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The Best Proximity Pair Focusing on Monotonicity and T-Absolutely Direct Sets

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Abstract. In this article, we mostly pay attention to the existence and uniqueness of the best proximity pair for T-absolutely direct sets. This investigation is based on some interesting relations existing in Banach lattices, in which $T : A \to B$ is an arbitrary map. The current provides a new view of the best proximity pair in Banach lattices by introducing T-absolutely direct sets.

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1. Introduction

The uniform monotonicity was introduced and studied by Birkhoff in [4]. Hudzik and Narloch in [9] studied the relationships between monotonicity and complex rotundity properties. We say that the Banach lattice X has the STM property (X is an strictly monotone space) if for

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any $u, v \in X$ such that $u \geqslant v \geqslant 0$ and $\|u\| = \|v\|$, it can be concluded that $u = v$. Also the Banach lattice X has the UM property (X is a uniformly monotone space) if for all $u_n \geqslant v_n \geqslant 0$ such that $\lim_{n\to\infty} \|u_n\| = \lim_{n\to\infty} \|v_n\|$ implies $\|u_n - v_n\| \to 0$ and the Banach lattice X has the ULUM (LLUM) property (X is a upper (lower) locally uniformly monotone space) if for each $u, v_n \in X$, $v_n \geqslant u \geqslant 0$ ($u \geqslant v_n \geqslant 0$) such that $v_n \to u$ then $\|v_n - u\| \to 0$. In 1992 Kurc [13] stated that the relation between the UM and UR (uniformly rotund) property as well as between the STM and R (rotund) property. Further relations can be found in [7]. Kurc in [13] introduced the dominated best approximation problem and in [8] Hudzik and Kurc generalized this problem on LLUM and ULUM spaces. In [5] the authors has discussed more general forms of the best approximation problem in Banach lattices by means of monotonicities. More details about Banach lattices and monotonicity could be found in [1, 11, 14, 15].

Let A and B be nonempty subsets of a normed space $(X, \|.\|)$ and $T : A \to B$ be a map. If $\|x - Tx\| = d(A, B)$ for some $x \in A$, in which $d(A, B) = \inf\{\|x - y\| : (x, y) \in A \times B\}$ then (x, Tx) is called the best proximity pair and x is called the best proximity point. The set of all the best proximity points is denoted by $P_T(A, B)$, i.e., $P_T(A, B) = \{x \in A : \|x - Tx\| = d(A, B)\}$.

The best proximity pair problem in Banach spaces has already been examined by considering some special conditions. In [12] Kirk et al introduced cyclic mapping with a restriction condition and Eldred and Veeramani in [6] introduced cyclic contraction maps and discussed the best proximity problem for cyclic contraction maps on uniformly convex Banach spaces. In [17] this problem is examined for relatively nonexpansive maps. Also proximinal pointwise contraction maps are defined by Anuradha and Veeramani in [3] and they proved the existence of best proximity points on a pair of weakly compact convex subsets of a Banach space. You can refer to [2, 10, 16] for some other maps. In this paper we will connect between monotonicity properties and the best proximity pair problem.

2. Preliminaries

Let X be a Banach lattice with a lattice norm $\|\cdot\|$. The norm $\|\cdot\|$ has the strictly monotone property if for all $x, y \in X^+$, the conditions $x \geqslant y$, $y \neq 0$ and $\|x\| = \|y\|$ implies $x = y$, in this case we say that X is an STM space or $X \in$ STM. Also we say that the norm is uniformly monotone ($X \in$ UM) if for any $y_n \geqslant x_n \geqslant 0$, $\lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \|y_n\|$ implies $\|y_n - x_n\| \to 0$.

A Banach lattice X is said to be upper (lower) locally uniformly monotone, $X \in$ ULUM ($X \in$ LLUM), if for any $x, y_n \in X$, $y_n \geqslant x \geqslant 0$ ($x \geqslant y_n \geqslant 0$), and $\|y_n\| \to \|x\|$ imply $\|y_n - x\| \to 0$.

