

On the existence of a solution for a bi-dealing singular fractional intergro-differential equation

A. Mansouri*

Department of Mathematics, Islamic Azad University, Arak Branch, Arak,
Iran

M. Shabibi

Department of Mathematics, Meharn Branch, Islamic Azad University,
Mehran, Iran

Abstract. In this paper, we investigate the existence of a solution for a pointwise defined, multi-singular fractional differential equation in some regions with some boundary conditions.

AMS Subject Classification: 34A08; 37C25; 46F30

Keywords and Phrases: Caputo derivative, Point-wise defined equation, singularity, fractional differential equation

1 Introduction

An evaluation of the mathematical model of a scientific observation leads into a differential equation that sometimes equations are of the form fractional order. We can mention engineering sciences, dynamic, chemistry and physics among which the equation occurs ([2]). Many studies have investigated the existence and behavior of these equations in recent decats ([3], [4]). Sometimes we lead to a differential system that have singularity in some points, recently many works has been published on

*Corresponding Author

the existence of the solutions for these singular systems ([9]).

In 2010, Agarwal, O'Regan and Stanek ([1]) studied the existence of solutions for the problem $D^\alpha u(t) + f(t, u(t)) = 0$ with boundary conditions $u'(0) = \dots = u^{(n-1)}(0) = 0$ and $u(1) = \int_0^1 u(s) d\mu(s)$, where $n \geq 2$, $\alpha \in (n-1, n)$, $\mu(s)$ is a functional of bounded variation with $\int_0^1 d\mu(s) < 1$, and f may have singularity at $t = 0$.

In 2015, Y. Liu and P. J. Y. Wong investigated the existence of solution for the fractional problem ${}^c D^\alpha x(t) = f(t, x(t), D^\beta x(t))$ with boundary conditions $x(0) + x'(0) = y(x)$, $\int_0^1 x(t) dt = m$ and $x''(0) = x^{(3)}(0) = \dots = x^{(n-1)}(0) = 0$, where $0 < t < 1$, m is a real number, $n \geq 2$, $\alpha \in (n-1, n)$, $0 < \beta < 1$, D^α and D^β are the Caputo fractional derivatives, $y \in C([0, 1], \mathbb{R}) \rightarrow \mathbb{R}$ and $f : (0, 1] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous with $f(t, x, y)$ may be singular at $t = 0$ ([7]).

In 2016 M. Shabibi, M. Postolache, Sh. Rezapour and S. M. Vaezpour investigated the solution of the multi-singular pointwise defined fractional integro-differential equation $D^\mu x(t) + f(t, x(t), x'(t), D^\beta x(t), I^p x(t)) = 0$ with boundary conditions $x'(0) = x(\xi)$ and $x(1) = \int_0^\eta x(s) ds$ when $\mu \in [2, 3)$ and $x'(0) = x(\xi)$, $x(1) = \int_0^\eta x(s) ds$ and $x^{(j)}(0) = 0$ for $j = 2, \dots, [\mu] - 1$ when $\mu \in [3, \infty)$, where $0 \leq t \leq 1$, $x \in C^1[0, 1]$, $\mu \in [2, \infty)$, $\beta, \xi, \eta \in (0, 1)$, $p > 1$, D^μ is the Caputo fractional derivative of order μ and $f : [0, 1] \times \mathbb{R}^5 \rightarrow \mathbb{R}$ is a function such that $f(t, \dots, \dots)$ is singular at some points $t \in [0, 1]$ ([8]).

In 2018 D. Baleanu, Kh. Ghafarnezhad, Sh. Rezapour and M. Shabibi reviewed the existence of solution for the pointwise defined three steps crisis integro-differential equation

$$D^\alpha x(t) + f(t, x(t), x'(t), D^\beta x(t), \int_0^t h(\xi) x(\xi) d\xi, \phi(x(t))) = 0$$

with boundary conditions $x(1) = x(0) = x''(0) = x^n(0) = 0$, where $\alpha \geq 2$, $\lambda, \mu, \beta \in (0, 1)$, $\phi : X \rightarrow X$ is a mapping such that $\|\phi(x) - \phi(y)\| \leq \theta_0 \|x - y\| + \theta_1 \|x' - y'\|$ for some non-negative real numbers θ_0 and $\theta_1 \in [0, \infty)$ and all $x, y \in X$, D^α is the Caputo fractional derivative of order α , $f(t, x_1(t), \dots, x_5(t)) = f_1(t, x_1(t), \dots, x_5(t))$ for all $t \in [0, \lambda)$, $f(t, x_1(t), \dots, x_5(t)) = f_2(t, x_1(t), \dots, x_5(t))$ for all $t \in [\lambda, \mu]$ and $f(t, x_1(t), \dots, x_5(t)) = f_3(t, x_1(t), \dots, x_5(t))$ for all $t \in (\mu, 1]$, $f_1(t, \dots, \dots)$ and $f_3(t, \dots, \dots)$ are continuous on $[0, \lambda)$ and $(\mu, 1]$ and $f_2(t, \dots, \dots)$ is

multi-singular ([5]).

Using idea of these papers, we investigate the existence of solutions for the folowing bi-dealing singular fractional intergro-differential equation

$$D^\alpha x(t) + f(t, x(t), x'(t), D^\beta x(t), \int_0^t h(s)x(s)ds, \phi x(t)) = 0 \quad (1)$$

with boundary conditions $x'(\mu) = x'(0) + \int_0^\lambda g(s)x(s)ds$ and $x(1) = x^{(j)}(0) = 0$ for $j \geq 2$ where $\alpha \geq 2$, $\mu, \lambda \in (0, 1)$, $g \in L^1[0, \lambda]$, $g(t) > 0$ for a.e. $t \in [0, \lambda]$, $\phi : [0, 1] \rightarrow \mathbb{R}^+$ is such that for all $x, y \in C^1[0, 1]$, $|\phi x(t) - \phi y(t)| \leq b_1|x(t) - y(t)| + b_2|x'(t) - y'(t)|$ for some $b_1, b_2 \in [0, \infty)$, $h \in L^1[0, 1]$, D^α is the Caputo fractional derivative of order α and $f(t, \cdot, \cdot, \cdot, \cdot, \cdot)$ deals as a singular function on some set $E \subset [0, 1]$ and deals as a continuous function on $E^c \subset [0, 1]$. Recall that $D^\alpha x(t) = f(t)$ is a pointwise defined equation on $[0, 1]$ if there exists set $D \subset [0, 1]$ such that the measure of D^c is zero and the equation holds on D (see [8]). Here we use $\|\cdot\|_1$ for the norm of $L^1[0, 1]$, $\|\cdot\|$ for the sup norm of $Y = C[0, 1]$ and $\|x\|_* = \max\{\|x\|, \|x'\|\}$ for the norm of $X = C^1[0, 1]$. The Riemann-Liouville integral of order p with the lower limit $a \geq 0$ for a function $f : (a, \infty) \rightarrow \mathbb{R}$ is defined by $I_{a+}^p f(t) = \frac{1}{\Gamma(p)} \int_a^t (t-s)^{p-1} f(s)ds$, provided that the right-hand side is pointwise define on (a, ∞) ([12]). we denote $I_{0+}^p f(t)$ by $I^p f(t)$. The Caputo fractional derivative of order $\alpha > 0$ is defined by ${}^c D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^n(s)}{(t-s)^{\alpha+1-n}} ds$, where $n = [\alpha] + 1$ and function $f : (a, \infty) \rightarrow \mathbb{R}$ is a function ([12]). Let Ψ be the family of nondecreasing functions $\psi : [0, \infty) \rightarrow [0, \infty)$ such that $\sum_{n=1}^\infty \psi^n(t) < \infty$ for all $t > 0$ (see [11]). One can check that $\psi(t) < t$ for all $t > 0$ ([11]). Let $T : X \rightarrow X$ and $\alpha : X \times X \rightarrow [0, \infty)$ be two maps. Then T is called an α -admissible map whenever $\alpha(x, y) \geq 1$ implies $\alpha(Tx, Ty) \geq 1$ ([11]). Let (X, d) be a metric space, $\psi \in \Psi$ and $\alpha : X \times X \rightarrow [0, \infty)$ a map. A self-map $T : X \rightarrow X$ is called an α - ψ -contraction whenever $\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y))$ for all $x, y \in X$ ([11]). We need next results.

Lemma 1.1. ([11]) *Let (X, d) be a complete metric space, $\psi \in \Psi$, $\alpha : X \times X \rightarrow [0, \infty)$ a map and $T : X \rightarrow X$ an α -admissible α - ψ -contraction. If T is continuous and there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$, then T has a fixed point.*

Lemma 1.2. ([14]) Let X be a Banach space with $C \subseteq X$ closed and convex. Let Ω be a relatively open subset of C with $0 \in \Omega$ and let $F : \Omega \rightarrow C$ be a continuous and compact mapping. Then either
i) the mapping F has a fixed point in $\bar{\Omega}$, or
ii) there exist $y \in \partial\Omega$ and $\lambda \in (0, 1)$ with $y = \lambda Fy$.

Lemma 1.3. ([6]) Let $n - 1 \leq \alpha < n$ and $x \in C(0, 1) \cap L^1(0, 1)$. Then, we have $I^\alpha D^\alpha x(t) = x(t) + \sum_{i=0}^{n-1} c_i t^i$ for some real constants $c_0, \dots, c_{n-1}$.

2 Main Results

Lemma 2.1. Let $\alpha \geq 2$, $n = [\alpha] + 1$, $\mu, \lambda \in (0, 1)$, $g \in L^1[0, \lambda]$, $g(t) > 0$ for a.e. $t \in [0, \lambda]$ and $y \in L^1[0, 1]$. A map x is a solution for the pointwise defined equation $D^\alpha x(t) + f(t) = 0$ with boundary conditions $x'(\mu) = x'(0) + \int_0^\lambda g(s)x(s)ds$ and $x(1) = x^{(j)}(0) = 0$ for $2 \leq j \leq n$, if and only if $x(t) = \int_0^1 G(t, s)y(s)ds$ for all $t \in [0, 1]$, where

$$G(t, s) = \frac{-(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-t)H(s) + [(A_\lambda + B_\lambda)t - B_\lambda](1-s)^{\alpha-1} + \alpha(t-1)(\mu-s)^{\alpha-2}}{A_\lambda \Gamma(\alpha)}, \text{ when}$$

$$0 \leq s \leq t \leq 1 \text{ and } s \leq \mu, \lambda,$$

$$G(t, s) = \frac{-(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{[(A_\lambda + B_\lambda)t - B_\lambda](1-s)^{\alpha-1} + \alpha(t-1)(\mu-s)^{\alpha-2}}{A_\lambda \Gamma(\alpha)}, \text{ when}$$

$$0 \leq s \leq t \leq 1 \text{ and } \lambda \leq s \leq \mu,$$

$$G(t, s) = \frac{-(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-t)H(s) + [(A_\lambda + B_\lambda)t - B_\lambda](1-s)^{\alpha-1}}{A_\lambda \Gamma(\alpha)}, \text{ when } 0 \leq s \leq t \leq 1$$

$$\text{and } \mu \leq s \leq \lambda,$$

$$G(t, s) = \frac{-(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{[(A_\lambda + B_\lambda)t - B_\lambda](1-s)^{\alpha-1}}{A_\lambda \Gamma(\alpha)}, \text{ when } 0 \leq s \leq t \leq 1 \text{ and}$$

$$s \geq \mu, \lambda,$$

$$G(t, s) = \frac{[(A_\lambda + B_\lambda)t - B_\lambda](1-s)^{\alpha-1} + (t-1)(\mu-s)^{\alpha-1}}{A_\lambda \Gamma(\alpha)}, \text{ when } 0 \leq t \leq s \leq 1 \text{ and}$$

$$s \leq \mu, \lambda,$$

$$G(t, s) = \frac{(1-t)H(s) + [(A_\lambda + B_\lambda)t - B_\lambda](1-s)^{\alpha-1}}{A_\lambda \Gamma(\alpha)}, \text{ when } 0 \leq t \leq s \leq 1 \text{ and}$$

$$s \geq \mu, \lambda,$$

$$G(t, s) = \frac{[(A_\lambda + B_\lambda)t - B_\lambda](1-s)^{\alpha-1}}{A_\lambda \Gamma(\alpha)}, \text{ when } 0 \leq t \leq s \leq 1 \text{ and } s \geq \mu, \lambda,$$

$$H(s) = \int_s^\lambda (t-s)^{\alpha-1} g(t)dt, \quad A_\lambda = \int_0^\lambda (1-t)g(t)dt \text{ and}$$

$$B_\lambda = \int_0^\lambda t g(t)dt.$$

Proof. First by similar manner as [5] we conclude that lemma (1.3) is valid on $L^1[0, 1]$. Now let $x(t)$ be a solution for the problem, since

$x^{(j)}(0) = 0$ for $j \geq 2$, by using Lemma (1.3) we have $x(t) = \frac{-1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds + c_0 + c_1 t$. By boundary condition $x(1) = 0$, we have so $c_1 = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s) ds - c_0$. On the other hand $g(t)x(t) = \frac{-g(t)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds + c_0 g(t) + c_1 t g(t)$, so

$$\begin{aligned} \int_0^\lambda g(t)x(t)dt &= \frac{-1}{\Gamma(\alpha)} \int_0^\lambda \int_0^t g(t)(t-s)^{\alpha-1} y(s) ds dt \\ &\quad + c_0 \int_0^\lambda g(t)dt + c_1 \int_0^\lambda t g(t)dt. \end{aligned}$$

Now since

$$\begin{aligned} \int_0^\lambda \int_0^t g(t)(t-s)^{\alpha-1} y(s) ds dt &= \int_0^\lambda \int_s^\lambda g(t)(t-s)^{\alpha-1} y(s) dt ds \\ &= \int_0^\lambda \left(\int_s^\lambda (t-s)^{\alpha-1} g(t) dt \right) y(s) ds, \end{aligned}$$

hence

$$\begin{aligned} \int_0^\lambda g(t)x(t)dt &= \frac{-1}{\Gamma(\alpha)} \int_0^\lambda \left(\int_s^\lambda (t-s)^{\alpha-1} g(t) dt \right) y(s) ds \\ &\quad + c_0 \int_0^\lambda g(t)dt + c_1 \int_0^\lambda t g(t)dt. \end{aligned}$$

Also we have

$$x'(\mu) = \frac{-1}{\Gamma(\alpha)} \int_0^\mu (\mu-s)^{\alpha-2} y(s) ds + c_1$$

and $x'(0) = c_1$, so by $x'(\mu) = x'(0) + \int_0^\lambda g(s)x(s)ds$, we conclude that

$$\begin{aligned} \frac{-1}{\Gamma(\alpha-1)} \int_0^\mu (\mu-s)^{\alpha-2} y(s) ds &= \frac{-1}{\Gamma(\alpha)} \int_0^\lambda \left(\int_s^\lambda (t-s)^{\alpha-1} g(t) dt \right) y(s) ds \\ &\quad + c_0 \int_0^\lambda g(t)dt + c_1 \int_0^\lambda t g(t)dt \\ &= \frac{-1}{\Gamma(\alpha)} \int_0^\lambda \left(\int_s^\lambda (t-s)^{\alpha-1} g(t) dt \right) y(s) ds + c_0 \int_0^\lambda g(t)dt \\ &\quad + \frac{1}{\Gamma(\alpha)} \left(\int_0^\lambda t g(t)dt \right) \int_0^1 (1-s)^{\alpha-1} y(s) ds - c_0 \int_0^\lambda t g(t)dt, \end{aligned}$$

so

$$c_0 \left(\int_0^\lambda g(t) dt - \int_0^\lambda t g(t) dt \right) = \frac{1}{\Gamma(\alpha)} \int_0^\lambda \left(\int_s^\lambda (t-s)^{\alpha-1} g(t) dt \right) y(s) ds \\ - \frac{1}{\Gamma(\alpha)} \left(\int_0^\lambda t g(t) dt \right) \int_0^1 (1-s)^{\alpha-1} y(s) ds - \frac{1}{\Gamma(\alpha-1)} \int_0^\mu (\mu-s)^{\alpha-2} y(s) ds,$$

therefore

$$c_0 = \frac{1}{\Gamma(\alpha) \int_0^\lambda (1-t) g(t) dt} \int_0^\lambda \left(\int_s^\lambda (t-s)^{\alpha-1} g(t) dt \right) y(s) ds \\ - \frac{\int_0^\lambda t g(t) dt}{\Gamma(\alpha) \int_0^\lambda (1-t) g(t) dt} \int_0^1 (1-s)^{\alpha-1} y(s) ds \\ - \frac{1}{\Gamma(\alpha-1) \int_0^\lambda t g(t) dt} \int_0^\mu (\mu-s)^{\alpha-2} y(s) ds,$$

hence

$$c_1 = \int_0^1 (1-s)^{\alpha-1} y(s) ds - c_0 \\ = \int_0^1 (1-s)^{\alpha-1} y(s) ds \\ - \frac{1}{\Gamma(\alpha) \int_0^\lambda (1-t) g(t) dt} \int_0^\lambda \left(\int_s^\lambda (t-s)^{\alpha-1} g(t) dt \right) y(s) ds \\ + \frac{\int_0^\lambda t g(t) dt}{\Gamma(\alpha) \int_0^\lambda (1-t) g(t) dt} \int_0^1 (1-s)^{\alpha-1} y(s) ds \\ + \frac{1}{\Gamma(\alpha-1) \int_0^\lambda t g(t) dt} \int_0^\mu (\mu-s)^{\alpha-2} y(s) ds.$$

