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# Asymmetric Two-Piece Multiple Linear Regression Model Based on the Scale Mixture of Normal Family; Bayesian Framework

**B. Moravveji**
Marvdasht Branch, Islamic Azad University

**Z. Khodadadi***
Marvdasht Branch, Islamic Azad University

**M. Maleki**
University of Isfahan

**Abstract.** The main object of this article is to discuss Bayesian methodology for linear regression model according to the class of two-piece scale mixture of normal distribution. This model is appropriate for capturing departure from the usual normal assumption of error such as heavy tails, asymmetric and types of heteroscedasticity. Linear regression model is used to analyze data based on the normality assumption. The robust inference for normality assumption as a way to replace the Gaussian assumption for the residual errors with two-piece scale mixture of normal distribution is a Bayesian framework. An efficient way for applying Bayesian methodology is introduced using Markov chain Monte Carlo (MCMC) algorithm as a way to specify the posterior inference which has been used.

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**Keywords and Phrases:** Bayesian estimates, linear regression, scale mixtures of normal family, two-piece distributions

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*Corresponding author

# 1. Introduction

Recently non-Gaussian statistical models have been reasonably extended on many applicable sciences such as economics, biology, hydrology and physic. In fact, there are many situations in which the statistical models and their skewed structures can be more useful compared to the models where the real data with outliers and asymmetrical behavior exist. Some of the statistical models which have recently considered the non-Gaussian structures are as follows.

Maleki and Nematollahi (2017a), Maleki et al. (2017) and Maleki and Arellano-Valle (2017) considered the autoregressive model with finite mixture of scale mixture of normal and skew-normal (FM-SMSN) innovations. Maleki et al. (2019a,b) and Maleki and Wraith (2018) have considered a new flexible family of asymmetric family in the structure of the linear mixed effect model, finite mixture model and mixture of factor analyzer model (see also Zarrin et al., 2019; Hajrajabi and Maleki, 2019; Ghasami et al., 2019; Maleki et al. 2019c,d; Hoseinzadeh et al., 2019; Contreras et al., 2019; Maleki et al., 2020).

In statistical modeling, regression analysis is a set of statistical processes for describing the relationships among dependent and predictor variables in various fields. Some recently researches in this field are as follows. Mahmoudi et al. (2016) considered the testing between two independent regression models. Testing the equality of two independent regressions and also comparing and classifying of several independent regression models have considered by Mahmoudi et al. (2018) and Ji-Jun et al. (2019). Gray et al. (2015) discussed Bayesian analysis of regression model based on scale mixture of normal ($SMN$) distributions.

It is assumed that the residual errors in linear regression models ($LR$ hereafter) follow the normal distributions. This model is usually applied to symmetric data and its analysis. When normality assumption is questionable, this analysis might not provide robust induce for the errors. So, several approaches have been proposed so far to overcome any probable shortcoming and replacing by normality assumption. For instance, Gray et al. (2015) discussed a Bayesian approach to regression model which errors follow the flexible class of distribution which called the SMN family. This class of distributions was presented by Andrews and Mallows (1974). This family of distributions is thick tailed and symmetric. In addition, it includes distributions such as, normal($N$), student-t(T), Pearson type VII(P. II), slash($SL$) and contaminated normal ($CN$). Cysneiros and Paula (2005), Villegas et al. (2012) and Zhu et al. (2013) discussed estimation and diagnostic analysis for the model in which the distributions of the error belong to the *SMN* class. Fernández and Steel (1999) discussed inferential procedures in regression models with student-t distribution for the errors. Rubio and Genton (2016) proposed non-informative prior structure for *LR* model with skew-symmetric error distributions. This class contains the skew-normal distributions, the skew logistic distributions and the skew-t distributions. We are interested in fitting regression models when the errors have the two-piece scale mixtures of normal (*TP-SMN*) distributions. The *TP-SMN* family has received attention from researchers in many fields. In fact, it is a rich family of light/heavy tailed symmetric/asymmetric distributions and is flexible to modeling the symmetric and asymmetric regression model. These types of distributional suppositions have several benefits, for example it was used for the description of outliers and types of heteroscedastic. Furthermore, *TP-SMN* distributions can be utilized to the unobserved heterogeneity that produces asymmetry of residual errors.