Obviously,

$$\begin{array}{ccc} UM & \Rightarrow & LLUM \\ \Downarrow & & \Downarrow \\ ULUM & \Rightarrow & STM \end{array}$$

For Example, L_p-spaces with $1 \leqslant p < \infty$ are UM spaces, but the space L_∞ is not even an STM space.

Recall that $(X, \|.\|)$ has order continuous norm if $0 \leqslant x_\alpha \downarrow 0$ implies $\|x_\alpha\| \to 0$.

From here on, A and B are two nonempty subsets of X and $T : A \to B$ is a map.

Definition 2.1. *Let $x \in A$. A sequence $\{x_n\} \subseteq A$ is said to be a Tx-minimizing sequence in A if $\lim_{n\to\infty} \|x_n - Tx\| = d(A, B)$ and $\{x_n\} \subseteq A$ is said to be a T-minimizing sequence in A if $\lim_{n\to\infty} \|x_n - Tx_n\| = d(A, B)$.*

Definition 2.2. *Let $x \in A$. A subset $A \subseteq X$ is said to be a Tx-absolutely direct set if for any $y, z \in A$ there exists $w \in A$ such that $|w - Tx| \leqslant |y - Tx| \wedge |z - Tx|$.*

Definition 2.3. *A subset $A \subseteq X$ is said to be a T-absolutely direct set if for any $x, y \in A$ there exists $z \in A$ such that $|z - Tx| \leqslant |x - Tx| \wedge |y - Tx|$ and $|z - Ty| \leqslant |x - Ty| \wedge |y - Ty|$.*

Example 2.4. Suppose that X is a Banach lattice, $x \in X$ and $A =$

$\{\alpha x : \alpha \in \mathbb{R}\}$. If T is a identity map then A is a Tx-absolutely direct set for any $x \in A$, but A is not a T-absolutely direct set.

Remark 2.5. *If A is a sublattice of a Banach lattice X (i.e., A is closed with respect to the finite infimum and supremum) and $A \geqslant B$ (or $B \geqslant A$) then A is a T-absolutely direct set and also A is a Tx-absolutely direct set for any $x \in A$. Recall that $A \geqslant B$ means $x \geqslant y$ for any $x \in A$ and $y \in B$.*

Definition 2.6. *The best proximity pair problem is said to be*

1. *T-solvable if $P_T(A, B) \neq \emptyset$,*
2. *T-uniquely solvable if card$(P_T(A, B)) = 1$ (or Tx-solvable if $P_T(A, B) = \{x\}$),*
3. *T-stable if for every T-minimizing sequence $\{x_n\}$ in A, dist$(x_n, P_T(A, B)) \to 0$ as $n \to \infty$,*
4. *T-strongly solvable if it is T-stable and T-uniquely solvable.*
5. *Tx-stable if for every Tx-minimizing sequence $\{x_n\}$ in A, dist$(x_n, P_T(A, B)) \to 0$ as $n \to \infty$ and*
6. *Tx-strongly solvable if it is Tx-stable and Tx-solvable.*

Example 2.7. Consider the space $X = \mathbb{R}^2$ with Euclidean norm and coordinatewise ordering, i.e., if $x = (x_1, x_2)$ and $y = (y_1, y_2)$, then $x \leqslant y$ if and only if $x_1 \leqslant y_1$ and $x_2 \leqslant y_2$ hold in $\mathbb{R}$. Let $A = \{(0, y) : y \in \mathbb{R}\}$, $B = \{(1, y) : y \in \mathbb{R}\}$ and $T : A \to B$ be defined as

$$T((0, y)) = \begin{cases} (1, 1), & y \in \mathbb{Q} \\ (1, 0), & y \notin \mathbb{Q} \end{cases}.$$

It is easy to see that $P_T(A, B) = \{(0, 1)\}$, A is not a T-absolutely direct set and it is Tx-absolutely direct set for any $x \in A$. Moreover the best proximity pair problem is neither T-stable nor Tx-stable for any $x = (0, y)$ such that $y \notin \mathbb{Q}$. Note that if $y \in \mathbb{Q}$, then this problem is Tx-stable.