So we have

$$x(t) = \frac{-1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds \\ + \frac{1}{\Gamma(\alpha) \int_0^\lambda (1-t) g(t) dt} \int_0^\lambda \left(\int_s^\lambda (t-s)^{\alpha-1} g(t) dt \right) y(s) ds$$

$$\begin{aligned}
& - \frac{\int_0^\lambda tg(t)dt}{\Gamma(\alpha) \int_0^\lambda (1-t)g(t)dt} \int_0^1 (1-s)^{\alpha-1} y(s)ds \\
& - \frac{1}{\Gamma(\alpha) \int_0^\lambda tg(t)dt} \int_0^\mu (\mu-s)^{\alpha-1} y(s)ds, \\
& + \frac{t}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s)ds \\
& - \frac{t}{\Gamma(\alpha) \int_0^\lambda (1-t)g(t)dt} \int_0^\lambda \left(\int_s^\lambda (t-s)^{\alpha-1} g(t)dt \right) y(s)ds \\
& + \frac{t \int_0^\lambda tg(t)dt}{\Gamma(\alpha) \int_0^\lambda (1-t)g(t)dt} \int_0^1 (1-s)^{\alpha-1} y(s)ds \\
& + \frac{t}{\Gamma(\alpha-1) \int_0^\lambda tg(t)dt} \int_0^\mu (\mu-s)^{\alpha-2} y(s)ds.
\end{aligned}$$

Put $H(s) = \int_s^\lambda (t-s)^{\alpha-1} g(t)dt$, $A_\lambda = \int_0^\lambda (1-t)g(t)dt$ and $B_\lambda = \int_0^\lambda tg(t)dt$,
so

$A_\lambda + B_\lambda = \int_0^\lambda g(t)dt$. Hence

$$\begin{aligned}
& x(t) \\
& = \frac{-1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)ds + \frac{1}{A_\lambda \Gamma(\alpha)} \int_0^\lambda H(s)y(s)ds \\
& - \frac{B_\lambda}{A_\lambda \Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s)ds - \frac{1}{A_\lambda \Gamma(\alpha)} \int_0^\mu (\mu-s)^{\alpha-1} y(s)ds, \\
& + \frac{t}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s)ds - \frac{t}{A_\lambda \Gamma(\alpha)} \int_0^\lambda H(s)y(s)ds \\
& + \frac{tB_\lambda}{A_\lambda \Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s)ds + \frac{t}{A_\lambda \Gamma(\alpha-1)} \int_0^\mu (\mu-s)^{\alpha-2} y(s)ds.
\end{aligned}$$

Therefore

$$\begin{aligned}
x(t) & = \frac{-1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)ds + \frac{1-t}{A_\lambda \Gamma(\alpha)} \int_0^\lambda H(s)y(s)ds \\
& + \frac{(A_\lambda + B_\lambda)t - B_\lambda}{A_\lambda \Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s)ds \\
& + \frac{t-1}{A_\lambda \Gamma(\alpha-1)} \int_0^\mu (\mu-s)^{\alpha-2} y(s)ds.
\end{aligned}$$

If $t \leq \lambda \leq \mu < 1$ then

$$\begin{aligned} x(t) &= \frac{-1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds + \frac{1-t}{A_\lambda \Gamma(\alpha)} \left(\int_0^t + \int_t^\lambda \right) H(s) y(s) ds \\ &\quad + \frac{(A_\lambda + B_\lambda)t - B_\lambda}{A_\lambda \Gamma(\alpha)} \left(\int_0^t + \int_t^\lambda + \int_\lambda^\mu + \int_\mu^1 \right) (1-s)^{\alpha-1} y(s) ds \\ &\quad + \frac{t-1}{A_\lambda \Gamma(\alpha-1)} \left(\int_0^t + \int_t^\lambda + \int_\lambda^\mu \right) (\mu-s)^{\alpha-2} y(s) ds, \end{aligned}$$

if $t \leq \mu \leq \lambda < 1$ then we have

$$\begin{aligned} x(t) &= \frac{-1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds + \frac{1-t}{A_\lambda \Gamma(\alpha)} \left(\int_0^t + \int_t^\mu + \int_\mu^\lambda \right) H(s) y(s) ds \\ &\quad + \frac{(A_\lambda + B_\lambda)t - B_\lambda}{A_\lambda \Gamma(\alpha)} \left(\int_0^t + \int_t^\mu + \int_\mu^\lambda + \int_\lambda^1 \right) (1-s)^{\alpha-1} y(s) ds \\ &\quad + \frac{t-1}{A_\lambda \Gamma(\alpha-1)} \left(\int_0^t + \int_t^\mu \right) (\mu-s)^{\alpha-2} y(s) ds, \end{aligned}$$

when $0 < \mu \leq t \leq \lambda < 1$ we have

$$\begin{aligned} x(t) &= \frac{-1}{\Gamma(\alpha)} \left(\int_0^\mu + \int_\mu^t \right) (t-s)^{\alpha-1} y(s) ds \\ &\quad + \frac{1-t}{A_\lambda \Gamma(\alpha)} \left(\int_0^\mu + \int_\mu^t + \int_t^\lambda \right) H(s) y(s) ds \\ &\quad + \frac{(A_\lambda + B_\lambda)t - B_\lambda}{A_\lambda \Gamma(\alpha)} \left(\int_0^\mu + \int_\mu^t + \int_t^\lambda + \int_\lambda^1 \right) (1-s)^{\alpha-1} y(s) ds \\ &\quad + \frac{t-1}{A_\lambda \Gamma(\alpha-1)} \int_0^\mu (\mu-s)^{\alpha-2} y(s) ds, \end{aligned}$$

when $0 < \lambda \leq t \leq \mu < 1$ then

$$\begin{aligned} x(t) &= \frac{-1}{\Gamma(\alpha)} \left(\int_0^\lambda + \int_\lambda^t \right) (t-s)^{\alpha-1} y(s) ds + \frac{1-t}{A_\lambda \Gamma(\alpha)} \int_0^\lambda H(s) y(s) ds \\ &\quad + \frac{(A_\lambda + B_\lambda)t - B_\lambda}{A_\lambda \Gamma(\alpha)} \left(\int_0^\lambda + \int_\lambda^t + \int_t^\mu + \int_\mu^1 \right) (1-s)^{\alpha-1} y(s) ds \\ &\quad + \frac{t-1}{A_\lambda \Gamma(\alpha-1)} \left(\int_0^\lambda + \int_\lambda^t + \int_t^\mu \right) (\mu-s)^{\alpha-2} y(s) ds. \end{aligned}$$

In the case $0 < \lambda \leq \mu \leq t < 1$ we have

$$\begin{aligned} x(t) &= \frac{-1}{\Gamma(\alpha)} \left(\int_0^\lambda + \int_\lambda^\mu + \int_\mu^t \right) (t-s)^{\alpha-1} y(s) ds \\ &+ \frac{1-t}{A_\lambda \Gamma(\alpha)} \int_0^\lambda H(s) y(s) ds \\ &+ \frac{(A_\lambda + B_\lambda)t - B_\lambda}{A_\lambda \Gamma(\alpha)} \left(\int_0^\lambda + \int_\lambda^\mu + \int_\mu^t + \int_t^1 \right) (1-s)^{\alpha-1} y(s) ds \\ &+ \frac{t-1}{A_\lambda \Gamma(\alpha-1)} \left(\int_0^\lambda + \int_\lambda^\mu \right) (\mu-s)^{\alpha-2} y(s) ds. \end{aligned}$$

Finally if $0 < \mu \leq \lambda \leq t < 1$ we can write $x(t)$ as

$$\begin{aligned} x(t) &= \frac{-1}{\Gamma(\alpha)} \left(\int_0^\mu + \int_\mu^\lambda + \int_\lambda^t \right) (t-s)^{\alpha-1} y(s) ds \\ &+ \frac{1-t}{A_\lambda \Gamma(\alpha)} \left(\int_0^\mu + \int_\mu^\lambda \right) H(s) y(s) ds \\ &+ \frac{(A_\lambda + B_\lambda)t - B_\lambda}{A_\lambda \Gamma(\alpha)} \left(\int_0^\mu + \int_\mu^\lambda + \int_\lambda^t + \int_t^1 \right) (1-s)^{\alpha-1} y(s) ds \\ &+ \frac{t-1}{A_\lambda \Gamma(\alpha-1)} \int_0^\mu (\mu-s)^{\alpha-2} y(s) ds. \end{aligned}$$

So we have $x(t) = \int_0^1 G(t, s) y(s) ds$ where

$$\begin{aligned} G(t, s) &= \frac{-(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-t)H(s) + [(A_\lambda + B_\lambda)t - B_\lambda](1-s)^{\alpha-1} + \alpha(t-1)(\mu-s)^{\alpha-2}}{A_\lambda \Gamma(\alpha)}, \text{ when } \\ &0 \leq s \leq t \leq 1 \text{ and } s \leq \mu, \lambda, \\ G(t, s) &= \frac{-(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{[(A_\lambda + B_\lambda)t - B_\lambda](1-s)^{\alpha-1} + \alpha(t-1)(\mu-s)^{\alpha-2}}{A_\lambda \Gamma(\alpha)}, \text{ when } \\ &0 \leq s \leq t \leq 1 \text{ and } \lambda \leq s \leq \mu, \\ G(t, s) &= \frac{-(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-t)H(s) + [(A_\lambda + B_\lambda)t - B_\lambda](1-s)^{\alpha-1}}{A_\lambda \Gamma(\alpha)}, \text{ when } 0 \leq s \leq t \leq 1 \\ &\text{and } \mu \leq s \leq \lambda, \\ G(t, s) &= \frac{-(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{[(A_\lambda + B_\lambda)t - B_\lambda](1-s)^{\alpha-1}}{A_\lambda \Gamma(\alpha)}, \text{ when } 0 \leq s \leq t \leq 1 \text{ and } \\ &s \geq \mu, \lambda, \\ G(t, s) &= \frac{[(A_\lambda + B_\lambda)t - B_\lambda](1-s)^{\alpha-1} + (t-1)(\mu-s)^{\alpha-1}}{A_\lambda \Gamma(\alpha)}, \text{ when } 0 \leq t \leq s \leq 1 \text{ and } \\ &s \leq \mu, \lambda, \\ G(t, s) &= \frac{(1-t)H(s) + [(A_\lambda + B_\lambda)t - B_\lambda](1-s)^{\alpha-1}}{A_\lambda \Gamma(\alpha)}, \text{ when } 0 \leq t \leq s \leq 1 \text{ and } \\ &s \geq \mu, \lambda \text{ and } \end{aligned}$$

$$G(t, s) = \frac{[(A_\lambda + B_\lambda)t - B_\lambda](1-s)^{\alpha-1}}{A_\lambda \Gamma(\alpha)}, \text{ when } 0 \leq t \leq s \leq 1 \text{ and } s \geq \mu, \lambda, \quad \square$$

Note that, for the Green function $G(t, s)$, $\frac{\partial G}{\partial t}(t, s)$ is given as

$$\frac{\partial G}{\partial t}(t, s) = \frac{-(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{-H(s) + (A_\lambda + B_\lambda)(1-s)^{\alpha-1} + \alpha(\mu-s)^{\alpha-2}}{A_\lambda \Gamma(\alpha)},$$

when $0 \leq s \leq t \leq 1$ and $s \leq \mu, \lambda$,

$$\frac{\partial G}{\partial t}(t, s) = \frac{-(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{(A_\lambda + B_\lambda)(1-s)^{\alpha-1} + \alpha(\mu-s)^{\alpha-2}}{A_\lambda \Gamma(\alpha)},$$

when $0 \leq s \leq t \leq 1$ and $\lambda \leq s \leq \mu$,

$$\frac{\partial G}{\partial t}(t, s) = \frac{-(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{-H(s) + (A_\lambda + B_\lambda)(1-s)^{\alpha-1}}{A_\lambda \Gamma(\alpha)},$$

when $0 \leq s \leq t \leq 1$ and $\mu \leq s \leq \lambda$,

$$\frac{-(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{(A_\lambda + B_\lambda)(1-s)^{\alpha-1}}{A_\lambda \Gamma(\alpha)},$$

when $0 \leq s \leq t \leq 1$ and $s \geq \mu, \lambda$,

$$\frac{(A_\lambda + B_\lambda)(1-s)^{\alpha-1} + (\mu-s)^{\alpha-1}}{A_\lambda \Gamma(\alpha)},$$

when $0 \leq t \leq s \leq 1$ and $s \leq \mu, \lambda$,

$$\frac{-H(s) + (A_\lambda + B_\lambda)(1-s)^{\alpha-1}}{A_\lambda \Gamma(\alpha)},$$

when $0 \leq t \leq s \leq 1$ and $s \geq \mu, \lambda$ and

$$\frac{\partial G}{\partial t}(t, s) = \frac{(A_\lambda + B_\lambda)(1-s)^{\alpha-1}}{A_\lambda \Gamma(\alpha)}, \text{ when } 0 \leq t \leq s \leq 1 \text{ and } s \geq \mu, \lambda,$$

Also we can see G and $\frac{\partial}{\partial t}G$ are continuous respect to t . Consider the space $X = C^1[0, 1]$ with the norm $\|\cdot\|_*$, where $\|x\|_* = \max\{\|x\|, \|x'\|\}$ and $\|\cdot\|$ is the supremum norm on $C[0, 1]$. Let f be a map on $[0, 1] \times X^5$ such that f is singular at some points of $[0, 1]$. Define the map $F : X \rightarrow X$ by

$$\begin{aligned} & F_x(t) \\ &= \int_0^1 G(t, s) f(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi) x(\xi) d\xi, \phi x(s)) ds \\ &= \frac{-1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi) x(\xi) d\xi, \phi x(s)) ds \\ &\quad + \frac{1-t}{A_\lambda \Gamma(\alpha)} \int_0^\lambda H(s) f(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi) x(\xi) d\xi, \phi x(s)) ds \\ &\quad + \frac{(A_\lambda + B_\lambda)t - B_\lambda}{A_\lambda \Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s, x(s), x'(s), D^\beta x(s), \\ &\quad \int_0^s h(\xi) x(\xi) d\xi, \phi x(s)) ds \end{aligned}$$

$$+ \frac{t-1}{A_\lambda \Gamma(\alpha-1)} \int_0^\mu (\mu-s)^{\alpha-2} f(s, x(s), x'(s), D^\beta x(s), \\ \int_0^s h(\xi) x(\xi) d\xi, \phi x(s)) ds,$$

for all $t \in [0, 1]$. Also $F'_x(t)$ is given as

$$\begin{aligned} & F'_x(t) \\ &= \int_0^1 \frac{\partial G}{\partial t}(t, s) f(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi) x(\xi) d\xi, \phi x(s)) ds \\ &= \frac{-1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} f(s, x(s), x'(s), D^\beta x(s), \\ & \quad \int_0^s h(\xi) x(\xi) d\xi, \phi x(s)) ds \\ & \quad - \frac{1}{A_\lambda \Gamma(\alpha)} \int_0^\lambda H(s) f(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi) x(\xi) d\xi, \phi x(s)) ds \\ & \quad + \frac{A_\lambda + B_\lambda}{A_\lambda \Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s, x(s), x'(s), D^\beta x(s), \\ & \quad \int_0^s h(\xi) x(\xi) d\xi, \phi x(s)) ds \\ & \quad + \frac{1}{A_\lambda \Gamma(\alpha-1)} \int_0^\mu (\mu-s)^{\alpha-2} f(s, x(s), x'(s), D^\beta x(s), \\ & \quad \int_0^s h(\xi) x(\xi) d\xi, \phi x(s)) ds. \end{aligned}$$

Note that, the singular pointwise defined equation (1) has a solution $x_0 \in X$ if and only if x_0 a fixed point of the map F .

