1\beta) \\ &= (d\alpha)\gamma'(1\alpha) \\ &= (d \otimes 1)\tilde{\psi} \quad \text{(by 3)} \\ &= (1 \otimes d)\tilde{\psi} \\ &= (1\alpha)\gamma' d\beta \\ &= d\beta. \end{aligned}$$

Therefore $d\alpha = d\beta \Rightarrow d \in \Gamma\text{-}Dom(U, S)$.
Conversely suppose $d \in \Gamma\text{-}Dom(U, S)$. Let $A = S \otimes_{\Gamma_U} S$. Then A is a (Γ_S, Γ_S)-bisystem as

$$s\gamma(x \otimes y) = (s\gamma x) \otimes y \text{ and } (x \otimes y)\gamma s = x \otimes (y\gamma s).$$

Let $(Z(A), +)$ be a free abelian group on A *i.e.*

$$Z(A) = \{\Sigma z_i a_i \ : \ z_i \in Z, a_i \in A\}.$$

Then $Z(A)$ is also a (Γ_S, Γ_S)-bisystem defined by

$$s\gamma(\Sigma z_i a_i) = \Sigma z_i(s\gamma a_i) \text{ and } (\Sigma z_i a_i)\gamma s = \Sigma z_i(a_i\gamma s).$$

Now we show that $S \times Z(A)$ is a Γ-semigroup. Define a map $\phi : (S \times Z(A)) \times \Gamma \times (S \times Z(A)) \longrightarrow S \times Z(A)$ by $((p, x)\gamma(q, y))\phi = (p\gamma y, x\gamma q + p\gamma y)$, where $p, q \in S, x, y \in Z(A)$ and $\gamma \in \Gamma$. Then $S \times Z(A)$ is a Γ-semigroup.

Define $\alpha : S \longrightarrow S \times Z(A)$ by $s\alpha = (s, 0)$ and $\beta : S \longrightarrow S \times Z(A)$ by $s\beta = (s, s \otimes 1 - 1 \otimes s)$. Since $u \otimes 1 = 1 \otimes u$, therefore

$$u\alpha = (u, 0) = (1u, u \otimes 1 - 1 \otimes u) = u\beta \ (\forall u \in U).$$

This implies that $d\alpha = d\beta$ (since $d \in \Gamma\text{-}Dom(U, S)$). So

$$(d, 0) = (d, d \otimes 1 - 1 \otimes d) \Rightarrow d \otimes 1 = 1 \otimes d.$$

To complete the proof of the theorem, take any $d \in \Gamma\text{-}Dom(U,S)$. By above lemma $d \otimes 1 = 1 \otimes d$. Now, by Proposition 2.10, the proof of the *zigzag* theorem is completed. □

Now we show that Isbell's theorem is also aplicable to Γ-semigroups as well as to Γ-monoids.

Theorem 3.3. *Let U be a Γ-subsemigroup of a Γ-semigroup S and let $d \in S$. Then $d \in \Gamma\text{-}Dom(U,S)$ if and only if $d \in U$ or there exists a series of factorizations of d as follows:*

$$d = a_0\gamma t_1 = y_1\gamma a_1\gamma t_1 = y_1\gamma a_2\gamma t_2 = y_2\gamma a_3\gamma t_2 = \cdots = y_m\gamma a_{2m-1}\gamma t_m = y_m\gamma a_{2m}$$

where $m \geq 1,\ a_i \in U\ (i = 0, 1, \ldots, 2m),\ y_i, t_i \in S\ (i = 1, 2, \ldots, m)$, and

$$a_0 = y_1\gamma a_1, \qquad a_{2m-1}\gamma t_m = a_{2m},$$
$$a_{2i-1}\gamma t_i = a_{2i}\gamma t_{i+1}, \qquad y_i\gamma a_{2i} = y_{i+1}\gamma a_{2i+1} \quad (1 \leq i \leq m-1).$$

Such a series of factorization is called a zigzag in S over U with value d, length m and spine $a_0, a_1, \ldots, a_{2m}$.

Proof. We begin by adjoining an identity element 1 to the Γ-semigroup S whether or not it already has one. If we call the resultant Γ-monoid and write $U \cup 1$ as $U^\star$, then we can easily verify that an element $d \in S \setminus U$ is in $\Gamma\text{-}Dom(U,S)$ iff $d \in \Gamma\text{-}Dom(U^\star, S^\star)$ and hence, if $d \otimes 1 = 1 \otimes d$ in the tensor product of $S^\star \otimes S^\star$. Then to obtain Isbell's theorem, we simply observe the *zigzag* in which the element 1 appears can be shorten. For example, if $a_{2i} = 1$, then the equalities

$$a_{2i-3}\gamma t_{i-1} = a_{2i-2}\gamma t_i, \qquad y_{i-1}\gamma a_{2i-2} = y_i\gamma a_{2i-1}$$
$$a_{2i-1}\gamma t_{i+1} = a_{2i}\gamma t_{i+1}, \qquad y_i\gamma a_{2i} = y_{i+1}\gamma a_{2i+1}$$
$$a_{2i+1}\gamma t_{i+1} = a_{2i+2}\gamma t_{i+2}, \qquad y_{i+1}\gamma a_{2i+2} = y_{i+2}\gamma a_{2i+3}$$

can be collapsed to

$$a_{2i-3}\gamma t_{i-1} = a_{2i-2}\gamma t_i, \qquad y_{i-1}\gamma a_{2i-2} = y_{i+1}\gamma a_{2i+1}\gamma a_{2i-1}$$
$$a_{2i+1}a_{2i-1}\gamma t_i = a_{2i+1}\gamma t_{i+1} = a_{2i+2}\gamma t_{i+2}, \quad y_{i+1}\gamma a_{2i+2} = y_{i+2}\gamma a_{2i+3}.$$

Similarly, if $a_{2i+1} = 1$, the length of the zigzag can be reduced.

Similarly $t_i \neq 1$, for if $t_1 = 1$, then $d = a_0\gamma 1 = a_0 \in U$ which is a contradiction. Now suppose that t_k(k>1) is the first variable such that $t_k = 1$, then

$$\begin{aligned} d &= a_0\gamma t_1 \\ &= y_1\gamma a_1\gamma t_1 \\ &= y_1\gamma a_2\gamma t_2 \\ &= y_2\gamma a_3\gamma t_3 \\ &. \\ &. \\ &. \\ &= y_k\gamma a_{2k-1}\gamma t_k \\ &= y_k\gamma a_{2k-1}. \end{aligned}$$

is a *zigzag* by length k with value d in which every t_i is in S. Therefore all t_i's and a_{2i}'s are members of S. Similarly, all y_i's and a_{2i-1}'s are in S. Hence the theorem follows. $\square$

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