3. Main Results

In this section we will examine, in two separate parts, monotonicity and proximity pair for T-absolutely and Tx-absolutely direct sets. In these parts we are going to give some relations between the best proximity pair problem and monotonicity properties in Banach lattices.

3.1 Monotonicity and the best proximity pair for T-absolutely direct sets

In this part we prove some uniqueness and existence theorems about the best proximity pair for T-absolutely direct sets and its relation with Monotonicity in Banach lattices.

Theorem 3.1.1. *A Banach lattice X is an STM space if and only if card$(P_T(A,B)) \leqslant 1$, for any convex subset A of X, which A is a T-absolutely direct set.*

Proof. Necessity. Suppose there exist $x, y \in A$ such that $\|x - Tx\| = \|y - Ty\| = d(A,B) = d$. Since A is a T-absolutely direct set, there exists $z \in A$ such that $|z - Tx| \leqslant |x - Tx| \wedge |y - Tx|$ and $|z - Ty| \leqslant |x - Ty| \wedge |y - Ty|$. Thus, $d \leqslant \|z - Tx\| \leqslant \|x - Tx\| = d$, and by the STM property of X, we obtain $|z - Tx| = |x - Tx|$. Moreover since A is convex, we have $\frac{x+z}{2} \in A$. Therefore,

$$d = \|x - Tx\| \leqslant \left\| \frac{x+z}{2} - Tx \right\| \leqslant \frac{\|x - Tx\| + \|z - Tx\|}{2} = d.$$

Thus $|x - Tx| = \left|\frac{x+z}{2} - Tx\right| = |z - Tx|$ according to which $X \in STM$ and $\left|\frac{x+z}{2} - Tx\right| \leqslant \frac{|x-Tx|+|z-Tx|}{2} = |x - Tx|$. Notice that any Banach lattice X has the following property

$$|f + g| + |f - g| = 2(|f| \vee |g|), \quad (\forall f, g \in X) \tag{1}$$

which leads to

$$|x - z| = 2(|x - Tx| \vee |z - Tx|) - |x + z - 2Tx| = 0,$$

i.e., $x = z$. By the relation $|z - Ty| \leqslant |x - Ty| \wedge |y - Ty|$, similarly we get $y = z$, and thus $x = y$.

Sufficiency. If $X \notin$ STM then there exist $x_1, x_2 \in X$ such that $x_1 \geqslant x_2 > 0$ and $\|x_1 + x_2\| = \|x_1\|$. Let $A = \{\lambda x_2 : \lambda \in [0,1]\}$, $B = \{-x_1\}$ and $T : A \to B$ be a constant function. Then A is a convex T-absolutely direct set and for any $\lambda x_2 \in A$, $|\lambda x_2 - (-x_1)| = |x_1 + \lambda x_2| \geqslant x_1 = |x_1|$. Therefore, $\|x_1\| = \|x_1 + x_2\| \geqslant \|x_1 + \lambda x_2\| \geqslant \|x_1\|$, which yields $\|x_1 + \lambda x_2\| = \|x_1\| = d(A,B)$ for all $\lambda \in [0,1]$. Hence $P_T(A,B) = A$, which is a contradiction. $\square$

Theorem 3.1.2. *Assume a Banach lattice X has the UM property, and $A \subseteq X$ is a closed convex T-absolutely direct set. Then any T-minimizing sequence in A is convergent. Moreover if T is also a continuous map then card$(P_T(A,B)) = 1$.*