Theorem 2.2. *Let $\alpha \geq 2$, $\beta, \mu, \lambda \in (0, 1)$ $g \in L^1[0, \lambda]$, $g(t) > 0$ for a.e. $t \in [0, \lambda]$, $\phi : [0, 1] \rightarrow R^+$ is such that for all $x, y \in C^1[0, 1]$, $|\phi x(t) - \phi y(t)| \leq b_1|x(t) - y(t)| + b_2|x'(t) - y'(t)|$ for some $b_1, b_2 \in [0, \infty)$, $h \in L^1[0, 1]$ and $f : [0, 1] \times X^5 \rightarrow \mathbb{R}$ be a mapping which is singular on some set $E \subset [0, 1]$ such that for $t \in E$ we have*

$$|f(t, x_1, x_2, \dots, x_5) - f(t, y_1, y_2, \dots, y_5)| \leq \sum_{i=1}^{k_0} a_i(t) \Lambda_i(|x_1 - y_1|, \dots, |x_5 - y_5|)$$

and is continuous on $E^c \subset [0, 1]$ such that for $t \in E^c$ we have

$$|f(t, x_1, x_2, \dots, x_5) - f(t, y_1, y_2, \dots, y_5)| \leq \sum_{j=1}^5 l_j |x_j - y_j|$$

for all $x_1, \dots, x_5, y_1, \dots, y_5 \in X$, where $k_0 \in \mathbb{N}$, $a_i : [0, 1] \rightarrow \mathbb{R}^+$, $\hat{a}_i \in L^1(E)$, $\hat{a}_i(s) = (1-s)^{\alpha-2} a_i(s)$, $l_i \in [0, \infty)$, $\Lambda_i : X^5 \rightarrow \mathbb{R}^+$ is a nondecreasing mapping respect all their components such that $\lim_{z \rightarrow 0^+} \frac{\Lambda_i(z, \dots, z)}{z} = q_i$ for some $q_i \in [0, \infty)$, and all $1 \leq i \leq k_0$. Also let for almost all $t \in [0, 1]$, $f(t, 0, 0, 0, 0, 0) = 0$, if

$$\begin{aligned} & \max\left\{\left[\sum_{i=1}^{k_0} (q_i + \epsilon) \Delta \|\hat{a}_i\|_E\right] \left[\frac{2}{\Gamma(\alpha)} + \frac{\|g\|_{[0, \lambda]}}{A_\lambda \Gamma(\alpha)} + \frac{1}{A_\lambda \Gamma(\alpha - 1)}\right] \right. \\ & + M(E^c) \left[l_1 + l_2 + \frac{l_3}{\Gamma(2 - \beta)} + l_4 m + l_5(b_1 + b_2)\right] \times \\ & \left. \left(\frac{2}{\Gamma(\alpha + 1)} + \frac{\|g\|_{[0, \lambda]}}{A_\lambda \Gamma(\alpha + 1)} + \frac{1}{A_\lambda \Gamma(\alpha)}\right), \right. \\ & \left. \left(\sum_{i=1}^{k_0} (q_i + \epsilon) \Delta \|\hat{a}_i\|_E\right) \left(\frac{1}{\Gamma(\alpha - 1)} + \frac{2\|g\|_{[0, \lambda]}}{A_\lambda \Gamma(\alpha)} + \frac{1}{A_\lambda \Gamma(\alpha - 1)}\right) \right. \\ & + M(E^c) \left(l_1 + l_2 + \frac{l_3}{\Gamma(2 - \beta)} + l_4 m + l_5 b_1 + l_5 b_2\right) \times \\ & \left. \left(\frac{1}{\Gamma(\alpha)} + \frac{2\|g\|_{[0, \lambda]}}{A_\lambda \Gamma(\alpha + 1)} + \frac{1}{A_\lambda \Gamma(\alpha)}\right)\right\} < 1, \end{aligned}$$

then

$$D^\alpha x(t) + f(t, x(t), x'(t), D^\beta x(t), \int_0^t h(s)x(s)ds, \phi x(t)) = 0$$

with boundary conditions $x'(\mu) = x'(0) + \int_0^\lambda g(s)x(s)ds$ and $x(1) = x^{(j)}(0) = 0$ for $j \geq 2$ has a solution.

Proof. First we show that F is continuous on X . Let $x_1, x_2 \in X$ and

$t \in [0, 1]$, then we have

$$\begin{aligned}
& |F_{x_1}(t) - F_{x_2}(t)| \\
\leq & \left| \int_0^1 G(t, s) f(s, x_1(s), x_1'(s), D^\beta x_1(s), \int_0^s h(\xi) x_1(\xi) d\xi, \phi x_1(s)) ds \right. \\
& \left. - \int_0^1 G(t, s) f(s, x_2(s), x_2'(s), D^\beta x_2(s), \int_0^s h(\xi) x_2(\xi) d\xi, \phi x_2(s)) ds \right| \\
\leq & \left| \int_0^1 G(t, s) (f(s, x_1(s), x_1'(s), D^\beta x_1(s), \int_0^s h(\xi) x_1(\xi) d\xi, \phi x_1(s)) \right. \\
& \left. - f(s, x_2(s), x_2'(s), D^\beta x_2(s), \int_0^s h(\xi) x_2(\xi) d\xi, \phi x_2(s))) ds \right| \\
\leq & \frac{1}{\Gamma(\alpha)} \int_{E \cap [0, t]} (t-s)^{\alpha-1} |f(s, x_1(s), x_1'(s), D^\beta x_1(s), \\
& \int_0^s h(\xi) x_1(\xi) d\xi, \phi x_1(s)) \\
& - f(s, x_2(s), x_2'(s), D^\beta x_2(s), \int_0^s h(\xi) x_2(\xi) d\xi, \phi x_2(s))| ds \\
+ & \frac{1-t}{A_\lambda \Gamma(\alpha)} \int_{E \cap [0, \lambda]} H(s) |f(s, x_1(s), x_1'(s), D^\beta x_1(s), \\
& \int_0^s h(\xi) x_1(\xi) d\xi, \phi x_1(s)) \\
& - f(s, x_2(s), x_2'(s), D^\beta x_2(s), \int_0^s h(\xi) x_2(\xi) d\xi, \phi x_2(s))| ds \\
+ & \frac{|t \int_0^\lambda g(\xi) d\xi - \int_0^\lambda \xi g(\xi) d\xi|}{A_\lambda \Gamma(\alpha)} \int_E (1-s)^{\alpha-1} |f(s, x_1(s), x_1'(s), D^\beta x_1(s), \\
& \int_0^s h(\xi) x_1(\xi) d\xi, \phi x_1(s)) \\
& - f(s, x_2(s), x_2'(s), D^\beta x_2(s), \int_0^s h(\xi) x_2(\xi) d\xi, \phi x_2(s))| ds \\
+ & \frac{1-t}{A_\lambda \Gamma(\alpha-1)} \int_{E \cap [0, \mu]} (\mu-s)^{\alpha-2} |f(s, x_1(s), x_1'(s), D^\beta x_1(s), \\
& \int_0^s h(\xi) x_1(\xi) d\xi, \phi x_1(s)) \\
& - f(s, x_2(s), x_2'(s), D^\beta x_2(s), \int_0^s h(\xi) x_2(\xi) d\xi, \phi x_2(s))| ds
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{\Gamma(\alpha)} \int_{E^c \cap [0, t]} (t-s)^{\alpha-1} |f(s, x_1(t), x'_1(s), D^\beta x_1(s), \\
& \quad \int_0^s h(\xi) x_1(\xi) d\xi, \phi x_1(s)) \\
& \quad - f(s, x_2(s), x'_2(s), D^\beta x_2(s), \int_0^s h(\xi) x_2(\xi) d\xi, \phi x_2(s))| ds \\
& + \frac{1-t}{A_\lambda \Gamma(\alpha)} \int_{E^c \cap [0, \lambda]} H(s) |f(s, x_1(t), x'_1(s), D^\beta x_1(s), \\
& \quad \int_0^s h(\xi) x_1(\xi) d\xi, \phi x_1(s)) \\
& \quad - f(s, x_2(s), x'_2(s), D^\beta x_2(s), \int_0^s h(\xi) x_2(\xi) d\xi, \phi x_2(s))| ds \\
& + \frac{|t \int_0^\lambda g(\xi) d\xi - \int_0^\lambda \xi g(\xi) d\xi|}{A_\lambda \Gamma(\alpha)} \int_{E^c} (1-s)^{\alpha-1} |f(s, x_1(t), x'_1(s), D^\beta x_1(s), \\
& \quad \int_0^s h(\xi) x_1(\xi) d\xi, \phi x_1(s)) \\
& \quad - f(s, x_2(s), x'_2(s), D^\beta x_2(s), \int_0^s h(\xi) x_2(\xi) d\xi, \phi x_2(s))| ds \\
& + \frac{1-t}{A_\lambda \Gamma(\alpha-1)} \int_{E^c \cap [0, \mu]} (\mu-s)^{\alpha-2} |f(s, x_1(t), x'_1(s), D^\beta x_1(s) \\
& \quad , \int_0^s h(\xi) x_1(\xi) d\xi, \phi x_1(s)) \\
& \quad - f(s, x_2(s), x'_2(s), D^\beta x_2(s), \int_0^s h(\xi) x_2(\xi) d\xi, \phi x_2(s))| ds \\
& \leq \frac{1}{\Gamma(\alpha)} \int_{E \cap [0, t]} (t-s)^{\alpha-1} \left[\sum_{i=1}^{k_0} a_i(s) \Lambda_i(|x_1(s) - x_2(s)|, \right. \\
& \quad |x'_1(s) - x'_2(s)|, |D^\beta(x_1 - x_2)(s)|, \left| \int_0^s h(\xi)(x_1(\xi) - x_2(\xi)) d\xi \right|, \\
& \quad \left. |\phi x_1(s) - \phi x_2(s)| \right] ds \\
& + \frac{1-t}{A_\lambda \Gamma(\alpha)} \int_{E \cap [0, \lambda]} H(s) \left[\sum_{i=1}^{k_0} a_i(s) \Lambda_i(|x_1(s) - x_2(s)|, |x'_1(s) - x'_2(s)|, \right. \\
& \quad \left. |D^\beta(x_1 - x_2)(s)|, \left| \int_0^s h(\xi)(x_1(\xi) - x_2(\xi)) d\xi \right|, |\phi x_1(s) - \phi x_2(s)| \right] ds
\end{aligned}$$

$$\begin{aligned}
& + \frac{\int_0^\lambda |t - \xi| g(\xi) d\xi}{A_\lambda \Gamma(\alpha)} \int_E (1 - s)^{\alpha-1} \left[\sum_{i=1}^{k_0} a_i(s) \Lambda_i(|x_1(s) - x_2(s)|, \right. \\
& \quad |x'_1(s) - x'_2(s)|, |D^\beta(x_1 - x_2)(s)|, \left| \int_0^s h(\xi)(x_1(\xi) - x_2(\xi)) d\xi \right|, \\
& \quad \left. |\phi x_1(s) - \phi x_2(s)| \right] ds \\
& + \frac{1 - t}{A_\lambda \Gamma(\alpha - 1)} \int_{E \cap [0, \mu]} (\mu - s)^{\alpha-2} \left[\sum_{i=1}^{k_0} a_i(s) \Lambda_i(|x_1(s) - x_2(s)|, \right. \\
& \quad |x'_1(s) - x'_2(s)|, |D^\beta(x_1 - x_2)(s)|, \left| \int_0^s h(\xi)(x_1(\xi) - x_2(\xi)) d\xi \right|, \\
& \quad \left. |\phi x_1(s) - \phi x_2(s)| \right] ds \\
& + \frac{1}{\Gamma(\alpha)} \int_{E^c \cap [0, t]} (t - s)^{\alpha-1} [l_1 |x_1(s) - x_2(s)| + l_2 |x'_1(s) - x'_2(s)| \\
& \quad + l_3 |D^\beta(x_1 - x_2)(s)| + l_4 \left| \int_0^s h(\xi)(x_1(\xi) - x_2(\xi)) d\xi \right| \\
& \quad + l_5 |\phi x_1(s) - \phi x_2(s)|] ds \\
& + \frac{1 - t}{A_\lambda \Gamma(\alpha)} \int_{E^c \cap [0, \lambda]} H(s) [l_1 |x_1(s) - x_2(s)| + l_2 |x'_1(s) - x'_2(s)| \\
& \quad + l_3 |D^\beta(x_1 - x_2)(s)| + l_4 \left| \int_0^s h(\xi)(x_1(\xi) - x_2(\xi)) d\xi \right| \\
& \quad + l_5 |\phi x_1(s) - \phi x_2(s)|] ds \\
& + \frac{\int_0^\lambda |t - \xi| g(\xi) d\xi}{A_\lambda \Gamma(\alpha)} \int_{E^c} (1 - s)^{\alpha-1} [l_1 |x_1(s) - x_2(s)| + l_2 |x'_1(s) - x'_2(s)| \\
& \quad + l_3 |D^\beta(x_1 - x_2)(s)| + l_4 \left| \int_0^s h(\xi)(x_1(\xi) - x_2(\xi)) d\xi \right| \\
& \quad + l_5 |\phi x_1(s) - \phi x_2(s)|] ds \\
& + \frac{1 - t}{A_\lambda \Gamma(\alpha - 1)} \int_{E^c \cap [0, \mu]} (\mu - s)^{\alpha-2} [l_1 |x_1(s) - x_2(s)| + l_2 |x'_1(s) - x'_2(s)| \\
& \quad + l_3 |D^\beta(x_1 - x_2)(s)| + l_4 \left| \int_0^s h(\xi)(x_1(\xi) - x_2(\xi)) d\xi \right| \\
& \quad + l_5 |\phi x_1(s) - \phi x_2(s)|] ds
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k_0} \int_{E \cap [0,t]} (t-s)^{\alpha-1} a_i(s) \Lambda_i(\|x_1 - x_2\|, \|x'_1 - x'_2\|, \\
&\quad \frac{\|x'_1 - x'_2\|}{\Gamma(2-\beta)}, \|x_1 - x_2\| \int_0^s |h(\xi)| d\xi, b_1 \|x_1 - x_2\| + b_2 \|x'_1 - x'_2\|) ds \\
&+ \frac{1-t}{A_\lambda \Gamma(\alpha)} \sum_{i=1}^{k_0} \int_{E \cap [0,\lambda]} H(s) a_i(s) \Lambda_i(\|x_1 - x_2\|, \|x'_1 - x'_2\|, \\
&\quad \frac{\|x'_1 - x'_2\|}{\Gamma(2-\beta)}, \|x_1 - x_2\| \int_0^s |h(\xi)| d\xi, b_1 \|x_1 - x_2\| + b_2 \|x'_1 - x'_2\|) ds \\
&+ \frac{\int_0^\lambda |t-\xi| g(\xi) d\xi}{A_\lambda \Gamma(\alpha)} \sum_{i=1}^{k_0} \int_E (1-s)^{\alpha-1} a_i(s) \Lambda_i(\|x_1 - x_2\|, \|x'_1 - x'_2\|, \\
&\quad \frac{\|x'_1 - x'_2\|}{\Gamma(2-\beta)}, \|x_1 - x_2\| \int_0^s |h(\xi)| d\xi, b_1 \|x_1 - x_2\| + b_2 \|x'_1 - x'_2\|) ds \\
&+ \frac{1-t}{A_\lambda \Gamma(\alpha-1)} \sum_{i=1}^{k_0} \int_{E \cap [0,\mu]} (\mu-s)^{\alpha-2} a_i(s) \Lambda_i(\|x_1 - x_2\|, \|x'_1 - x'_2\|, \\
&\quad \frac{\|x'_1 - x'_2\|}{\Gamma(2-\beta)}, \|x_1 - x_2\| \int_0^s |h(\xi)| d\xi, b_1 \|x_1 - x_2\| + b_2 \|x'_1 - x'_2\|) ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{E^c \cap [0,t]} (t-s)^{\alpha-1} [l_1 \|x_1 - x_2\| + l_2 \|x'_1 - x'_2\| + \frac{l_3 \|x'_1 - x'_2\|}{\Gamma(2-\beta)} \\
&\quad + l_4 \|x_1 - x_2\| \int_0^s |h(\xi)| d\xi + l_5 b_1 \|x_1 - x_2\| + l_5 b_2 \|x'_1 - x'_2\|] ds \\
&+ \frac{1-t}{A_\lambda \Gamma(\alpha)} \int_{E^c \cap [0,\lambda]} H(s) [l_1 \|x_1 - x_2\| + l_2 \|x'_1 - x'_2\| + \frac{l_3 \|x'_1 - x'_2\|}{\Gamma(2-\beta)} \\
&\quad + l_4 \|x_1 - x_2\| \int_0^s |h(\xi)| d\xi + l_5 b_1 \|x_1 - x_2\| + l_5 b_2 \|x'_1 - x'_2\|] ds \\
&+ \frac{\int_0^\lambda |t-\xi| g(\xi) d\xi}{A_\lambda \Gamma(\alpha)} \int_{E^c} (1-s)^{\alpha-1} [l_1 \|x_1 - x_2\| + l_2 \|x'_1 - x'_2\| \\
&\quad + \frac{l_3 \|x'_1 - x'_2\|}{\Gamma(2-\beta)} + l_4 \|x_1 - x_2\| \int_0^s |h(\xi)| d\xi + l_5 b_1 \|x_1 - x_2\| \\
&\quad + l_5 b_2 \|x'_1 - x'_2\|] ds
\end{aligned}$$

$$\begin{aligned}
 & + \frac{1-t}{A_\lambda \Gamma(\alpha-1)} \int_{E^c \cap [0, \mu]} (\mu-s)^{\alpha-2} [l_1 \|x_1 - x_2\| + l_2 \|x'_1 - x'_2\| \\
 & + \frac{l_3 \|x'_1 - x'_2\|}{\Gamma(2-\beta)} + l_4 \|x_1 - x_2\| \int_0^s |h(\xi)| d\xi + l_5 b_1 \|x_1 - x_2\| \\
 & + l_5 b_2 \|x'_1 - x'_2\|] ds.
 \end{aligned}$$