The rest of this paper is organized as follows. In Section 2, we review some main properties of the *SMN* and *TP-MSN* families. The proposed Bayesian method for estimating the linear regression based on the TP-SMN model parameters is also provided in Section 3. In Section 4, we illustrate an application of the proposed model and method to real data.

## 2. Two-Piece Distributions Based on the Scale Mixtures of Normal Family)

In order to define the linear regression model based on the *TP-SMN* family (*TP-SMN-LR*) model, we briefly consider and describe the *TP-SMN* family of distribution. We refer the readers to Maleki and Mahmoudi

(2017), Maleki and Nematollahi (2017b) and Morevveji et al. (2019) for extensive details on these models. This family is useful for some statistical models including the one proposed based on the viewpoint of Bayesian analysis. Before this point, we introduce the *SMN* family. This family proposed by Andrews and Mallows (1974) is a rich family of symmetric distribution given by

$$f_{SMN}\left(y|\mu,\sigma,\nu\right)=\int_0^{\infty}\phi\left(y|\mu,u^{-1}\sigma^2\right)dH\left(u|\boldsymbol{\nu}\right),\qquad y\in R,\tag{1}$$

where $\phi\left(\cdot|\mu,\sigma^2\right)$ denotes the density of univariate normal distribution with location $\mu\in R$, scale parameter $\sigma^2>0$ and $H\left(\cdot|\boldsymbol{\nu}\right)$ is the cumulative distribution function (cdf) of the mixing distribution which can be indexed by a parameter vector $\boldsymbol{\nu}$. This family is denoted by $SMN\left(\mu\ ,\sigma,\boldsymbol{\nu}\right)$ and contains the normal, Student-t, Cauchy, Contaminated normal and Slash distributions. A random variable by *SMN* distribution with location $\mu\in R$, scale parameter $\sigma^2>0$ has the stochastic representation in form

$$Y=\mu+\sigma U^{-1/2}Z,\tag{2}$$

where $Z$ is a unit normal random variable that is independent of $U$, the mixing random variable has cdf $H\left(\cdot|\boldsymbol{\nu}\right)$. Arellano-Valle et al. (2005) introduced a more extensive discussion on the two- piece distributions. A random variable $Y$ follows the two-piece ($TP$) distribution denoted by $Y\sim TP\left(\mu\ ,\sigma,\gamma\right)$, if its probability density function can be written as:

$$\begin{aligned}\mathrm{g}\left(y|\mu,\sigma,\gamma\right)=&\frac{2}{\sigma\left[a\left(\gamma\right)+b\left(\gamma\right)\right]}\\&\times\left[f\left(\frac{y-\mu}{\sigma b\left(\gamma\right)}\right)I(y<\mu)\quad+f\left(\frac{y-\mu}{\sigma a\left(\gamma\right)}\right)I\left(y\geqslant\mu\right)\right],y\in R,\end{aligned}\tag{3}$$

where $f\left(\cdot\right)$ is a symmetric function around zero and $\gamma$ is the skewness parameter in (0,1) $\{\ a\left(\gamma\right),b(\gamma)\}$ are positive differentiable functions, $a\left(\gamma\right)\ =\ \gamma$, $\mathrm{b}(\gamma)\ =\ 1-\gamma$. The family in equation (2) involves the light/heavy-tails symmetric/asymmetric distributions as a special case. The random variable $Y\sim TP\left(\mu,\sigma,\gamma\right)$ has the stochastic representation in form of $Y\ =\ \mu+\ \ \sigma W\left|X\right|$for which W is a discrete random variable, a dependent factor in relation to $X$, and is a symmetric random variable with density $f(\cdot)$. The *pmf* of $W$ is following

$$P(W=w\,|\gamma)=\frac{a(\gamma)}{a(\gamma)+b(\gamma)}I_{\{a(\gamma)\}}(w)+\frac{b(\gamma)}{a(\gamma)+b(\gamma)}I_{\{-b(\gamma)\}}(w) \quad (4)$$