Proof. Suppose $\{z_n\} \subseteq A$ is a T-minimizing sequence in A, and so $\lim\limits_{n\to\infty} \|z_n - Tz_n\| = d(A,B) = d$. We show that $\{z_n\}$ is a Cauchy sequence. Otherwise, there are subsequences $\{z_{n_k}\}$ and $\{z_{m_k}\}$ of $\{z_n\}$ and $\varepsilon > 0$ such that $\|z_{n_k} - z_{m_k}\| \geqslant \varepsilon$ for large k's. Now since A is a closed convex T-absolutely direct set, there exists $x_k \in A$ such that

$$|x_k - Tz_{n_k}| \leqslant |z_{n_k} - Tz_{n_k}| \wedge |z_{m_k} - Tz_{n_k}|$$

and

$$|x_k - Tz_{m_k}| \leqslant |z_{n_k} - Tz_{m_k}| \wedge |z_{m_k} - Tz_{m_k}|.$$

Therefore $d \leqslant \|x_k - Tz_{n_k}\| \leqslant \|z_{n_k} - Tz_{n_k}\| \to d$. In view of the fact that A is convex, then $\frac{z_{n_k}+x_k}{2} \in A$ and

$$d \leqslant \left\|\frac{z_{n_k} + x_k}{2} - Tz_{n_k}\right\| \leqslant \frac{\|z_{n_k} - Tz_{n_k}\| + \|x_k - Tz_{n_k}\|}{2} \leqslant \|z_{n_k} - Tz_{n_k}\| \to d.$$

And since $X \in$ UM and

$$\left|\frac{z_{n_k} + x_k}{2} - Tz_{n_k}\right| \leqslant \frac{|z_{n_k} - Tz_{n_k}| + |x_k - Tz_{n_k}|}{2} \leqslant |z_{n_k} - Tz_{n_k}|,$$

we have

$$\left\| |z_{n_k} - Tz_{n_k}| - \left|\frac{z_{n_k} + x_k}{2} - Tz_{n_k}\right| \right\| \to 0. \tag{2}$$

Then by (1) and (2), we obtain

$\|z_{n_k} - x_k\| = \|2\left(|z_{n_k} - Tz_{n_k}| \vee |x_k - Tz_{n_k}|\right) - |z_{n_k} + x_k - 2Tz_{n_k}|\| \to 0.$

Likewise, we get $\|z_{m_k} - x_k\| \to 0$. Hence $\|z_{n_k} - z_{m_k}\| \to 0$, which is a contradiction.

Finally let $z_n \to z \in A$ and T is a continuous map. Then $Tz_n \to Tz$ and as a result $\|z - Tz\| = d(A, B)$. On the other hand X has the STM property, so by Theorem 3.1.1, $\operatorname{card}(P_T(A, B)) \leqslant 1$, therefore $\operatorname{card}(P_T(A, B)) = 1$. $\square$

Example 3.1.3. Let $X = \mathbb{R}^2$ be a Banach lattice with Euclidean norm and coordinatewise ordering. Put $A = \{(x, y) : x + y = 1, 0 \leqslant x \leqslant 1\}$ and $B = \{(x, y) : x + y = 2\}$. We define the continuous map $T : A \to B$ by $T(x, y) = (x + 1/2, y + 1/2)$. X is a UM space and A a closed convex subset of X, but $P_T(A, B) = A$. We can see the condition T-absolutely in Theorem 3.1.2 is necessary. Note that all sequences in A are T-minimizing.

Theorem 3.1.4 *Let $X \in$ UM and $P_T(A, B) \neq \emptyset$. If A is a convex T-absolutely direct set Then the best proximity pair problem is T-strongly solvable.*

Proof. Since $X \in$ UM implies $X \in$ STM, by Theorem 3.1.1, $\operatorname{card}(P_T(A, B)) \leqslant 1$ for any convex subset A of X and any mapping $T : A \to B$ where A is a T-absolutely direct set. Let $\{x_n\}$ be a T-minimizing sequence in A and $x_0 \in P_T(A, B)$. Since A is a T-absolutely direct set, there exist $y_n \in A$ such that $|y_n - Tx_0| \leqslant |x_0 - Tx_0| \wedge |x_n - Tx_0|$ and $|y_n - Tx_n| \leqslant |x_0 - Tx_n| \wedge |x_n - Tx_n|$.