Note that $\|D^\beta x\| \leq \frac{\|x'\|}{\Gamma(2-\beta)}$, also we have

$$H(s) = \int_s^\lambda (t-s)^{\alpha-1} g(t) dt \leq \int_s^\lambda (\lambda-s)^{\alpha-1} g(t) dt \leq (\lambda-s)^{\alpha-1} \|g\|_{[0, \lambda]}$$

where $\|g\|_{[0, \lambda]} = \int_0^\lambda g(t) dt$. Let $\Delta = \max\{1, \frac{1}{\Gamma(2-\beta)}, m, b_1 + b_2\}$, then by the last inequality for all $x_1, x_2 \in X$ and $t \in [0, 1]$ we have

$$\begin{aligned}
 & |F_{x_1}(t) - F_{x_2}(t)| \\
 \leq & \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k_0} \int_E (1-s)^{\alpha-1} a_i(s) \Lambda_i(\|x_1 - x_2\|, \|x'_1 - x'_2\|, \\
 & \frac{\|x'_1 - x'_2\|}{\Gamma(2-\beta)}, \|x_1 - x_2\| \int_0^s |h(\xi)| d\xi, b_1 \|x_1 - x_2\| + b_2 \|x'_1 - x'_2\|) ds \\
 & + \frac{1-t}{A_\lambda \Gamma(\alpha)} \sum_{i=1}^{k_0} \int_E (\lambda-s)^{\alpha-1} \|g\|_{[0, \lambda]} a_i(s) \Lambda_i(\|x_1 - x_2\|, \|x'_1 - x'_2\|, \\
 & \frac{\|x'_1 - x'_2\|}{\Gamma(2-\beta)}, \|x_1 - x_2\| \int_0^s |h(\xi)| d\xi, b_1 \|x_1 - x_2\| + b_2 \|x'_1 - x'_2\|) ds \\
 & + \frac{\int_0^\lambda |t-\xi| g(\xi) d\xi}{A_\lambda \Gamma(\alpha)} \sum_{i=1}^{k_0} \int_E (1-s)^{\alpha-1} a_i(s) \Lambda_i(\|x_1 - x_2\|, \|x'_1 - x'_2\|, \\
 & \frac{\|x'_1 - x'_2\|}{\Gamma(2-\beta)}, \|x_1 - x_2\| \int_0^s |h(\xi)| d\xi, b_1 \|x_1 - x_2\| + b_2 \|x'_1 - x'_2\|) ds \\
 & + \frac{1-t}{A_\lambda \Gamma(\alpha-1)} \sum_{i=1}^{k_0} \int_E (\mu-s)^{\alpha-2} a_i(s) \Lambda_i(\|x_1 - x_2\|, \|x'_1 - x'_2\|, \\
 & \frac{\|x'_1 - x'_2\|}{\Gamma(2-\beta)}, \|x_1 - x_2\| \int_0^s |h(\xi)| d\xi, b_1 \|x_1 - x_2\| + b_2 \|x'_1 - x'_2\|) ds
 \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{\Gamma(\alpha)} \int_{E^c} (1-s)^{\alpha-1} [l_1 \|x_1 - x_2\| + l_2 \|x'_1 - x'_2\| + \frac{l_3 \|x'_1 - x'_2\|}{\Gamma(2-\beta)} \\
& \quad + l_4 \|x_1 - x_2\| \int_0^s |h(\xi)| d\xi + l_5 b_1 \|x_1 - x_2\| + l_5 b_2 \|x'_1 - x'_2\|] ds \\
& + \frac{1-t}{A_\lambda \Gamma(\alpha)} \int_{E^c} (\lambda-s)^{\alpha-1} \|g\|_{[0,\lambda]} [l_1 \|x_1 - x_2\| + l_2 \|x'_1 - x'_2\| \\
& \quad + \frac{l_3 \|x'_1 - x'_2\|}{\Gamma(2-\beta)} + l_4 \|x_1 - x_2\| \int_0^s |h(\xi)| d\xi + l_5 b_1 \|x_1 - x_2\| \\
& \quad + l_5 b_2 \|x'_1 - x'_2\|] ds \\
& + \frac{\int_0^\lambda |t-\xi| g(\xi) d\xi}{A_\lambda \Gamma(\alpha)} \int_{E^c} (1-s)^{\alpha-1} [l_1 \|x_1 - x_2\| + l_2 \|x'_1 - x'_2\| \\
& \quad + \frac{l_3 \|x'_1 - x'_2\|}{\Gamma(2-\beta)} + l_4 \|x_1 - x_2\| \int_0^s |h(\xi)| d\xi + l_5 b_1 \|x_1 - x_2\| \\
& \quad + l_5 b_2 \|x'_1 - x'_2\|] ds \\
& + \frac{1-t}{A_\lambda \Gamma(\alpha-1)} \int_{E^c} (\mu-s)^{\alpha-2} [l_1 \|x_1 - x_2\| + l_2 \|x'_1 - x'_2\| \\
& \quad + \frac{l_3 \|x'_1 - x'_2\|}{\Gamma(2-\beta)} + l_4 \|x_1 - x_2\| \int_0^s |h(\xi)| d\xi + l_5 b_1 \|x_1 - x_2\| \\
& \quad + l_5 b_2 \|x'_1 - x'_2\|] ds \\
& \leq \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k_0} \Lambda_i(\Delta \|x_1 - x_2\|_*, \dots, \Delta \|x_1 - x_2\|_*) \int_E (1-s)^{\alpha-1} a_i(s) ds \\
& + \frac{(1-t) \|g\|_{[0,\lambda]}}{A_\lambda \Gamma(\alpha)} \sum_{i=1}^{k_0} \Lambda_i(\Delta \|x_1 - x_2\|_*, \dots, \Delta \|x_1 - x_2\|_*) \times \\
& \quad \int_E (1-s)^{\alpha-1} a_i(s) ds \\
& + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k_0} \Lambda_i(\Delta \|x_1 - x_2\|_*, \dots, \Delta \|x_1 - x_2\|_*) \int_E (1-s)^{\alpha-1} a_i(s) ds \\
& + \frac{1-t}{A_\lambda \Gamma(\alpha-1)} \sum_{i=1}^{k_0} \Lambda_i(\Delta \|x_1 - x_2\|_*, \dots, \Delta \|x_1 - x_2\|_*) \times \\
& \quad \int_E (1-s)^{\alpha-2} a_i(s) ds
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{\Gamma(\alpha)} \|x_1 - x_2\|_* [l_1 + l_2 + \frac{l_3}{\Gamma(2-\beta)} + l_4 m + l_5(b_1 + b_2)] \times \\
& \quad \int_{E^c} (1-s)^{\alpha-1} ds \\
& + \frac{(1-t)\|g\|_{[0,\lambda]}}{A_\lambda \Gamma(\alpha)} \|x_1 - x_2\|_* [l_1 + l_2 + \frac{l_3}{\Gamma(2-\beta)} + l_4 m + l_5(b_1 + b_2)] \times \\
& \quad \int_{E^c} (1-s)^{\alpha-1} ds \\
& + \frac{1}{\Gamma(\alpha)} \|x_1 - x_2\|_* [l_1 + l_2 + \frac{l_3}{\Gamma(2-\beta)} + l_4 m + l_5(b_1 + b_2)] \times \\
& \quad \int_{E^c} (1-s)^{\alpha-1} ds \\
& + \frac{1-t}{A_\lambda \Gamma(\alpha-1)} \|x_1 - x_2\|_* [l_1 + l_2 + \frac{l_3}{\Gamma(2-\beta)} + l_4 m + l_5(b_1 + b_2)] \times \\
& \quad \int_{E^c} (1-s)^{\alpha-2} ds.
\end{aligned} \tag{2}$$

Now let $\epsilon > 0$ be arbitrary and $x_1 \rightarrow x_2$ in X . Since $\lim_{z \rightarrow 0^+} \frac{\Lambda_i(z, \dots, z)}{z} = q_i$ for $i = 1, \dots, k_0$, so $\lim_{z \rightarrow 0^+} \frac{\Lambda_i(\Delta z, \dots, \Delta z)}{\Delta z} = q_i$, so for $\epsilon > 0$ there exists $0 < \delta \leq \epsilon$ such that $0 < z \leq \delta$ implies that

$$0 < \Lambda_i(\Delta z, \dots, \Delta z) < (q_i + \epsilon) \Delta z,$$

for all $i = 1, \dots, k_0$, in particular

$$0 < \Lambda_i(\Delta \delta, \dots, \Delta \delta) < (q_i + \epsilon) \Delta \delta < (q_i + \epsilon) \Delta \epsilon,$$

so when $\|x_1 - x_2\| < \delta$, by (2) we have

$$\begin{aligned}
& |F_{x_1}(t) - F_{x_2}(t)| \\
& \leq \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k_0} \Lambda_i(\Delta \delta, \dots, \Delta \delta) \|\hat{a}_i\|_E \\
& + \frac{(1-t)\|g\|_{[0,\lambda]}}{A_\lambda \Gamma(\alpha)} \sum_{i=1}^{k_0} \Lambda_i(\Delta \delta, \dots, \Delta \delta) \|\hat{a}_i\|_E
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k_0} \Lambda_i(\Delta\delta, \dots, \Delta\delta) \|\hat{a}_i\|_E \\
& + \frac{1-t}{A_\lambda \Gamma(\alpha-1)} \sum_{i=1}^{k_0} \Lambda_i(\Delta\delta, \dots, \Delta\delta) \|\hat{a}_i\|_E \\
& + \frac{1}{\Gamma(\alpha)} \delta[l_1 + l_2 + \frac{l_3}{\Gamma(2-\beta)} + l_4 m + l_5(b_1 + b_2)] \frac{M(E^c)}{\alpha} \\
& + \frac{(1-t)\|g\|_{[0,\lambda]}}{A_\lambda \Gamma(\alpha)} \delta[l_1 + l_2 + \frac{l_3}{\Gamma(2-\beta)} + l_4 m + l_5(b_1 + b_2)] \frac{M(E^c)}{\alpha} \\
& + \frac{1}{\Gamma(\alpha)} \delta[l_1 + l_2 + \frac{l_3}{\Gamma(2-\beta)} + l_4 m + l_5(b_1 + b_2)] \frac{M(E^c)}{\alpha} \\
& + \frac{1-t}{A_\lambda \Gamma(\alpha-1)} \delta[l_1 + l_2 + \frac{l_3}{\Gamma(2-\beta)} + l_4 m + l_5(b_1 + b_2)] \frac{M(E^c)}{\alpha-1} \\
& \leq \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k_0} (q_i + \epsilon) \Delta\epsilon \|\hat{a}_i\|_E + \frac{(1-t)\|g\|_{[0,\lambda]}}{A_\lambda \Gamma(\alpha)} \sum_{i=1}^{k_0} (q_i + \epsilon) \Delta\epsilon \|\hat{a}_i\|_E \\
& + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k_0} (q_i + \epsilon) \Delta\epsilon \|\hat{a}_i\|_E + \frac{1-t}{A_\lambda \Gamma(\alpha-1)} \sum_{i=1}^{k_0} (q_i + \epsilon) \Delta\epsilon \|\hat{a}_i\|_E \\
& + \frac{M(E^c)}{\Gamma(\alpha+1)} [l_1 + l_2 + \frac{l_3}{\Gamma(2-\beta)} + l_4 m + l_5(b_1 + b_2)] \epsilon \\
& + \frac{(1-t)\|g\|_{[0,\lambda]} M(E^c)}{A_\lambda \Gamma(\alpha+1)} [l_1 + l_2 + \frac{l_3}{\Gamma(2-\beta)} + l_4 m + l_5(b_1 + b_2)] \epsilon \\
& + \frac{M(E^c)}{\Gamma(\alpha+1)} [l_1 + l_2 + \frac{l_3}{\Gamma(2-\beta)} + l_4 m + l_5(b_1 + b_2)] \epsilon \\
& + \frac{(1-t)M(E^c)}{A_\lambda \Gamma(\alpha)} [l_1 + l_2 + \frac{l_3}{\Gamma(2-\beta)} + l_4 m + l_5(b_1 + b_2)] \epsilon
\end{aligned}$$

where M is the Lebesgue measure. Hence

$$\begin{aligned}
& \|F_{x_1} - F_{x_2}\| \\
\leq & \left(\left[\sum_{i=1}^{k_0} (q_i + \epsilon) \Delta \|\hat{a}_i\|_E \right] \left[\frac{2}{\Gamma(\alpha)} + \frac{\|g\|_{[0,\lambda]}}{A_\lambda \Gamma(\alpha)} + \frac{1}{A_\lambda \Gamma(\alpha - 1)} \right] \right. \\
& + M(E^c) [l_1 + l_2 + \frac{l_3}{\Gamma(2 - \beta)} + l_4 m + l_5 (b_1 + b_2)] \times \\
& \left. \left[\frac{2}{\Gamma(\alpha + 1)} + \frac{\|g\|_{[0,\lambda]}}{A_\lambda \Gamma(\alpha + 1)} + \frac{1}{A_\lambda \Gamma(\alpha)} \right] \right) \epsilon.
\end{aligned}$$