This *pmf* also can be represents in form

$$P(W=w|\gamma)=\gamma^{\frac{1+s}{2}}(1-\gamma)^{\frac{1-s}{2}}, \quad w=\gamma,-(1-\gamma), \quad (5)$$

where the s is sign(w) and defined as

$$s=sign\ (w) \ = \begin{cases} +1 & , \quad w \geqslant 0 \\ -1 & , \quad w<0 \end{cases}$$

In the equation of (3), if the function $f(\cdot)$ belongs to the *SMN* family, $Y$ is said to follow a *TP-SMN* distribution and denoted by $Y \sim TP(\mu\ ,\sigma,\boldsymbol{\nu},\gamma)$. Its density takes in form

$$g(y|\mu,\sigma,\gamma)=2[(1-\gamma)f_{SMN}(y|\mu,\sigma^2(1-\gamma)^2,\nu)I_{(-\infty,\mu]}(y) \quad (6)$$

$$+\gamma f_{SMN}(y|\mu,\sigma^2\gamma^2,\nu)I_{(\mu,+\infty)}(y),$$

Let $Y\ \sim$ *TP-SMN*$(\mu,\sigma,\boldsymbol{\nu},\gamma)$, then $Y$ has the following stochastic representation

$$Y=\mu+\sigma W U^{-1/2}|Z|, \quad (7)$$

where $Z\ \sim\ SMN(0\ ,1,\boldsymbol{\nu})$ and $W$, $U$, $Z$ are independent latent random variables, for which $Wp(.|\gamma)$ is the probability mass function (*pmf*) defined by (6). Also, the following hierarchical representation is held,

$$\begin{aligned} &Y\,|U=u\ ,\ W=w\ \sim\ \ \phi\left(y|\mu,\sigma^2w^2/u\right)I_A(y)^{(1+s)/2}I_{A^c}(y)^{(1-s)/2}, \\ &W\,|U\ \sim\ \ P(w|\gamma)\,, \\ &U\ \sim\ \ H(u|\,\boldsymbol{\nu})\,, \end{aligned}$$

where $s=\mathrm{sign}(w)$ takes values in $\{-1,+1\}$, and $A{=}(\mu\ ,\ +\infty)$, $HN(\cdot)$ denotes the half-normal distribution on the interval $A$ and $P(w|\gamma)$ is defined by (5).

# 3. The TP-SMN Linear Regression

In this section, we introduce the *TP-SMN* linear regression (*TP-SMN-LR*) and obtain the *ML* estimates of this model.

## 3.1 TP-SMN linear regression model

The *TP-SMN-LR* model is defined by

$$Y_i = \boldsymbol{x}_i^\top \boldsymbol{\beta} + \varepsilon_i, \quad i = 1, \ldots, n, \tag{8}$$

where $\boldsymbol{\beta} = (\beta_0, \beta_1, \ldots, \beta_p)^\top$ is a vector of regression parameters, $Y_i$ is a response variable and $\boldsymbol{x}_i = (x_{i1}, \ldots, x_{ip})^\top$ is a vector of fixed explanatory variables for subject $i$.

Following Sahu et al. (2003), in this study we extend the normal regression model using the following assumption:

$$Y_i \underset{\sim}{ind} \; TP\text{-}SMN\left(\boldsymbol{x}_i^\top \boldsymbol{\beta}, \sigma, \boldsymbol{\nu}, \gamma\right), \quad i = 1, ..., n,$$

with i.i.d. observation $\boldsymbol{Y} = ((Y_1, \boldsymbol{x}_1), \ldots, (Y_n, \boldsymbol{x}_n))^\top$ of the *TP-SMN-LR* model and vector of parameters $\boldsymbol{\theta} = (\boldsymbol{\beta}, \sigma, \boldsymbol{\nu}, \gamma)^\top$