By a similar manner as in the proof of Theorem 3.1.1 we can prove that $y_n = x_0$ for any $n \in \mathbb{N}$. Therefore $|x_0 - Tx_n| \leqslant |x_n - Tx_n|$ and it follows that $\|x_0 - Tx_n\| \to d$.

Now since A is convex then $\frac{x_n + x_0}{2} \in A$, and therefore $d \leqslant \|\frac{x_n + x_0}{2} - Tx_n\| \leqslant \frac{\|x_n - Tx_n\| + \|x_0 - Tx_n\|}{2} \leqslant \|x_n - Tx_n\| \to d$ as $n \to \infty$. By $X \in$ UM and $|\frac{x_n + x_0}{2} - Tx_n| \leqslant |x_n - Tx_n|$, we have

$$\||x_n - Tx_n| - |\frac{x_n + x_0}{2} - Tx_n|\| \to 0 \text{ as } n \to \infty. \tag{3}$$

Hence, by (1) and (3) we obtain

$$\|x_n - x_0\| = \|2(|x_n - Tx_n| \vee |x_0 - Tx_n|) - |x_n + x_0 - 2Tx_n|\| \to 0.$$

i.e., $\mathrm{dist}(x_n, P_T(A, B)) \to 0$ as $n \to \infty$, and so the best proximity pair problem is T-strongly solvable. $\square$

Example 3.1.5. Assume that $M_2(\mathbb{R})$ be the vector space of 2×2 real matrices with the order relation $\leqslant$, defined by $A \leqslant B$ only if $a_{ij} \leqslant b_{ij}$ for $i, j \in \{1, 2\}$. We define a norm on $M_2(\mathbb{R})$ with $\|A\| = \sum_{i,j=1}^{2} |a_{ij}|$. Then $(M_2(\mathbb{R}), \|\cdot\|)$ is a Banach lattice with the STM property.

Let $\mathcal{A} = \{\begin{bmatrix} 0 & a \\ b & c \end{bmatrix} : a, b, c \geqslant \delta > 1\}$ and $\mathcal{B} = \{\begin{bmatrix} 0 & a \\ b & c \end{bmatrix} : a, b, c \leqslant 1\}$.

We defined $T : \mathcal{A} \to \mathcal{B}$ by $T(\begin{bmatrix} 0 & a \\ b & c \end{bmatrix}) = \begin{bmatrix} 0 & -\frac{a}{\delta} + 2 \\ -\frac{b}{\delta} + 2 & -\frac{c}{\delta} + 2 \end{bmatrix}$. It is clear that $\mathcal{A}$ is a convex T-absolutely direct set and $P_T(\mathcal{A}, \mathcal{B}) = \{\begin{bmatrix} 0 & \delta \\ \delta & \delta \end{bmatrix}\}$, because $\mathrm{card}(P_T(\mathcal{A}, \mathcal{B})) \leqslant 1$ by the STM property of $M_2(\mathbb{R})$ and

$$\| \begin{bmatrix} 0 & \delta \\ \delta & \delta \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \| = 3(\delta - 1) = d(\mathcal{A}, \mathcal{B}).$$

3.2 Monotonicity and the best proximity pair for Tx-absolutely direct sets

In this part we study the relationship between the best proximity pair for Tx-absolutely direct sets, and we investigate solvability of the best proximity pair for Tx-absolutely direct sets, in terms of, Monotonicity in Banach lattices.