Also we have

$$\begin{aligned}
& |F'_{x_1}(t) - F'_{x_2}(t)| \\
\leq & \left| \frac{-1}{\Gamma(\alpha - 1)} \int_0^t (t - s)^{\alpha-2} (f(s, x_1(t), x'_1(s), D^\beta x_1(s), \right. \\
& \left. \int_0^s h(\xi) x_1(\xi) d\xi, \phi x_1(s)) \right. \\
& \left. - f(s, x_2(s), x'_2(s), D^\beta x_2(s), \int_0^s h(\xi) x_2(\xi) d\xi, \phi x_2(s))) ds \right. \\
& - \frac{1}{A_\lambda \Gamma(\alpha)} \int_0^\lambda H(s) (f(s, x_1(t), x'_1(s), D^\beta x_1(s), \\
& \left. \int_0^s h(\xi) x_1(\xi) d\xi, \phi x_1(s)) \right. \\
& \left. - f(s, x_2(s), x'_2(s), D^\beta x_2(s), \int_0^s h(\xi) x_2(\xi) d\xi, \phi x_2(s))) ds \right. \\
& + \frac{A_\lambda + B_\lambda}{A_\lambda \Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-1} (f(s, x_1(t), x'_1(s), D^\beta x_1(s), \\
& \left. \int_0^s h(\xi) x_1(\xi) d\xi, \phi x_1(s)) \right. \\
& \left. - f(s, x_2(s), x'_2(s), D^\beta x_2(s), \int_0^s h(\xi) x_2(\xi) d\xi, \phi x_2(s))) ds \right. \\
& + \frac{1}{A_\lambda \Gamma(\alpha - 1)} \int_0^\mu (\mu - s)^{\alpha-2} (f(s, x_1(t), x'_1(s), D^\beta x_1(s), \\
& \left. \int_0^s h(\xi) x_1(\xi) d\xi, \phi x_1(s)) \right.
\end{aligned}$$

$$\begin{aligned}
& -f(s, x_2(s), x_2'(s), D^\beta x_2(s), \int_0^s h(\xi)x_2(\xi)d\xi, \phi x_2(s)))ds \\
\leq & \frac{1}{\Gamma(\alpha-1)} \int_{E \cap [0,t]} (t-s)^{\alpha-2} |f(s, x_1(t), x_1'(s), D^\beta x_1(s), \\
& \int_0^s h(\xi)x_1(\xi)d\xi, \phi x_1(s)) \\
& -f(s, x_2(s), x_2'(s), D^\beta x_2(s), \int_0^s h(\xi)x_2(\xi)d\xi, \phi x_2(s))| ds \\
+ & \frac{1}{A_\lambda \Gamma(\alpha)} \int_{E \cap [0,\lambda]} H(s) |f(s, x_1(t), x_1'(s), D^\beta x_1(s), \\
& \int_0^s h(\xi)x_1(\xi)d\xi, \phi x_1(s)) \\
& -f(s, x_2(s), x_2'(s), D^\beta x_2(s), \int_0^s h(\xi)x_2(\xi)d\xi, \phi x_2(s))| ds \\
+ & \frac{A_\lambda + B_\lambda}{A_\lambda \Gamma(\alpha)} \int_E (1-s)^{\alpha-1} |f(s, x_1(t), x_1'(s), D^\beta x_1(s), \\
& \int_0^s h(\xi)x_1(\xi)d\xi, \phi x_1(s)) \\
& -f(s, x_2(s), x_2'(s), D^\beta x_2(s), \int_0^s h(\xi)x_2(\xi)d\xi, \phi x_2(s))| ds \\
+ & \frac{1}{A_\lambda \Gamma(\alpha-1)} \int_{E \cap [0,\mu]} (\mu-s)^{\alpha-2} |f(s, x_1(t), x_1'(s), D^\beta x_1(s), \\
& \int_0^s h(\xi)x_1(\xi)d\xi, \phi x_1(s)) \\
& -f(s, x_2(s), x_2'(s), D^\beta x_2(s), \int_0^s h(\xi)x_2(\xi)d\xi, \phi x_2(s))| ds \\
+ & \frac{1}{\Gamma(\alpha-1)} \int_{E^c \cap [0,t]} (t-s)^{\alpha-2} |f(s, x_1(t), x_1'(s), D^\beta x_1(s), \\
& \int_0^s h(\xi)x_1(\xi)d\xi, \phi x_1(s)) \\
& -f(s, x_2(s), x_2'(s), D^\beta x_2(s), \int_0^s h(\xi)x_2(\xi)d\xi, \phi x_2(s))| ds
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{A_\lambda \Gamma(\alpha)} \int_{E^c \cap [0, \lambda]} H(s) |f(s, x_1(t), x'_1(s), D^\beta x_1(s), \\
& \quad \int_0^s h(\xi) x_1(\xi) d\xi, \phi x_1(s)) \\
& \quad - f(s, x_2(s), x'_2(s), D^\beta x_2(s), \int_0^s h(\xi) x_2(\xi) d\xi, \phi x_2(s))| ds \\
& + \frac{A_\lambda + B_\lambda}{A_\lambda \Gamma(\alpha)} \int_{E^c} (1-s)^{\alpha-1} |f(s, x_1(t), x'_1(s), D^\beta x_1(s), \\
& \quad \int_0^s h(\xi) x_1(\xi) d\xi, \phi x_1(s)) \\
& \quad - f(s, x_2(s), x'_2(s), D^\beta x_2(s), \int_0^s h(\xi) x_2(\xi) d\xi, \phi x_2(s))| ds \\
& + \frac{1}{A_\lambda \Gamma(\alpha-1)} \int_{E^c \cap [0, \mu]} (\mu-s)^{\alpha-2} |f(s, x_1(t), x'_1(s), D^\beta x_1(s), \\
& \quad \int_0^s h(\xi) x_1(\xi) d\xi, \phi x_1(s)) \\
& \quad - f(s, x_2(s), x'_2(s), D^\beta x_2(s), \int_0^s h(\xi) x_2(\xi) d\xi, \phi x_2(s))| ds \\
& \leq \frac{1}{\Gamma(\alpha-1)} \int_{E \cap [0, t]} (t-s)^{\alpha-2} \left[\sum_{i=1}^{k_0} a_i(s) \Lambda_i(|x_1(s) - x_2(s)|, \right. \\
& \quad |x'_1(s) - x'_2(s)|, |D^\beta(x_1 - x_2)(s)|, \left| \int_0^s h(\xi)(x_1(\xi) - x_2(\xi)) d\xi \right|, \\
& \quad \left. |\phi x_1(s) - \phi x_2(s)|) \right] ds \\
& + \frac{1}{A_\lambda \Gamma(\alpha)} \int_{E \cap [0, \lambda]} H(s) \left[\sum_{i=1}^{k_0} a_i(s) \Lambda_i(|x_1(s) - x_2(s)|, |x'_1(s) - x'_2(s)|, \right. \\
& \quad |D^\beta(x_1 - x_2)(s)|, \left| \int_0^s h(\xi)(x_1(\xi) - x_2(\xi)) d\xi \right|, |\phi x_1(s) - \phi x_2(s)|) \right] ds \\
& + \frac{\int_0^\lambda g(\xi) d\xi}{A_\lambda \Gamma(\alpha)} \int_E (1-s)^{\alpha-1} \left[\sum_{i=1}^{k_0} a_i(s) \Lambda_i(|x_1(s) - x_2(s)|, |x'_1(s) - x'_2(s)|, \right. \\
& \quad |D^\beta(x_1 - x_2)(s)|, \left| \int_0^s h(\xi)(x_1(\xi) - x_2(\xi)) d\xi \right|, |\phi x_1(s) - \phi x_2(s)|) \right] ds
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{A_\lambda \Gamma(\alpha - 1)} \int_{E \cap [0, \mu]} (\mu - s)^{\alpha-2} \left[\sum_{i=1}^{k_0} a_i(s) \Lambda_i(|x_1(s) - x_2(s)|, \right. \\
& \quad |x'_1(s) - x'_2(s)|, |D^\beta(x_1 - x_2)(s)|, \left| \int_0^s h(\xi)(x_1(\xi) - x_2(\xi)) d\xi \right|, \\
& \quad \left. |\phi x_1(s) - \phi x_2(s)| \right] ds \\
& + \frac{1}{\Gamma(\alpha - 1)} \int_{E^c \cap [0, t]} (t - s)^{\alpha-2} [l_1|x_1(s) - x_2(s)| + l_2|x'_1(s) - x'_2(s)| \\
& \quad + l_3|D^\beta(x_1 - x_2)(s)| + l_4 \left| \int_0^s h(\xi)(x_1(\xi) - x_2(\xi)) d\xi \right| \\
& \quad + l_5|\phi x_1(s) - \phi x_2(s)|] ds \\
& + \frac{1}{A_\lambda \Gamma(\alpha)} \int_{E^c \cap [0, \lambda]} H(s) [l_1|x_1(s) - x_2(s)| + l_2|x'_1(s) - x'_2(s)| \\
& \quad + l_3|D^\beta(x_1 - x_2)(s)| + l_4 \left| \int_0^s h(\xi)(x_1(\xi) - x_2(\xi)) d\xi \right| \\
& \quad + l_5|\phi x_1(s) - \phi x_2(s)|] ds \\
& + \frac{\int_0^\lambda g(\xi) d\xi}{A_\lambda \Gamma(\alpha)} \int_{E^c} (1 - s)^{\alpha-1} [l_1|x_1(s) - x_2(s)| + l_2|x'_1(s) - x'_2(s)| \\
& \quad + l_3|D^\beta(x_1 - x_2)(s)| + l_4 \left| \int_0^s h(\xi)(x_1(\xi) - x_2(\xi)) d\xi \right| \\
& \quad + l_5|\phi x_1(s) - \phi x_2(s)|] ds \\
& + \frac{1}{A_\lambda \Gamma(\alpha - 1)} \int_{E^c \cap [0, \mu]} (\mu - s)^{\alpha-2} [l_1|x_1(s) - x_2(s)| \\
& \quad + l_2|x'_1(s) - x'_2(s)| + l_3|D^\beta(x_1 - x_2)(s)| + \\
& \quad l_4 \left| \int_0^s h(\xi)(x_1(\xi) - x_2(\xi)) d\xi \right| + l_5|\phi x_1(s) - \phi x_2(s)|] ds \\
& \leq \frac{1}{\Gamma(\alpha - 1)} \sum_{i=1}^{k_0} \int_E (1 - s)^{\alpha-2} a_i(s) \Lambda_i(\|x_1 - x_2\|, \\
& \quad \|x'_1 - x'_2\|, \frac{\|x'_1 - x'_2\|}{\Gamma(2 - \beta)}, m\|x_1 - x_2\|, b_1\|x_1 - x_2\| + b_2\|x'_1 - x'_2\|) ds \\
& + \frac{1}{A_\lambda \Gamma(\alpha)} \sum_{i=1}^{k_0} \int_E \|g\|_{[0, \lambda]} (\lambda - s)^{\alpha-1} a_i(s) \Lambda_i(\|x_1 - x_2\|, \|x'_1 - x'_2\|,
\end{aligned}$$

$$\begin{aligned}
& \frac{\|x'_1 - x'_2\|}{\Gamma(2 - \beta)}, m\|x_1 - x_2\|, b_1\|x_1 - x_2\| + b_2\|x'_1 - x'_2\|)ds \\
& + \frac{\int_0^\lambda g(\xi)d\xi}{A_\lambda\Gamma(\alpha)} \sum_{i=1}^{k_0} \int_E (1-s)^{\alpha-1} a_i(s) \Lambda_i(\|x_1 - x_2\|, \|x'_1 - x'_2\|, \\
& \frac{\|x'_1 - x'_2\|}{\Gamma(2 - \beta)}, m\|x_1 - x_2\|, b_1\|x_1 - x_2\| + b_2\|x'_1 - x'_2\|)ds \\
& + \frac{1}{A_\lambda\Gamma(\alpha-1)} \sum_{i=1}^{k_0} \int_E (1-s)^{\alpha-2} a_i(s) \Lambda_i(\|x_1 - x_2\|, \|x'_1 - x'_2\|, \\
& \frac{\|x'_1 - x'_2\|}{\Gamma(2 - \beta)}, m\|x_1 - x_2\|, b_1\|x_1 - x_2\| + b_2\|x'_1 - x'_2\|)ds \\
& + \frac{1}{\Gamma(\alpha-1)} \int_{E^c} (1-s)^{\alpha-2} [l_1\|x_1 - x_2\| + l_2\|x'_1 - x'_2\| + \frac{l_3\|x'_1 - x'_2\|}{\Gamma(2 - \beta)} \\
& + l_4 m\|x_1 - x_2\| + l_5 b_1\|x_1 - x_2\| + l_5 b_2\|x'_1 - x'_2\|]ds \\
& + \frac{1}{A_\lambda\Gamma(\alpha)} \int_{E^c} \|g\|_{[0,\lambda]} (\lambda-s)^{\alpha-1} [l_1\|x_1 - x_2\| \\
& + l_2\|x'_1 - x'_2\| + \frac{l_3\|x'_1 - x'_2\|}{\Gamma(2 - \beta)} + l_4 m\|x_1 - x_2\| + l_5 b_1\|x_1 - x_2\| \\
& + l_5 b_2\|x'_1 - x'_2\|]ds \\
& + \frac{\int_0^\lambda g(\xi)d\xi}{A_\lambda\Gamma(\alpha)} \int_{E^c} (1-s)^{\alpha-1} [l_1\|x_1 - x_2\| + l_2\|x'_1 - x'_2\| + \frac{l_3\|x'_1 - x'_2\|}{\Gamma(2 - \beta)} \\
& + l_4 m\|x_1 - x_2\| + l_5 b_1\|x_1 - x_2\| + l_5 b_2\|x'_1 - x'_2\|]ds \\
& + \frac{1}{A_\lambda\Gamma(\alpha-1)} \int_{E^c} (1-s)^{\alpha-2} [l_1\|x_1 - x_2\| + l_2\|x'_1 - x'_2\| \\
& + \frac{l_3\|x'_1 - x'_2\|}{\Gamma(2 - \beta)} + l_4 m\|x_1 - x_2\| + l_5 b_1\|x_1 - x_2\| + l_5 b_2\|x'_1 - x'_2\|]ds \\
& \leq \frac{1}{\Gamma(\alpha-1)} \sum_{i=1}^{k_0} \Lambda_i(\Delta\|x_1 - x_2\|_*, \dots, \Delta\|x_1 - x_2\|_*) \int_E (1-s)^{\alpha-2} a_i(s)ds \\
& + \frac{\|g\|_{[0,\lambda]}}{A_\lambda\Gamma(\alpha)} \sum_{i=1}^{k_0} \Lambda_i(\Delta\|x_1 - x_2\|_*, \dots, \Delta\|x_1 - x_2\|_*) \times \\
& \int_E (1-s)^{\alpha-1} a_i(s)ds
\end{aligned}$$