## 3.2 Bayesian inference for the TP-SMN-LR model

In the Bayesian approach, we use the *MCMC* type algorithm (Gamerman and Lopes, 2006) to infer the parameters for the *TP-SMN-LR* model. (To do this algorithm, the stochastic representation has a major role). The *MCMC* technique can be developed using a data augmentation scheme in which we assume the latent variable in the model given by the vector of $\boldsymbol{U} = (U_1, \ldots, U_n)^\top$ and $\boldsymbol{W} = (W_1, \ldots, W_n)^\top$. So, we have that:

$$Y_i \,|U_i\,,\, W_i \underset{\sim}{ind}\, HN\left(\mu + \boldsymbol{x}_i^\top \boldsymbol{\beta}, \sigma^2 w_i^2 / u_i\right) I_A(Y_i)^{(1+s_i)/2} I_{A^c}(Y_i)^{(1-s_i)/2},$$

$$W_i \,|U_i \;\underset{\sim}{ind}\; P\left(W_i = w_i |\, \gamma\right),$$

$$U_i \underset{\sim}{ind}\, H\left(u_i \,|\, \boldsymbol{\nu}\right)$$

such that $A=(-\infty,\mu)$ and $s_i=\operatorname{sign}(w_i)$. So the augmented-likelihood function is given by

$$L_{comp}(\boldsymbol{\theta}\,|\boldsymbol{Y}_{obs})=\prod_{\mathrm{i=1}}^{\mathrm{n}} f\left(Y_i-\boldsymbol{x}_i^\top\boldsymbol{\beta}\,|\mu,\sigma\right)h\left(u_i|\,\boldsymbol{\nu}\right)p\left(w_i|\,\gamma\right) \tag{9}$$

$$=\left(\sigma^{-n}\prod_{i=1}^{n}h\left(U_i|\nu\right)\right)\exp\left(\frac{-1}{2\sigma^2}\sum_{i=1}^{n}U_iW_i^{-2}\left(Y_i-\boldsymbol{x}_i^\top\boldsymbol{\beta}-\mu\right)^2\right)$$
$$\times\,\gamma^{\frac{n}{2}+\frac{1}{2}\sum_1^n S_i}(1-\gamma)^{\frac{n}{2}-\frac{1}{2}\sum_1^n S_i}.$$

## 3.3 Priors and posteriors of the parameters

On a logical account of utilizing conditional analysis and Bayesian methodology, when prior information is not available, what we used in such situations is the weakly informative prior, which means a prior that involves no information about $\boldsymbol{\theta}=(\boldsymbol{\beta},\sigma,\boldsymbol{\nu},\gamma)^\top$. We apply the weakly informative for unknown quantities. Recently, to drive weakly informative prior for the parameters in the *TP-SMN-LR* model given by

1) Prior for $\gamma$: we can use the prior the skewness parameter $\gamma$. the prior is appropriated given by $\gamma\sim Beta(\alpha,\beta)$,

2) Prior for $\mu$: The chosen prior for can be presented as

$$\mu\sim N\left(\eta,\xi^2\right),$$

3) Prior for $\sigma^2$: The prior distribution for $\sigma^2$ is written by

$$\sigma^2\sim IG\left(\frac{\vartheta}{2},\frac{\vartheta}{2}\right),$$

that *IG* denotes the Inverse-Gaussian distribution. Selecting the prior distribution for $\boldsymbol{\nu}$, a parameter relying on mixing distribution $H$ differ from *TP-SMN-LR* model as indicated in the following way:

4) The prior distributions for the parameter of $\boldsymbol{\nu}$:

i) In the *TP-T-LR*, the proper prior distribution is

$$\nu = \ \nu \ \sim exp\left(\frac{\kappa}{2}\right) \ I\left(2, +\infty\right),$$

ii) In the *TP-CN-LR* for each component of the non-information and independent $U(0,1)$ are are followed,

iii) In the *TP-SL-LR* the proper prior distribution is

$$\boldsymbol{\nu} = \nu \sim \ \Gamma\left(a, b\right),$$

where a, b are positive values. $(b \ll a)$

The prior distribution used to for the vector of regression parameters $\beta$ is given by: $\beta \sim N_{p+1}(\mu_\beta, \Sigma_\beta)$. That the prior structure is:

$$\pi\left(\boldsymbol{\theta}\right) = \pi(\boldsymbol{\beta})\pi\left(\mu\right)\pi\left(\sigma^2\right)\pi\left(\gamma\right)\pi\left(\boldsymbol{\nu}\right).$$