Theorem 3.2.1. *Assume that X is a Banach lattice with the UM property and A is a closed convex Tx-absolutely direct set for some $x \in A$. Then all the Tx-minimizing sequences in A, are convergent to a unique point in A.*

Proof. Suppose $\{x_n\}$ is a Tx-minimizing sequence in A, i.e., $\lim_{n\to\infty} \|x_n -$

$Tx\| = d(A, B) = d$. Set $z_1 = x_1$. For z_1 and x_2, choose $z_2 \in A$ such that

$$|z_2 - Tx| \leqslant |z_1 - Tx| \wedge |x_2 - Tx|.$$

For z_n and x_{n+1}, choose $z_{n+1} \in A$ such that

$$|z_{n+1} - Tx| \leqslant |z_n - Tx| \wedge |x_{n+1} - Tx|.$$

Then $|z_n - Tx|$ is monotonically decreasing and $d \leqslant \|z_n - Tx\| \leqslant \|x_n - Tx\ | \to d$, i.e.,

$$\|z_n - Tx\| \to d, \tag{4}$$

so $\{z_n\}$ is also a minimizing sequence in A. By Zorn's lemma, there exists $z_0 \in X^+$ such that $z_0 = \inf_n |z_n - Tx|$. Since $X \in$ UM, X has order continous norm by Theorem 3.1.1 in [5]. Therefore, $\||z_n - Tx| - z_0\| \to 0$, as $n \to \infty$, i.e., $\{|z_n - Tx|\}$ is a Cauchy sequence and $\|z_0\| = d$. We will prove that $\{z_n\}$ is also a Cauchy sequence.

Since A is convex, for any $m, n \in \mathbb{N}$ and $n > m$, we have $\frac{z_n + z_m}{2} \in A$. Therefore, $d \leqslant \|\frac{z_n + z_m}{2} - Tx\| \leqslant \frac{\|z_n - Tx\| + \|z_m - Tx\|}{2} \to d$, i.e.,

$$\|\frac{z_n + z_m}{2} - Tx\| \to d \ \text{ as } m, n \to \infty. \tag{5}$$

Moreover,

$$|\frac{z_n + z_m}{2} - Tx| \leqslant \frac{|z_n - Tx| + |z_m - Tx|}{2} \leqslant |z_m - Tx|,$$

so (4), (5) and the UM property of X imply that

$$\||z_m - Tx| - |\frac{z_n + z_m}{2} - Tx|\| \to 0 \ \text{ as } m, n \to \infty. \tag{6}$$

Hence, by (1) and (6), we get

$$\begin{aligned} \|z_n - z_m\| &= \|2(z_n - Tx| \vee |z_m - Tx|) - |z_n + z_m - 2Tx|\| \\ &= \|2|z_m - Tx| - |z_n + z_m - 2Tx|\| \to 0 \ \text{ as } m, n \to \infty \end{aligned}$$

whence it follows that $\{z_n\}$ is a Tx-minimizing Cauchy sequence in A. As A is closed, there exists $z \in A$ such that $z_n \to z$. Therefore $\|z - Tx\| =$

$d(A,B)$. Since $|z_n - Tx| \leqslant |x_n - Tx|$, by the same argument as before, we can show $\|x_n - z_n\| \to 0$. Hence $x_n \to z$.

Now, if $\{y_n\}$ is another Tx-minimizing sequence then $y_n \to w$, for some $w \in A$, and $\|w - Tx\| = d(A,B)$. We prove that $z = w$.

Since A is a Tx-absolutely direct set, there exists $y \in A$ such that $|y - Tx| \leqslant |z - Tx| \wedge |w - Tx|$. Thus, $d \leqslant \|y - Tx\| \leqslant \|z - Tx\| = d$, i.e., $\|y - Tx\| = \|z - Tx\|$. UM property implies STM property, and so, $|y - Tx| = |z - Tx|$. Similar argument shows that $|y - Tx| = |w - Tx|$. Thus $|z - Tx| = |w - Tx|$, and since A is convex, we have $\frac{z+w}{2} \in A$. Therefore, $d = \|z - Tx\| \leqslant \|\frac{z+w}{2} - Tx\| \leqslant \frac{\|z-Tx\|+\|w-Tx\|}{2} = d$. Moreover, since $|\frac{z+w}{2} - Tx| \leqslant \frac{|z-Tx|+|w-Tx|}{2} = |z - Tx| = |w - Tx|$ and $X \in$ STM, we get $|z - Tx| = |\frac{z+w}{2} - Tx| = |w - Tx|$. By (1) we obtain