$$\begin{aligned}
& + \frac{\int_0^\lambda g(\xi) d\xi}{A_\lambda \Gamma(\alpha)} \sum_{i=1}^{k_0} \Lambda_i(\Delta \|x_1 - x_2\|_*, \dots, \Delta \|x_1 - x_2\|_*) \times \\
& \quad \int_E (1-s)^{\alpha-1} a_i(s) ds \\
& + \frac{1}{A_\lambda \Gamma(\alpha-1)} \sum_{i=1}^{k_0} \Lambda_i(\Delta \|x_1 - x_2\|_*, \dots, \Delta \|x_1 - x_2\|_*) \times \\
& \quad \int_E (1-s)^{\alpha-2} a_i(s) ds \\
& + \frac{1}{\Gamma(\alpha-1)} \|x_1 - x_2\|_* [l_1 + l_2 + \frac{l_3}{\Gamma(2-\beta)} + l_4 m + l_5 b_1 + l_5 b_2] \times \\
& \quad \int_{E^c} (1-s)^{\alpha-2} ds \\
& + \frac{\|g\|_{[0,\lambda]}}{A_\lambda \Gamma(\alpha)} \|x_1 - x_2\|_* [l_1 + l_2 + \frac{l_3}{\Gamma(2-\beta)} + l_4 m + l_5 b_1 + l_5 b_2] \times \\
& \quad \int_{E^c} (1-s)^{\alpha-1} ds \\
& + \frac{\int_0^\lambda g(\xi) d\xi}{A_\lambda \Gamma(\alpha)} \|x_1 - x_2\|_* [l_1 + l_2 + \frac{l_3}{\Gamma(2-\beta)} + l_4 m + l_5 b_1 + l_5 b_2] \times \\
& \quad \int_{E^c} (1-s)^{\alpha-1} ds \\
& + \frac{1}{A_\lambda \Gamma(\alpha-1)} \|x_1 - x_2\|_* [l_1 + l_2 + \frac{l_3}{\Gamma(2-\beta)} + l_4 m + l_5 b_1 + l_5 b_2] \times \\
& \quad \int_{E^c} (1-s)^{\alpha-2} ds \\
& \leq \frac{1}{\Gamma(\alpha-1)} \sum_{i=1}^{k_0} \Lambda_i(\Delta \delta, \dots, \Delta \delta) \|\hat{a}_i\|_E \\
& \quad + \frac{\|g\|_{[0,\lambda]}}{A_\lambda \Gamma(\alpha)} \sum_{i=1}^{k_0} \Lambda_i(\Delta \delta, \dots, \Delta \delta) \|\hat{a}_i\|_E \\
& + \frac{\int_0^\lambda g(\xi) d\xi}{A_\lambda \Gamma(\alpha)} \sum_{i=1}^{k_0} \Lambda_i(\Delta \delta, \dots, \Delta \delta) \|\hat{a}_i\|_E
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{A_\lambda \Gamma(\alpha - 1)} \sum_{i=1}^{k_0} \Lambda_i(\Delta\delta, \dots, \Delta\delta) \|\hat{a}_i\|_E \\
& + \frac{1}{A_\lambda \Gamma(\alpha - 1)} \delta[l_1 + l_2 + \frac{l_3}{\Gamma(2 - \beta)} + l_4 m + l_5 b_1 + l_5 b_2] \frac{M(E^c)}{\alpha - 1} \\
& + \frac{\|g\|_{[0, \lambda]}}{A_\lambda \Gamma(\alpha)} \delta[l_1 + l_2 + \frac{l_3}{\Gamma(2 - \beta)} + l_4 m + l_5 b_1 + l_5 b_2] \frac{M(E^c)}{\alpha} \\
& + \frac{\int_0^\lambda g(\xi) d\xi}{A_\lambda \Gamma(\alpha)} \delta[l_1 + l_2 + \frac{l_3}{\Gamma(2 - \beta)} + l_4 m + l_5 b_1 + l_5 b_2] \frac{M(E^c)}{\alpha} \\
& + \frac{1}{A_\lambda \Gamma(\alpha - 1)} \delta[l_1 + l_2 + \frac{l_3}{\Gamma(2 - \beta)} + l_4 m + l_5 b_1 + l_5 b_2] \frac{M(E^c)}{\alpha - 1} \\
& \leq \frac{1}{\Gamma(\alpha - 1)} \sum_{i=1}^{k_0} (q_i + \epsilon) \Delta\epsilon \|\hat{a}_i\|_E + \frac{\|g\|_{[0, \lambda]}}{A_\lambda \Gamma(\alpha)} \sum_{i=1}^{k_0} (q_i + \epsilon) \Delta\epsilon \|\hat{a}_i\|_E \\
& + \frac{\|g\|_{[0, \lambda]}}{A_\lambda \Gamma(\alpha)} \sum_{i=1}^{k_0} (q_i + \epsilon) \Delta\epsilon \|\hat{a}_i\|_E + \frac{1}{A_\lambda \Gamma(\alpha - 1)} \sum_{i=1}^{k_0} (q_i + \epsilon) \Delta\epsilon \|\hat{a}_i\|_E \\
& + \frac{[l_1 + l_2 + \frac{l_3}{\Gamma(2 - \beta)} + l_4 m + l_5 b_1 + l_5 b_2]}{A_\lambda \Gamma(\alpha)} M(E^c) \epsilon \\
& + \frac{\|g\|_{[0, \lambda]} [l_1 + l_2 + \frac{l_3}{\Gamma(2 - \beta)} + l_4 m + l_5 b_1 + l_5 b_2]}{A_\lambda \Gamma(\alpha + 1)} M(E^c) \epsilon \\
& + \frac{\|g\|_{[0, \lambda]} [l_1 + l_2 + \frac{l_3}{\Gamma(2 - \beta)} + l_4 m + l_5 b_1 + l_5 b_2]}{A_\lambda \Gamma(\alpha + 1)} M(E^c) \epsilon \\
& + \frac{[l_1 + l_2 + \frac{l_3}{\Gamma(2 - \beta)} + l_4 m + l_5 b_1 + l_5 b_2]}{A_\lambda \Gamma(\alpha)} M(E^c) \epsilon \\
& = [(\sum_{i=1}^{k_0} (q_i + \epsilon) \Delta\epsilon \|\hat{a}_i\|_E) (\frac{1}{\Gamma(\alpha - 1)} + \frac{2\|g\|_{[0, \lambda]}}{A_\lambda \Gamma(\alpha)} + \frac{1}{A_\lambda \Gamma(\alpha - 1)}) \\
& + M(E^c) (l_1 + l_2 + \frac{l_3}{\Gamma(2 - \beta)} + l_4 m + l_5 b_1 + l_5 b_2) \times \\
& (\frac{1}{\Gamma(\alpha)} + \frac{2\|g\|_{[0, \lambda]}}{A_\lambda \Gamma(\alpha + 1)} + \frac{1}{A_\lambda \Gamma(\alpha)})] \epsilon.
\end{aligned}$$

So

$$\begin{aligned}
& \|F'_{x_1} - F'_{x_2}\| \\
& \leq [(\sum_{i=1}^{k_0} (q_i + \epsilon) \Delta \epsilon \|\hat{a}_i\|_E) (\frac{1}{\Gamma(\alpha-1)} + \frac{2\|g\|_{[0,\lambda]}}{A_\lambda \Gamma(\alpha)} + \frac{1}{A_\lambda \Gamma(\alpha-1)}) \\
& + M(E^c)(l_1 + l_2 + \frac{l_3}{\Gamma(2-\beta)} + l_4 m + l_5 b_1 + l_5 b_2) \times \\
& (\frac{1}{\Gamma(\alpha)} + \frac{2\|g\|_{[0,\lambda]}}{A_\lambda \Gamma(\alpha+1)} + \frac{1}{A_\lambda \Gamma(\alpha)})] \epsilon.
\end{aligned}$$

Hence

$$\|F_{x_1} - F_{x_2}\|_* \leq Q\epsilon$$

where

$$\begin{aligned}
Q &= \max\{[\sum_{i=1}^{k_0} (q_i + \epsilon) \Delta \|\hat{a}_i\|_E] [\frac{2}{\Gamma(\alpha)} + \frac{\|g\|_{[0,\lambda]}}{A_\lambda \Gamma(\alpha)} + \frac{1}{A_\lambda \Gamma(\alpha-1)}] \\
& + M(E^c)[l_1 + l_2 + \frac{l_3}{\Gamma(2-\beta)} + l_4 m + l_5(b_1 + b_2)] \times \\
& [\frac{2}{\Gamma(\alpha+1)} + \frac{\|g\|_{[0,\lambda]}}{A_\lambda \Gamma(\alpha+1)} + \frac{1}{A_\lambda \Gamma(\alpha)}], \\
& (\sum_{i=1}^{k_0} (q_i + \epsilon) \Delta \|\hat{a}_i\|_E) (\frac{1}{\Gamma(\alpha-1)} + \frac{2\|g\|_{[0,\lambda]}}{A_\lambda \Gamma(\alpha)} + \frac{1}{A_\lambda \Gamma(\alpha-1)}) \\
& + M(E^c)(l_1 + l_2 + \frac{l_3}{\Gamma(2-\beta)} + l_4 m + l_5 b_1 + l_5 b_2) \times \\
& (\frac{1}{\Gamma(\alpha)} + \frac{2\|g\|_{[0,\lambda]}}{A_\lambda \Gamma(\alpha+1)} + \frac{1}{A_\lambda \Gamma(\alpha)})\} < \infty.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary then $\|F_{x_1} - F_{x_2}\|_* \rightarrow 0$ as $x_1 \rightarrow x_2$ in X , this shows that F is continuous in X . Since $\lim_{z \rightarrow 0^+} \frac{\Lambda_i(\Delta z, \dots, \Delta z)}{\Delta z} = q_i$, so for $\epsilon > 0$ there is $\delta > 0$ such that $z \in (0, \delta]$ implies that

$\Lambda_i(\Delta z, \dots, \Delta z) < (q_i + \epsilon)\Delta z$, for all $1 \leq i \leq k_0$. Also since

$$\begin{aligned} & \max\left\{\left[\sum_{i=1}^{k_0} q_i \Delta \|\hat{a}_i\|_E\right] \left[\frac{2}{\Gamma(\alpha)} + \frac{\|g\|_{[0,\lambda]}}{A_\lambda \Gamma(\alpha)} + \frac{1}{A_\lambda \Gamma(\alpha-1)}\right] \right. \\ & + M(E^c) \left[l_1 + l_2 + \frac{l_3}{\Gamma(2-\beta)} + l_4 m + l_5(b_1 + b_2)\right] \times \\ & \left. \left[\frac{2}{\Gamma(\alpha+1)} + \frac{\|g\|_{[0,\lambda]}}{A_\lambda \Gamma(\alpha+1)} + \frac{1}{A_\lambda \Gamma(\alpha)}\right], \right. \\ & \left. \left(\sum_{i=1}^{k_0} q_i \Delta \|\hat{a}_i\|_E\right) \left(\frac{1}{\Gamma(\alpha-1)} + \frac{2\|g\|_{[0,\lambda]}}{A_\lambda \Gamma(\alpha)} + \frac{1}{A_\lambda \Gamma(\alpha-1)}\right) \right. \\ & + M(E^c) \left(l_1 + l_2 + \frac{l_3}{\Gamma(2-\beta)} + l_4 m + l_5 b_1 + l_5 b_2\right) \times \\ & \left. \left(\frac{1}{\Gamma(\alpha)} + \frac{2\|g\|_{[0,\lambda]}}{A_\lambda \Gamma(\alpha+1)} + \frac{1}{A_\lambda \Gamma(\alpha)}\right)\right\} < 1, \end{aligned}$$

there is $\epsilon_0 > 0$ such that

$$\begin{aligned} & \max\left\{\left[\sum_{i=1}^{k_0} (q_i + \epsilon_0) \Delta \|\hat{a}_i\|_E\right] \left[\frac{2}{\Gamma(\alpha)} + \frac{\|g\|_{[0,\lambda]}}{A_\lambda \Gamma(\alpha)} + \frac{1}{A_\lambda \Gamma(\alpha-1)}\right] \right. \\ & + M(E^c) \left[l_1 + l_2 + \frac{l_3}{\Gamma(2-\beta)} + l_4 m + l_5(b_1 + b_2)\right] \times \\ & \left. \left[\frac{2}{\Gamma(\alpha+1)} + \frac{\|g\|_{[0,\lambda]}}{A_\lambda \Gamma(\alpha+1)} + \frac{1}{A_\lambda \Gamma(\alpha)}\right], \right. \\ & \left. \left(\sum_{i=1}^{k_0} (q_i + \epsilon_0) \Delta \|\hat{a}_i\|_E\right) \left(\frac{1}{\Gamma(\alpha-1)} + \frac{2\|g\|_{[0,\lambda]}}{A_\lambda \Gamma(\alpha)} + \frac{1}{A_\lambda \Gamma(\alpha-1)}\right) \right. \\ & + M(E^c) \left(l_1 + l_2 + \frac{l_3}{\Gamma(2-\beta)} + l_4 m + l_5 b_1 + l_5 b_2\right) \times \\ & \left. \left(\frac{1}{\Gamma(\alpha)} + \frac{2\|g\|_{[0,\lambda]}}{A_\lambda \Gamma(\alpha+1)} + \frac{1}{A_\lambda \Gamma(\alpha)}\right)\right\} < 1. \end{aligned}$$

Let $\delta_0 = \delta(\epsilon_0)$, $R = \min\{\delta_0, 1\}$, $C = \{x \in X : \|x\|_* \leq R\}$ and define $\alpha : X^2 \rightarrow [0, \infty)$ as $\alpha(x, y) = 1$ for $x, y \in X$, other wise let $\alpha(x, y) = 0$, so for all $1 \leq i \leq k_0$ $\Lambda_i(\Delta \|x\|_*, \dots, \Delta \|x\|_*) < (q_i + \epsilon_0)\Delta R$. Let $x \in C$, then for all $t \in [0, 1]$ we have

$$\begin{aligned}
& |F_x(t)| \\
\leq & \left| \int_0^1 G(t,s) f(s, x(t), x'(t), D^\beta x(t), \int_0^s h(\xi)x(\xi)d\xi, \phi x(s)) ds \right| \\
\leq & \int_0^1 |G(t,s)| |f(s, x(t), x'(t), D^\beta x(t), \int_0^s h(\xi)x(\xi)d\xi, \phi x(s))| ds \\
= & \int_0^1 |G(t,s)| |f(s, x(t), x'(t), D^\beta x(t), \int_0^s h(\xi)x(\xi)d\xi, \phi x(s)) \\
& - f(s, 0, \dots, 0)| ds \\
= & \int_E |G(t,s)| |f(s, x(t), x'(t), D^\beta x(t), \int_0^s h(\xi)x(\xi)d\xi, \phi x(s)) \\
& - f(s, 0, \dots, 0)| ds \\
+ & \int_{E^c} |G(t,s)| |f(s, x(t), x'(t), D^\beta x(t), \int_0^s h(\xi)x(\xi)d\xi, \phi x(s)) \\
& - f(s, 0, \dots, 0)| ds \\
\leq & \int_E |G(t,s)| \sum_{i=1}^{k_0} a_i(s) \Lambda_i(|x(s)|, |x'(s)|, |D^\beta x(s)|, \\
& |\int_0^s h(\xi)x(\xi)d\xi|, |\phi x(s) - \phi 0(s)|) ds \\
+ & \int_{E^c} |G(t,s)| [l_1|x(s)| + l_2|x'(s)| + l_3|D^\beta x(s)| \\
+ & l_4|\int_0^s h(\xi)x(\xi)d\xi| + l_5|\phi x(s) - \phi 0(s)|] ds \\
\leq & \frac{1}{\Gamma(\alpha)} \int_{E \cap [0,t]} (t-s)^{\alpha-1} \sum_{i=1}^{k_0} a_i(s) \Lambda_i(|x(s)|, |x'(s)|, |D^\beta x(s)|, \\
& |\int_0^s h(\xi)x(\xi)d\xi|, |\phi x(s) - \phi 0(s)|) ds \\
+ & \frac{1-t}{A_\lambda \Gamma(\alpha)} \int_{E \cap [0,\lambda]} H(s) \sum_{i=1}^{k_0} a_i(s) \Lambda_i(|x(s)|, |x'(s)|, |D^\beta x(s)|, \\
& |\int_0^s h(\xi)x(\xi)d\xi|, |\phi x(s) - \phi 0(s)|) ds
\end{aligned}$$

$$\begin{aligned}
& + \frac{\int_0^\lambda |t - \xi| g(\xi) d\xi}{A_\lambda \Gamma(\alpha)} \int_E (1-s)^{\alpha-1} \sum_{i=1}^{k_0} a_i(s) \Lambda_i(|x(s)|, |x'(s)|, |D^\beta x(s)|, \\
& \quad | \int_0^s h(\xi) x(\xi) d\xi |, |\phi x(s) - \phi 0(s)|) ds \\
& + \frac{1-t}{A_\lambda \Gamma(\alpha-1)} \int_{E \cap [0, \mu]} (\mu-s)^{\alpha-2} \sum_{i=1}^{k_0} a_i(s) \Lambda_i(|x(s)|, |x'(s)|, |D^\beta x(s)|, \\
& \quad | \int_0^s h(\xi) x(\xi) d\xi |, |\phi x(s) - \phi 0(s)|) ds \\
& + \frac{1}{\Gamma(\alpha)} \int_{E^c \cap [0, t]} (t-s)^{\alpha-1} [l_1 |x(s)| + l_2 |x'(s)| + l_3 |D^\beta x(s)| \\
& + l_4 | \int_0^s h(\xi) x(\xi) d\xi | + l_5 |\phi x(s) - \phi 0(s)|] ds \\
& + \frac{1-t}{A_\lambda \Gamma(\alpha)} \int_{E^c \cap [0, \lambda]} H(s) [l_1 |x(s)| + l_2 |x'(s)| + l_3 |D^\beta x(s)| \\
& + l_4 | \int_0^s h(\xi) x(\xi) d\xi | + l_5 |\phi x(s) - \phi 0(s)|] ds \\
& + \frac{\int_0^\lambda |t - \xi| g(\xi) d\xi}{A_\lambda \Gamma(\alpha)} \int_{E^c} (1-s)^{\alpha-1} [l_1 |x(s)| + l_2 |x'(s)| + l_3 |D^\beta x(s)| \\
& + l_4 | \int_0^s h(\xi) x(\xi) d\xi | + l_5 |\phi x(s) - \phi 0(s)|] ds \\
& + \frac{1-t}{A_\lambda \Gamma(\alpha-1)} \int_{E^c \cap [0, \mu]} (\mu-s)^{\alpha-2} [l_1 |x(s)| + l_2 |x'(s)| + l_3 |D^\beta x(s)| \\
& + l_4 | \int_0^s h(\xi) x(\xi) d\xi | + l_5 |\phi x(s) - \phi 0(s)|] ds \\
& \leq \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k_0} \int_{E \cap [0, t]} (t-s)^{\alpha-1} a_i(s) \Lambda_i(\|x\|, \|x'\|, \frac{\|x'\|}{\Gamma(2-\beta)}, \\
& \quad m\|x\|, b_1\|x\| + b_2\|x'\|) ds \\
& + \frac{1-t}{A_\lambda \Gamma(\alpha)} \sum_{i=1}^{k_0} \int_{E \cap [0, \lambda]} H(s) a_i(s) \Lambda_i(\|x\|, \|x'\|, \frac{\|x'\|}{\Gamma(2-\beta)}, \\
& \quad m\|x\|, b_1\|x\| + b_2\|x'\|) ds
\end{aligned}$$