To use the *MCMC* methods for instance Gibbs sample, then we draw a sample from these (full conditional) distribution for Gibbs sampling procedure, the latent variable $W_i$ , $i = 1, \ldots, n$ are defined by

$$W_i | \ \boldsymbol{\theta} \ ; Y_i - \boldsymbol{x}_i^\top \boldsymbol{\beta} \ = \begin{cases} -\left(1-\gamma\right), & y_i - \boldsymbol{x}_i^\top \boldsymbol{\beta} \ \leqslant \mu, \\ \gamma \quad , & y_i - \boldsymbol{x}_i^\top \boldsymbol{\beta} \ > \mu. \end{cases}$$

We suppose that $\boldsymbol{\theta}_{(-p)}$ is the vector of parameters except of P. Based on weakly informative prior distributions the conditional posterior distributions will be calculated in what follow:

$$\mu | \ \boldsymbol{\theta}_{(-\mu)}, \boldsymbol{U}, \boldsymbol{W} \ ; \boldsymbol{Y} \ \sim \ TN\left(\eta_\mu, \xi_\mu^2\right) I_{\left( \ y_i - \boldsymbol{x}_i^\top \boldsymbol{\beta} \ ^-, \ y_i - \boldsymbol{x}_i^\top \boldsymbol{\beta} \ ^+\right)}\left(\mu\right),$$

where

$$\eta_\mu = \frac{\xi^2 \sum_{i=1}^n u_i w_i^{-2} \left(y_i - \boldsymbol{x}_i^\top \boldsymbol{\beta} \ \right) + \sigma^2 \eta}{\xi^2 \sum_{i=1}^n u_i w_i^{-2} + \sigma^2},$$

$$\xi_\mu^2 = \frac{\sigma^2 \xi^2}{\xi^2 \sum_{i=1}^n u_i w_i^{-2} + \sigma^2},$$

for which $TN(\cdot)$ is a truncated normal distribution.

$$\sigma^2 \big| \boldsymbol{\theta}_{(-\sigma)}, \boldsymbol{U}, \boldsymbol{W}\,; \boldsymbol{Y} \sim IG\left(\frac{n+\vartheta}{2}, \frac{1}{2}\left(\sum_{i=1}^{n}\left(\frac{y_i - \boldsymbol{x}_i^{\top}\boldsymbol{\beta} - \eta}{\frac{w_i}{\sqrt{u_i}}}\right)^2 + \vartheta\right)\right).$$

$$\gamma \big| \boldsymbol{\theta}_{(-\gamma)}, \boldsymbol{U}, \boldsymbol{W}\,; \boldsymbol{Y} \sim \text{Beta}\left(\alpha + \frac{n}{2} + \frac{1}{2}\sum_{i=1}^{n} s_i,\ \beta + \frac{n}{2} - \frac{1}{2}\sum_{i=1}^{n} s_i\right).$$

There are relationships between the conditional posterior distribution for the latent variable $U_i, i = 1, \ldots, n$ and the specific members of the *SMN* family. To do this process, we define that

$c_i = \left(\frac{y_i - \mu}{\sigma W_i}\right)^2; i = 1, \ldots, n$, and these conditional posterior distribution are given by: *TP-T-LR* model:

$$U_i \big| \boldsymbol{\theta}, \boldsymbol{W}\,;\, \boldsymbol{Y} \sim \text{Gamma}\left(\nu/2 + 1/2, \nu/2 + c_i/2\right),$$

and

$$\pi\left(\nu \big| \boldsymbol{\theta}_{(-\nu)}, \boldsymbol{W}, \boldsymbol{U}\,; \boldsymbol{Y}\right) \propto \frac{1}{\left(2^{\nu/2}\Gamma\left(\nu/2\right)\right)^n} \times \text{Gamma}\left(\frac{n\nu}{2} + 1, \frac{1}{2}\left[\sum_{i=1}^{n}\left(u_i - \log u_i\right) + \kappa\right]\right) I_{(2,\infty)}\left(\nu\right),$$