$$|z-w| = 2(|z-Tx| \vee |w-Tx|) - |z+w-2Tx| = 2|z-Tx| - 2|z-Tx| = 0.$$

So, $z = w$. □

Theorem 3.2.2. *Let $X \in ULUM$ and $z_0 \in P_T(A,B)$. If A is a convex Tz_0-absolutely direct set then the best proximity pair problem is Tz_0-stable.*

Proof. Since A is a Tz_0-absolutely direct set, then for any $x \in A$, there exists $y \in A$ such that $|y - Tz_0| \leqslant |x - Tz_0| \wedge |z_0 - Tz_0|$. By a similar argument as in the proof of Theorem 3.2.2 we can prove that $y = z_0$. Therefore $|z_0 - Tz_0| \leqslant |x - Tz_0|$ for any $x \in A$. i.e., $|z_0 - Tz_0|$ is the infimum of $|A - Tz_0|$ in the given order. Suppose $\{z_n\}$ is a Tz_0-minimizing sequence in A, i.e., $\lim\limits_{n\to\infty} \|z_n - Tz_0\| = d(A,B) = d$. Since $|z_0 - Tz_0| \leqslant |z_n - Tz_0|$ and $\|z_n - Tz_0\| \to \|z_0 - Tz_0\| = d$, by the ULUM property of X, we have

$$\||z_n - Tz_0| - |z_0 - Tz_0|\| \to 0 \text{ as } n \to \infty. \tag{7}$$

Moreover since A is convex $\frac{z_n+z_0}{2} \in A$, and so $|z_0 - Tz_0| \leqslant |\frac{z_n+z_0}{2} - Tz_0|$. On the other hand, we have

$$d = \|z_0 - Tz_0\| \leqslant \|\frac{z_n + z_0}{2} - Tz_0\| \leqslant \frac{\|z_n - Tz_0\| + \|z_0 - Tz_0\|}{2} \to d$$

and $X \in$ ULUM. Therefore

$$\||\frac{z_n + z_0}{2} - Tz_0| - |z_0 - Tz_0|\| \to 0. \tag{8}$$

By (1), (7) and (8) we obtain

$$\begin{aligned}
\|z_n - z_0\| &= \|2(|z_n - Tz_0| \vee |z_0 - Tz_0|) - |z_n + z_0 - 2Tz_0|\| \\
&= \|2(|z_n - Tz_0| \vee |z_0 - Tz_0| - |z_0 - Tz_0|) \\
&+ (2|z_0 - Tz_0| - |z_n + z_0 - 2Tz_0|)\| \\
&\leqslant 2\||z_n - Tz_0| - |z_0 - Tz_0|\| + \|2|z_0 - Tz_0| - |z_n + z_0 - 2Tz_0|\| \\
&\to 0 \ \text{ as } n \to \infty,
\end{aligned}$$

i.e., $\mathrm{dist}(z_n, P_T(A, B)) \to 0$ as $n \to \infty$.

Therefore the best proximity pair problem is Tz_0-stable. □

Corollary 3.2.3. *Let $X \in ULUM$ and $z_0 \in P_T(A, B)$. If A is a convex T-absolutely direct set then the best proximity pair problem is Tz_0-strongly solvable.*

Proof. By Theorem 3.2.1, card $(P_T(A, B)) = 1$. Since A is a T-absolutely direct set, then for any $x \in A$, there exists $y \in A$ such that $|y - Tz_0| \leqslant |x - Tz_0| \wedge |z_0 - Tz_0|$ and with the same manner as in Theorem 3.2.2, the best proximity pair problem is Tz_0-stable. Therefore it is Tz_0-strongly solvable. □

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