$$\begin{aligned}
& + \frac{\int_0^\lambda |t - \xi| g(\xi) d\xi}{A_\lambda \Gamma(\alpha)} \sum_{i=1}^{k_0} \int_E (1-s)^{\alpha-1} a_i(s) \Lambda_i(\|x\|, \|x'\|, \frac{\|x'\|}{\Gamma(2-\beta)}, \\
& \quad m\|x\|, b_1\|x\| + b_2\|x'\|) ds \\
& + \frac{1-t}{A_\lambda \Gamma(\alpha-1)} \sum_{i=1}^{k_0} \int_{E \cap [0, \mu]} (\mu-s)^{\alpha-2} a_i(s) \Lambda_i(\|x\|, \|x'\|, \frac{\|x'\|}{\Gamma(2-\beta)}, \\
& \quad m\|x\|, b_1\|x\| + b_2\|x'\|) ds \\
& + \frac{1}{\Gamma(\alpha)} \int_{E^c \cap [0, t]} (t-s)^{\alpha-1} [l_1\|x\| + l_2\|x'\| + \frac{l_3\|x'\|}{\Gamma(2-\beta)} \\
& \quad + l_4 m\|x\| + l_5 b_1\|x\| + l_5 b_2\|x'\|] ds \\
& + \frac{1-t}{A_\lambda \Gamma(\alpha)} \int_{E^c \cap [0, \lambda]} H(s) [l_1\|x\| + l_2\|x'\| + \frac{l_3\|x'\|}{\Gamma(2-\beta)} \\
& \quad + l_4 m\|x\| + l_5 b_1\|x\| + l_5 b_2\|x'\|] ds \\
& + \frac{\int_0^\lambda |t - \xi| g(\xi) d\xi}{A_\lambda \Gamma(\alpha)} \int_{E^c} (1-s)^{\alpha-1} [l_1\|x\| + l_2\|x'\| + \frac{l_3\|x'\|}{\Gamma(2-\beta)} \\
& \quad + l_4 m\|x\| + l_5 b_1\|x\| + l_5 b_2\|x'\|] ds \\
& + \frac{1-t}{A_\lambda \Gamma(\alpha-1)} \int_{E^c \cap [0, \mu]} (\mu-s)^{\alpha-2} [l_1\|x\| + l_2\|x'\| + \frac{l_3\|x'\|}{\Gamma(2-\beta)} \\
& \quad + l_4 m\|x\| + l_5 b_1\|x\| + l_5 b_2\|x'\|] ds \\
& \leq \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k_0} \int_E (1-s)^{\alpha-1} a_i(s) \Lambda_i(\Delta\|x\|_*, \dots, \Delta\|x\|_*) ds \\
& + \frac{1-t}{A_\lambda \Gamma(\alpha)} \sum_{i=1}^{k_0} \int_E \|g\|_{[0, \lambda]} (\lambda-s)^{\alpha-1} a_i(s) \Lambda_i(\Delta\|x\|_*, \dots, \Delta\|x\|_*) ds \\
& + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k_0} \int_E (1-s)^{\alpha-1} a_i(s) \Lambda_i(\Delta\|x\|_*, \dots, \Delta\|x\|_*) ds \\
& + \frac{1-t}{A_\lambda \Gamma(\alpha-1)} \sum_{i=1}^{k_0} \int_E (1-s)^{\alpha-2} a_i(s) \Lambda_i(\Delta\|x\|_*, \dots, \Delta\|x\|_*) ds \\
& + \frac{1}{\Gamma(\alpha)} \int_{E^c} (1-s)^{\alpha-1} \|x\|_* [l_1 + l_2 + \frac{l_3}{\Gamma(2-\beta)} + l_4 m + l_5 b_1 + l_5 b_2] ds
\end{aligned}$$

$$\begin{aligned}
& + \frac{1-t}{A_\lambda \Gamma(\alpha)} \int_{E^c} \|g\|_{[0,\lambda]} (\lambda-s)^{\alpha-1} \|x\|_* [l_1 + l_2 + \frac{l_3}{\Gamma(2-\beta)} \\
& \quad + l_4 m + l_5 b_1 + l_5 b_2] ds \\
& + \frac{1}{\Gamma(\alpha)} \int_{E^c} (1-s)^{\alpha-1} \|x\|_* [l_1 + l_2 + \frac{l_3}{\Gamma(2-\beta)} + l_4 m + l_5 b_1 + l_5 b_2] ds \\
& + \frac{1-t}{A_\lambda \Gamma(\alpha-1)} \int_{E^c} (1-s)^{\alpha-2} \|x\|_* [l_1 + l_2 + \frac{l_3}{\Gamma(2-\beta)} \\
& \quad + l_4 m + l_5 b_1 + l_5 b_2] ds \\
& \leq \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k_0} (q_i + \epsilon_0) \Delta R \|\hat{a}_i\|_E + \frac{(1-t)\|g\|_{[0,\lambda]}}{A_\lambda \Gamma(\alpha)} \sum_{i=1}^{k_0} (q_i + \epsilon_0) \Delta R \|\hat{a}_i\|_E \\
& + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k_0} (q_i + \epsilon_0) \Delta R \|\hat{a}_i\|_E + \frac{1-t}{A_\lambda \Gamma(\alpha-1)} \sum_{i=1}^{k_0} (q_i + \epsilon_0) \Delta R \|\hat{a}_i\|_E \\
& + \frac{1}{\Gamma(\alpha+1)} [l_1 + l_2 + \frac{l_3}{\Gamma(2-\beta)} + l_4 m + l_5 (b_1 + b_2)] M(E^c) R \\
& + \frac{(1-t)\|g\|_{[0,\lambda]}}{A_\lambda \Gamma(\alpha+1)} [l_1 + l_2 + \frac{l_3}{\Gamma(2-\beta)} + l_4 m + l_5 (b_1 + b_2)] M(E^c) R \\
& + \frac{1}{\Gamma(\alpha+1)} [l_1 + l_2 + \frac{l_3}{\Gamma(2-\beta)} + l_4 m + l_5 (b_1 + b_2)] M(E^c) R \\
& + \frac{1-t}{A_\lambda \Gamma(\alpha)} [l_1 + l_2 + \frac{l_3}{\Gamma(2-\beta)} + l_4 m + l_5 (b_1 + b_2)] M(E^c) R.
\end{aligned}$$

Hence

$$\begin{aligned}
\|F_x\| & \leq ((q_i + \epsilon_0) \Delta \|\hat{a}_i\|_E (\frac{2}{\Gamma(\alpha)} + \frac{\|g\|_{[0,\lambda]}}{A_\lambda \Gamma(\alpha)} + \frac{1}{A_\lambda \Gamma(\alpha-1)}) \\
& \quad + M(E^c) [l_1 + l_2 + \frac{l_3}{\Gamma(2-\beta)} + l_4 m + l_5 (b_1 + b_2)] \times \\
& \quad (\frac{2}{\Gamma(\alpha+1)} + \frac{\|g\|_{[0,\lambda]}}{A_\lambda \Gamma(\alpha+1)} + \frac{1}{A_\lambda \Gamma(\alpha)})) R \leq R.
\end{aligned}$$

Using similar proof we conclude that

$$\begin{aligned}
\|F'_x\| &\leq \left(\left(\sum_{i=1}^{k_0} (q_i + \epsilon_0) \Delta \epsilon \|\hat{a}_i\|_E \right) \left(\frac{1}{\Gamma(\alpha-1)} + \frac{2\|g\|_{[0,\lambda]}}{A_\lambda \Gamma(\alpha)} + \frac{1}{A_\lambda \Gamma(\alpha-1)} \right) \right. \\
&\quad + \left. M(E^c) \left[l_1 + l_2 + \frac{l_3}{\Gamma(2-\beta)} + l_4 m + l_5(b_1 + b_2) \right] \times \right. \\
&\quad \left. \left(\frac{1}{\Gamma(\alpha)} + \frac{2\|g\|_{[0,\lambda]}}{A_\lambda \Gamma(\alpha+1)} + \frac{1}{A_\lambda \Gamma(\alpha)} \right) \right) R \leq R.
\end{aligned}$$

Therefore $\|F_x\|_* = \max\{\|F_x\|, \|F'_x\|\} \leq R$, so $\|F'_x\| \leq r$ so $\|F_x\|_* \leq r$ therefore $F_x \in C$. By similar manner we conclude that $F_y \in C$, hence $\alpha(F_x, F_y) \geq 1$, so F is α -admissible. It's obvious that $C \neq \phi$, hence there exists $x_0 \in C$ such that $F_{x_0} \in C$ and therefore $\alpha(x_0, F_{x_0}) \geq 1$. Let

$$\begin{aligned}
\gamma &:= \max \left\{ \left[\sum_{i=1}^{k_0} (q_i + \epsilon) \Delta \|\hat{a}_i\|_E \right] \left[\frac{2}{\Gamma(\alpha)} + \frac{\|g\|_{[0,\lambda]}}{A_\lambda \Gamma(\alpha)} + \frac{1}{A_\lambda \Gamma(\alpha-1)} \right] \right. \\
&\quad + \left. M(E^c) \left[l_1 + l_2 + \frac{l_3}{\Gamma(2-\beta)} + l_4 m + l_5(b_1 + b_2) \right] \times \right. \\
&\quad \left. \left[\frac{2}{\Gamma(\alpha+1)} + \frac{\|g\|_{[0,\lambda]}}{A_\lambda \Gamma(\alpha+1)} + \frac{1}{A_\lambda \Gamma(\alpha)} \right], \right. \\
&\quad \left(\sum_{i=1}^{k_0} (q_i + \epsilon) \Delta \|\hat{a}_i\|_E \right) \left(\frac{1}{\Gamma(\alpha-1)} + \frac{2\|g\|_{[0,\lambda]}}{A_\lambda \Gamma(\alpha)} + \frac{1}{A_\lambda \Gamma(\alpha-1)} \right) \\
&\quad + \left. M(E^c) \left(l_1 + l_2 + \frac{l_3}{\Gamma(2-\beta)} + l_4 m + l_5 b_1 + l_5 b_2 \right) \times \right. \\
&\quad \left. \left(\frac{1}{\Gamma(\alpha)} + \frac{2\|g\|_{[0,\lambda]}}{A_\lambda \Gamma(\alpha+1)} + \frac{1}{A_\lambda \Gamma(\alpha)} \right) \right\} < 1,
\end{aligned}$$

then define $\psi : [0, \infty) \rightarrow [0, \infty)$ as $\psi(t) = \gamma t$, then ψ is nondecreasing and

$$\sum_{n=1}^{\infty} \psi^n(t) = \frac{\gamma}{1-\gamma} < \infty,$$

so $\psi \in \Psi$. Also for $x, y \in C$ we have

$$\begin{aligned} \|F_x - F_y\| &\leq ([\sum_{i=1}^{k_0} (q_i + \epsilon) \Delta \|\hat{a}_i\|_E] [\frac{2}{\Gamma(\alpha)} + \frac{\|g\|_{[0,\lambda]}}{A_\lambda \Gamma(\alpha)} + \frac{1}{A_\lambda \Gamma(\alpha - 1)}] \\ &+ M(E^c)[l_1 + l_2 + \frac{l_3}{\Gamma(2 - \beta)} + l_4 m + l_5(b_1 + b_2)] \times \\ &[\frac{2}{\Gamma(\alpha + 1)} + \frac{\|g\|_{[0,\lambda]}}{A_\lambda \Gamma(\alpha + 1)} + \frac{1}{A_\lambda \Gamma(\alpha)}]) \|x - y\|_* \leq \gamma \|x - y\|_* \end{aligned}$$

and

$$\begin{aligned} \|F'_x - F'_y\| &\leq (\sum_{i=1}^{k_0} q_i \Delta \|\hat{a}_i\|_E) (\frac{1}{\Gamma(\alpha - 1)} + \frac{2\|g\|_{[0,\lambda]}}{A_\lambda \Gamma(\alpha)} + \frac{1}{A_\lambda \Gamma(\alpha - 1)}) \\ &+ M(E^c)(l_1 + l_2 + \frac{l_3}{\Gamma(2 - \beta)} + l_4 m + l_5 b_1 + l_5 b_2) \times \\ &(\frac{1}{\Gamma(\alpha)} + \frac{2\|g\|_{[0,\lambda]}}{A_\lambda \Gamma(\alpha + 1)} + \frac{1}{A_\lambda \Gamma(\alpha)}) \|x - y\|_* \\ &\leq \gamma \|x - y\|_* \end{aligned}$$

so

$$\|F_x - F_y\|_* \leq \gamma \|x - y\|_*.$$

Hence for all $x, y \in X$ we have

$$\alpha(x, y) \|F_x - F_y\|_* \leq \gamma \|x - y\|_* = \Psi(\|x - y\|_*).$$

Now, using lemma (1.3) we conclude that F has a fixed point in X which is a solution for the problem. $\square$

Example 2.3. Consider the problem

$$D^{\frac{5}{2}}x(t) + f(t, x(t), x'(t), D^{\frac{1}{2}}x(t), \int_0^t \xi x(\xi) d\xi, D^{\frac{1}{3}}x(t)) = 0, \quad (3)$$

with boundary conditions $x'(\frac{1}{3}) = x'(0) + \int_0^{\frac{1}{2}} sx(s) ds$ and $x(1) = x^{(j)}(0) = 0$ for $2 \leq j \leq 3$, where

$$f(t, x_1, \dots, x_5) = \begin{cases} \frac{1}{10\sqrt{1-tp(t)}} \sum_{i=1}^5 |x_i| & t \in E := [0, \frac{1}{3}] \\ \frac{1}{20} (1-t) \sum_{i=1}^5 |x_i| & t \in E^c := (\frac{1}{3}, 1] \end{cases}$$

and $p(t) = 0$ whenever $t \in E \cap \mathcal{Q}$ and $p(t) = 1$ whenever $t \in E \cap \mathcal{Q}^c$. Put $\alpha = \frac{5}{2}$, $\beta = \frac{1}{2}$, $\Lambda_i(x_1, \dots, x_5) = \frac{1}{10} \sum_{i=1}^5 |x_i|$, $a_i(t) = \frac{1}{\sqrt{1-tp(t)}}$, $g(t) = h(t) = t$, $\lambda = \frac{1}{2}$, $\mu = \frac{1}{2}$, $m = \frac{1}{2}$, $\phi x(t) = D^{\frac{1}{3}}x(t)$, $b_1 = 0$, $b_2 = \frac{1}{\Gamma(\frac{5}{3})}$, $q_i = 5$, $l_1 = \dots = l_5 = \frac{2}{60}$ and $k_0 = 1$ for all $1 \leq i \leq k_0$. Note that for $t \in E^c$

$$|f(t, x_1, \dots, x_5) - f(t, y_1, \dots, y_5)| = \frac{1}{20} (1-t) |\sum_{i=1}^5 |x_i| - |y_i|| \leq \frac{2}{60} \sum_{i=1}^5 |x_i - y_i|,$$

and for $t \in E$ we have

$$\begin{aligned} |f(t, x_1, \dots, x_5) - f(t, y_1, \dots, y_5)| &= \frac{1}{\sqrt{1-tp(t)}} |\sum_{i=1}^5 |x_i| - |y_i|| \\ &\leq \frac{1}{10\sqrt{1-tp(t)}} |x_i - y_i| = a_i(t) \sum_{i=1}^{k_0} \Lambda_i(|x_i - y_i|), \end{aligned}$$

$$\begin{aligned} \lim_{z \rightarrow 0^+} \frac{\Lambda_i(z, z, z, z, z)}{z} &= \frac{1}{2} = q_i, \quad (1-t)^{\alpha-2} a_i(t) \in L^1(E), \quad \|\hat{a}_i\| = \frac{1}{3}, \\ \|g\|_{[0, \lambda]} &= \frac{1}{18}, \quad \text{for all } 1 \leq i \leq k_0, \quad |\phi x(t) - \phi y(t)| \leq D^{\frac{1}{3}} |x(t) - y(t)| \leq \\ &\frac{1}{\Gamma(\frac{5}{3})} |x(t) - y(t)|, \quad \int_0^1 h(t) dt = \frac{1}{2} = m, \end{aligned}$$

$$\Delta = \max\{1, \frac{1}{\Gamma(2-\beta)}, m, b_1 + b_2\} = \max\{1, \frac{1}{\Gamma(\frac{3}{2})}, \frac{1}{2}, \frac{1}{\Gamma(\frac{5}{3})}\} = \frac{2}{\sqrt{\pi}},$$

$$A_\lambda = \int_0^\lambda (1-t)g(t)dt = \int_0^{\frac{1}{2}} (1-t)t dt = \frac{1}{12},$$