*TP-SL-LR* model:

$$U_i \big| \boldsymbol{\theta}, \boldsymbol{W};\, \boldsymbol{Y} \sim \text{TGamma}\left(\nu + 1/2, c_i/2\right) I_{(0,1)}(u_i),$$

where $\text{TGamma}\left(\cdot\right) I_A(\cdot)$ denotes the truncated Gamma distribution on the interval A, and

$$\nu \big| \boldsymbol{\theta}_{(-\nu)}, \boldsymbol{W}, \boldsymbol{U}\,; \boldsymbol{Y} \sim Gamma\left(n + a, b - \sum_{i=1}^{n} \log u_i\right),$$

*TP-CN-LR* model:

$$\pi\left(U_i \big| \theta, \boldsymbol{W}\,; \boldsymbol{Y}\right) = \frac{d_i}{d_i + e_i} I_{\{\tau\}}\left(u_i\right) + \frac{e_i}{d_i + e_i} I_{\{1\}}\left(u_i\right),$$

where $d_i = \nu\sqrt{\tau}\exp\left(-\tau c_i/2\right)$ and $e_i = (1-\nu)\exp\left(-c_i/2\right)$

$$\nu|\,\boldsymbol{\theta}_{(-\nu)}, \boldsymbol{U}, \boldsymbol{W}\ ; \boldsymbol{Y} \ \sim\ \text{Beta}\left(\frac{n-\sum_{i=1}^{n} u_i}{1-\tau}+1, \frac{\sum_{i=1}^{n} u_i - n\tau}{1-\tau}+1\right),$$

$$\pi\left(\tau|\,\boldsymbol{\theta}_{(-\tau)}, \boldsymbol{U}, \boldsymbol{W}\ ; \boldsymbol{Y}\right) \propto \nu^{\frac{n-\sum_{i=1}^{n} u_i}{1-\tau}} (1-\nu)^{\frac{\sum_{i=1}^{n} u_i - n\tau}{1-\tau}}.$$

The conditional posterior distribution for vector of regression coefficient $\boldsymbol{\beta}$ :

$$\boldsymbol{\beta} \ \sim\ N_{p+1}\left(\boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}_\beta\right),$$

where

$$\boldsymbol{\mu}_\beta = \boldsymbol{K}^{-1}\left[\ \boldsymbol{\Sigma}_\beta^{-1}\boldsymbol{\mu}_\beta + \frac{1}{\sigma^2}\ \left(\sum_{i=1}^{n} u_i y_i \boldsymbol{x}_i\right)\right],$$

$$\boldsymbol{\Sigma}_\beta = \left(\boldsymbol{\Sigma}^{-1}\ +\ \sum_{i=1}^{n} \frac{u_i}{\sigma^2 w_i^2}\boldsymbol{x}_i\boldsymbol{x}_i^\top\right)^{-1},$$

for which $\boldsymbol{K} = \boldsymbol{\Sigma}^{-1}\ +\ \sum_{i=1}^{n} \frac{u_i}{\sigma^2 w_i^2}\boldsymbol{X}_i\boldsymbol{X}_i^\top$ .

## 4. Application

In this section, we consider a road accident dataset for 26 *US* states. This data called "*Road Accident Deaths in US States*" is available in the "*MASS*" *R* package (see scatter plot in Figure 2). By considering the number of deaths (*deaths*) as the response variables, the number of drivers in miles (*drivers*), the population density per square mile (*popden*), the length of rural roads in miles (*rural*), the average of maximum daily temperature in January (*temp*) and the fuel consumption in ten million gallons per year (*fuel*) as the fixed explanatory variables, we fit the *TP-SMN-LR* model to the dataset. In the Bayesian approach, some model selection criteria based on the posterior mean of the deviance like the expected Akaike information criterion (*EAIC*) (Brooks, 2002) and the expected Bayesian information criterion (*EBIC*) (Carlin and Louis, 2001) can be considered. we consider the weakly informative prior distributions given by $\boldsymbol{\beta} \sim N_6\left(\mathbf{0}, 10^3\boldsymbol{I}_6\right)$, $\gamma \sim Beta\,(1,\ 1)$, $\mu \sim N\left(0, 10^3\right)$, $\sigma^2 \sim IG\,(0.001, 0.001)$, and $\nu \sim \exp(0.1)I_{(2,\infty)}$ for the *TP-T-LR*, $\nu \sim$ Gamma(1, 0.01) for the *TP-SL-LR*, and $\nu \sim U(0,\ 1)$ independent of $\tau \sim U(0,\ 1)$ for the *TP-CN-LR* models.

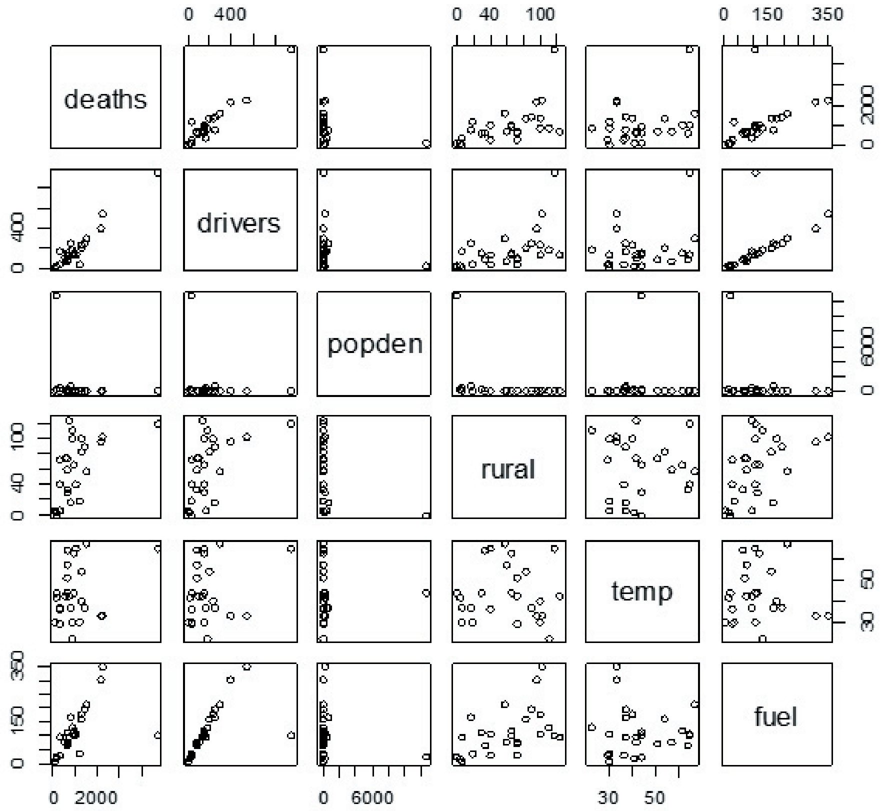


**Figure 1.** Road Accident Deaths in US States.

Table 1 presents the *Bayesian* estimates of the *TP-SMN-LR* parameters; Results shows that the robust *TP-T-LR* model is much better fitting to this data compared to other *TP-SMN-LR* counterparts. Also, the histograms of the residuals based on the *TP-SMN-LR* models are given in Figure 2. Histograms include the estimated density of residuals based on the *Bayesian* estimates of the model parameters.

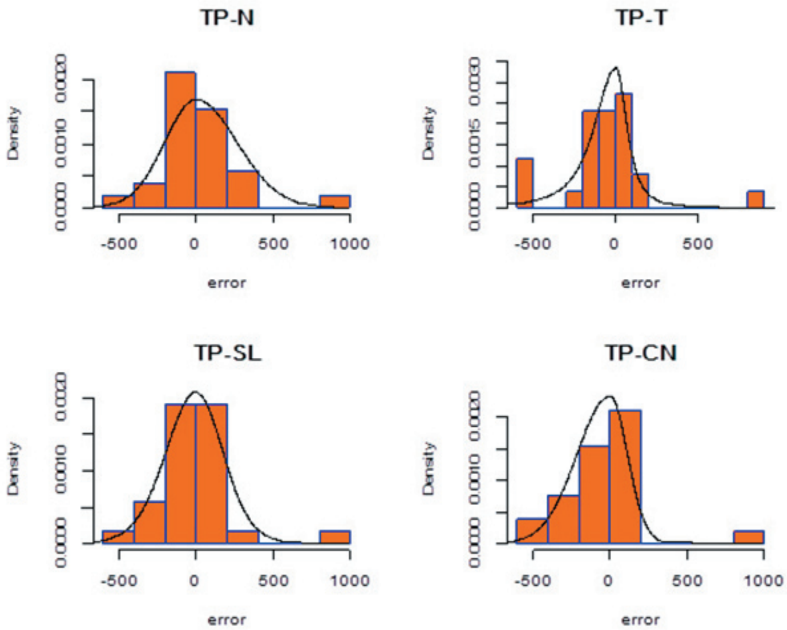


**Figure 2.** Bayesian estimated densities of the TP-SMN-LR models fitted on their corresponding estimated.errors of each linear regression using the “Road Accident Deaths in US States” dataset.

**Table 1:** Bayesian estimates of the TP-SMN-LR parameters for the ”Road Accident Deaths in US States” data.

| Parameters | *TP-N* | *TP-T* | *TP-SL* | *TP-CN* |
|---|---|---|---|---|
| $\beta_0$ | 0.05713741 | 0.04114499 | 0.04990076 | 0.04570802 |
| $\beta_1$ | 4.54202865 | 4.70166957 | 4.57606486 | 4.56499832 |
| $\beta_2$ | -0.01614764 | -0.01577284 | -0.01446356 | -0.01396913 |
| $\beta_3$ | 2.11850377 | 0.24049311 | 1.97724059 | 1.99364405 |
| $\beta_4$ | 3.26351725 | 3.17958383 | 3.00922262 | 2.95008406 |
| $\beta_5$ | -1.16613434 | 0.25157303 | -0.90497227 | -0.59052356 |
| $\sigma$ | 470.228500 | 213.083200 | 311.270600 | 325.919900 |
| $\gamma$ | 0.56732560 | 0.33540590 | 0.47287520 | 0.34901170 |
| $\nu$ | — | 2.10004600 | 2.10004600 | 0.05927957 |
| $\tau$ | — | — | — | 0.02508201 |
| *EAIC* | 375.301900 | 365.720000 | 373.430100 | 370.661000 |
| *EBIC* | 385.366700 | 377.042900 | 384.753000 | 383.242000 |

## 5. Conclusion

In this paper, we have proposed a new asymmetric and heavy-tailed class of two-piece distributions to linear regression model. Also, by considering the hierarchical representation of the proposed linear regression model, the *MCMC*-algorithm is used to find the Bayesian estimates of the model parameters. It is important to highlight the capacity of the *TP-SMN-LR* models to attenuate outlying observations. Our methodology applied to a real dataset indicates that a *TP-T* linear regression model with small value of degrees of freedom (high presence of heavy-tails), seems to fit the data better than the *TP-N* linear regression model as well as the other *TP-SMN-LR* models counterparts due to its robustness.

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**Behjat Moravveji**
Ph.D Student of Statistics
Department of Statistics
Marvdasht Branch, Islamic Azad University
Marvdasht, Iran
E-mail: bmoraveji1393@gmail.com

**Zahra Khodadadi**
Assistant Professor of Statistics
Department of Statistics
Marvdasht Branch, Islamic Azad University
Marvdasht, Iran
E-mail: zkhodadadi@miau.ac.ir

**Mohsen Maleki**
Assistant Professor of Statistics
Faculty of Mathematics and Statistics
Department of Statistics
University of Isfahan
Isfahan, Iran
E-mail: m.maleki.stat@gmail.com