$$M(E^c) = \frac{2}{3} \text{ and}$$

$$\begin{aligned}
& \max\{[\sum_{i=1}^{k_0}(q_i + \epsilon)\Delta\|\hat{a}_i\|_E][\frac{2}{\Gamma(\alpha)} + \frac{\|g\|_{[0,\lambda]}}{A_\lambda\Gamma(\alpha)} + \frac{1}{A_\lambda\Gamma(\alpha-1)}] \\
& + M(E^c)[l_1 + l_2 + \frac{l_3}{\Gamma(2-\beta)} + l_4m + l_5(b_1 + b_2)] \times \\
& [\frac{2}{\Gamma(\alpha+1)} + \frac{\|g\|_{[0,\lambda]}}{A_\lambda\Gamma(\alpha+1)} + \frac{1}{A_\lambda\Gamma(\alpha)}], \\
& (\sum_{i=1}^{k_0}(q_i + \epsilon)\Delta\|\hat{a}_i\|_E)(\frac{1}{\Gamma(\alpha-1)} + \frac{2\|g\|_{[0,\lambda]}}{A_\lambda\Gamma(\alpha)} + \frac{1}{A_\lambda\Gamma(\alpha-1)}) \\
& + M(E^c)(l_1 + l_2 + \frac{l_3}{\Gamma(2-\beta)} + l_4m + l_5b_1 + l_5b_2) \times \\
& (\frac{1}{\Gamma(\alpha)} + \frac{2\|g\|_{[0,\lambda]}}{A_\lambda\Gamma(\alpha+1)} + \frac{1}{A_\lambda\Gamma(\alpha)})\} \\
& \leq \max\{(\frac{1}{2} \cdot \frac{2}{\sqrt{\pi}} \cdot \frac{1}{3})[\frac{2}{\Gamma(\frac{5}{2})} + \frac{\frac{1}{18}}{\frac{1}{2}\Gamma(\frac{5}{2})} + \frac{1}{\frac{1}{2}\Gamma(\frac{3}{2})}] \\
& + \frac{2}{3}[\frac{2}{60} + \frac{2}{60} + \frac{\frac{2}{60}}{\Gamma(\frac{3}{2})} + \frac{2}{60} \cdot \frac{1}{2} + \frac{2}{60} \cdot \frac{1}{\Gamma(\frac{5}{3})}][\frac{2}{\Gamma(\frac{7}{2})} + \frac{\frac{1}{18}}{\frac{1}{2}\Gamma(\frac{7}{2})} + \frac{1}{\frac{1}{2}\Gamma(\frac{5}{2})}], \\
& (\frac{1}{2} \cdot \frac{2}{\sqrt{\pi}} \cdot \frac{1}{3})(\frac{1}{\Gamma(\frac{3}{2})} + \frac{2 \cdot \frac{1}{18}}{\frac{1}{2}\Gamma(\frac{5}{2})} + \frac{1}{\frac{1}{2}\Gamma(\frac{3}{2})}) \\
& + \frac{2}{3}[\frac{2}{60} + \frac{2}{60} + \frac{\frac{2}{60}}{\Gamma(\frac{3}{2})} + \frac{2}{60} \cdot \frac{1}{2} + \frac{2}{60} \cdot \frac{1}{\Gamma(\frac{5}{3})}] \times \\
& (\frac{1}{\Gamma(\frac{5}{2})} + \frac{2 \cdot \frac{1}{18}}{\frac{1}{2}\Gamma(\frac{7}{2})} + \frac{1}{\frac{1}{2}\Gamma(\frac{5}{2})})\} < 1.
\end{aligned}$$

Now by using Theorem (2.2), problem (3) has a solution.

Theorem 2.4. Let $\alpha \geq 2$, $\beta, \mu, \lambda \in (0, 1)$ $g \in L^1[0, \lambda]$, $g(t) > 0$ for a.e. $t \in [0, \lambda]$, $\phi : [0, 1] \rightarrow R^+$ is such that for all $x, y \in C^1[0, 1]$, $|\phi x(t) - \phi y(t)| \leq b_1|x(t) - y(t)| + b_2|x'(t) - y'(t)|$ for some $b_1, b_2 \in [0, \infty)$, $h \in L^1[0, 1]$ and $f : [0, 1] \times X^5 \rightarrow \mathbb{R}$ be a mapping which is singular on

some set $E \subset [0, 1]$ such that for $t \in E$ we have

$$|f(t, x_1, x_2, \dots, x_5) - f(t, y_1, y_2, \dots, y_5)| \leq \sum_{i=1}^5 a_i(t) \Lambda_i(|x_i - y_i|)$$

and is continuous on $E^c \subset [0, 1]$ for all $x_1, \dots, x_5, y_1, \dots, y_5 \in X$, where $a_i : [0, 1] \rightarrow \mathbb{R}^+$, $\hat{a}_i \in L^1(E)$, $\hat{a}_i(s) = (1-s)^{\alpha-2} a_i(s)$, $\Lambda_i : X \rightarrow \mathbb{R}^+$ is a nondecreasing mapping such that $\lim_{z \rightarrow 0^+} \frac{\Lambda_i(z)}{z^{\gamma_i}} = q_i$ for some $\gamma_i, q_i \in [0, \infty)$, and all $1 \leq i \leq 5$. Also let $q > 0$, $\Delta = \max\{1, \frac{2}{\Gamma(2-\beta)}, m, b_1 + b_2\}$ and for almost all $t \in [0, 1]$ and $(x_1, \dots, x_5) \in X^5$ we have

$$|f(t, x_1, x_2, \dots, x_5)| \leq b(t)L(x_1, x_2, \dots, x_5) + K(x_1, x_2, \dots, x_5)$$

where $b : [0, 1] \rightarrow \mathbb{R}^+$, $L, K : \mathbb{R}^5 \rightarrow [0, \infty)$ are such that $(1-t)^{\alpha-2}b(t) \in L^1[0, 1]$, L, K are such that

$$\lim_{z \rightarrow \infty} \frac{L(z, z, z, z, z)}{z} = q$$

and

$$\lim_{z \rightarrow \infty} K(z, z, z, z, z) < \infty.$$

If

$$\max\left\{\frac{2}{\Gamma(\alpha)} + \frac{\|g\|_{[0, \lambda]}}{A_\lambda \Gamma(\alpha)} + \frac{1}{A_\lambda \Gamma(\alpha - 1)}, \frac{1}{\Gamma(\alpha - 1)} + \frac{2\|g\|_{[0, \lambda]}}{A_\lambda \Gamma(\alpha)} + \frac{1}{A_\lambda \Gamma(\alpha - 1)}\right\} q \|\hat{b}\|_{[0, 1]} \in [0, \frac{1}{\Delta}),$$

then

$$D^\alpha x(t) + f(t, x(t), x'(t), D^\beta x(t), \int_0^t h(s)x(s)ds, \phi x(t)) = 0$$

with boundary conditions $x'(\mu) = x'(0) + \int_0^\lambda g(s)x(s)ds$ and $x(1) = x^{(j)}(0) = 0$ for $j \geq 2$ has a solution.

Proof. Define $F_1, F_2 : X \rightarrow \mathbb{R}$ as

$$F_1 x(t) = \int_{E^c} G(t, s) f(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)d\xi, \phi x(s))ds \text{ and}$$

$F_2x(t) = \int_E G(t, s)f(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)d\xi, \phi x(s))ds$,
so $Fx = F_1X + F_2x$. Now to show that F is continuous, we will prove
that F_1, F_2 are continuous. Let $\epsilon > 0$ be arbitrary and $t \in E^c$ be fixed
for a moment, then there exists $\delta > 0$ such that
 $\sqrt{(x_1 - y_1)^2 + \dots + (x_5 - y_5)^2} < \delta$ implies that

$$|f(t, x_1, \dots, x_5) - f(t, y_1, \dots, y_5)| < \epsilon. \quad (4)$$

Now let $\{x_n\}$ be a sequence such that $x_n \rightarrow x_0$ in X for some $x_0 \in X$,
then there exists $n_0 \in \mathbb{N}$ such that $n \geq n_0$ implies that $\|x_n - x_0\|_* < \frac{\delta}{\Delta\sqrt{5}}$
where $\Delta = \max\{1, \frac{2}{\Gamma(2-\beta)}, m, b_1 + b_2\}$, then $\|x_n - x_0\| < \frac{\delta}{\Delta\sqrt{5}} \leq \frac{\delta}{\sqrt{5}}$ and
 $\|x'_n - x'_0\| < \frac{\delta}{\Delta\sqrt{5}} \leq \frac{\delta}{\sqrt{5}}$, hence for $t \in E^c$, $|x_n(t) - x_0(t)| < \frac{\delta}{\sqrt{5}}$ and
 $|x'_n(t) - x'_0(t)| < \frac{\delta}{\sqrt{5}}$, so if $n \geq n_0$, then since $\|D^\beta x_n - D^\beta x_0\| \leq \frac{\|x'_n - x'_0\|}{\Gamma(2-\beta)}$
we have

$$|D^\beta x_n(t) - D^\beta x_0(t)| \leq \frac{\frac{\delta}{\Delta\sqrt{5}}}{\Gamma(2-\beta)} \leq \frac{\delta}{\sqrt{5}}$$

and

$$\begin{aligned} \left| \int_0^t h(\xi)x_n(\xi)d\xi - \int_0^t h(\xi)x_0(\xi)d\xi \right| &\leq \int_0^t |h(\xi)||x_n(\xi) - x_0(\xi)|d\xi \\ &< \frac{\delta}{\Delta\sqrt{5}} \int_0^t |h(\xi)|d\xi = m \frac{\delta}{\Delta\sqrt{5}} \leq \frac{\delta}{\sqrt{5}}. \end{aligned}$$

Also we have

$$\begin{aligned} |\phi x_n(t) - \phi x_0(t)| &\leq b_1|x_n(t) - x_0(t)| + b_2|x'_n(t) - x'_0(t)| \\ &< (b_1 + b_2) \frac{\delta}{\Delta\sqrt{5}} \leq \frac{\delta}{\sqrt{5}}, \end{aligned}$$

hence

$$\begin{aligned} &(|x_n(t) - x_0(t)|^2 + |x'_n(t) - x'_0(t)|^2 + |D^\beta x_n(t) - D^\beta x_0(t)|^2 \\ &+ \left| \int_0^t h(\xi)x_n(\xi)d\xi - \int_0^t h(\xi)x_0(\xi)d\xi \right|^2 + |\phi x_n(t) - \phi x_0(t)|^2)^{\frac{1}{2}} \\ &< \sqrt{\frac{\delta^2}{5} + \dots + \frac{\delta^2}{5}} = \sqrt{\delta^2} = \delta \end{aligned}$$

so by (4) we conclude that

$$|f(t, x_n(t), x'_n(t), D^\beta x_n(t), \int_0^t h(\xi)x_n(\xi)d\xi, \phi x_n(t)) \\ - f(t, x_0(t), x'_0(t), D^\beta x_0(t), \int_0^t h(s)x_0(s)ds, \phi x_0(t))| < \epsilon$$

as $n \geq n_0$. So

$$f(t, x_n(s), x'_n(s), D^\beta x_n(s), \int_0^s h(\xi)x_n(\xi)d\xi, \phi x_n(s)) \rightarrow \\ f(t, x_0(s), x'_0(s), D^\beta x_0(s), \int_0^s h(\xi)x_0(\xi)d\xi, \phi x_0(s))$$

as $x_n(t) \rightarrow x_0(t)$ for $s \in E^c$. In other hand $G(t, s)$ and $\frac{\partial G(t, s)}{\partial t}$ are bonded and in $L^1(E^c)$ respect to s , hence
 $F_1 x_n(t) = \int_{E^c} G(t, s) f(s, x_n(s), x'_n(s), D^\beta x_n(s), \int_0^s h(\xi)x_n(\xi)d\xi, \phi x_n(s)) ds$
tends to
 $F_1 x_0(t) = \int_{E^c} G(t, s) f(s, x_0(s), x'_0(s), D^\beta x_0(s), \int_0^s h(\xi)x_0(\xi)d\xi, \phi x_0(s)) ds$
and
 $F'_1 x_n(t) = \int_{E^c} \frac{\partial G(t, s)}{\partial t} f(s, x_n(s), x'_n(s), D^\beta x_n(s), \int_0^s h(\xi)x_n(\xi)d\xi, \phi x_n(s)) ds$
tends to
 $F'_1 x_0(t) = \int_{E^c} \frac{\partial G(t, s)}{\partial t} f(s, x_0(s), x'_0(s), D^\beta x_0(s), \int_0^s h(\xi)x_0(\xi)d\xi, \phi x_0(s)) ds$
as $n \rightarrow \infty$, so F_1 is continuous in X . Now we will prove that F_2 is con-

tinuous in X . Let $x, y \in X$, then for all $t \in [0, 1]$ we have

$$\begin{aligned}
 & |F_2x(t) - F_2y(t)| \\
 \leq & \left| \int_E G(t, s) [f(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)d\xi, \phi x(s)) \right. \\
 & \left. - f(s, y(s), y'(s), D^\beta y(s), \int_0^s h(\xi)y(\xi)d\xi, \phi y(s))] ds \right| \\
 \leq & \frac{1}{\Gamma(\alpha)} \int_{E \cap [0, t]} (t-s)^{\alpha-1} |f(s, x(s), x'(s), D^\beta x(s), \\
 & \int_0^s h(\xi)x(\xi)d\xi, \phi x(s)) \\
 & - f(s, y(s), y'(s), D^\beta y(s), \int_0^s h(\xi)y(\xi)d\xi, \phi y(s))| ds \\
 + & \frac{1-t}{A_\lambda \Gamma(\alpha)} \int_{E \cap [0, \lambda]} H(s) |f(s, x(s), x'(s), D^\beta x(s), \\
 & \int_0^s h(\xi)x(\xi)d\xi, \phi x(s)) \\
 & - f(s, y(s), y'(s), D^\beta y(s), \int_0^s h(\xi)y(\xi)d\xi, \phi y(s))| ds \\
 + & \frac{|t \int_0^\lambda g(\xi)d\xi - \int_0^\lambda \xi g(\xi)d\xi|}{A_\lambda \Gamma(\alpha)} \int_E (1-s)^{\alpha-1} |f(s, x(s), x'(s), D^\beta x(s), \\
 & \int_0^s h(\xi)x(\xi)d\xi, \phi x(s)) \\
 & - f(s, y(s), y'(s), D^\beta y(s), \int_0^s h(\xi)y(\xi)d\xi, \phi y(s))| ds \\
 + & \frac{1-t}{A_\lambda \Gamma(\alpha-1)} \int_{E \cap [0, \mu]} (\mu-s)^{\alpha-2} |f(s, x(s), x'(s), D^\beta x(s), \\
 & \int_0^s h(\xi)x(\xi)d\xi, \phi x(s)) \\
 & - f(s, y(s), y'(s), D^\beta y(s), \int_0^s h(\xi)y(\xi)d\xi, \phi y(s))| ds \\
 \leq & \frac{1}{\Gamma(\alpha)} \int_{E \cap [0, t]} (t-s)^{\alpha-1} [a_1(s) \Lambda_1(|x(s) - y(s)|) \\
 & + a_2(s) \Lambda_2(|x'(s) - y'(s)|) + a_3(s) \Lambda_3(|D^\beta(x-y)(s)|)]
 \end{aligned}$$

$$\begin{aligned}
& +a_4(s)\Lambda_4(|\int_0^s h(\xi)(x(\xi)-y(\xi))d\xi|)+a_5(s)\Lambda_5(|\phi x(s)-\phi y(s)|)]ds \\
+ & \frac{1-t}{A_\lambda\Gamma(\alpha)}\int_{E\cap[0,\lambda]}H(s)[a_1(s)\Lambda_1(|x(s)-y(s)|) \\
& +a_2(s)\Lambda_2(|x'(s)-y'(s)|)+a_3(s)\Lambda_3(|D^\beta(x-y)(s)|) \\
& +a_4(s)\Lambda_4(|\int_0^s h(\xi)(x(\xi)-y(\xi))d\xi|)+a_5(s)\Lambda_5(|\phi x(s)-\phi y(s)|)]ds \\
+ & \frac{\int_0^\lambda |t-\xi|g(\xi)d\xi}{A_\lambda\Gamma(\alpha)}\int_E(1-s)^{\alpha-1}[a_1(s)\Lambda_1(|x(s)-y(s)|) \\
& +a_2(s)\Lambda_2(|x'(s)-y'(s)|)+a_3(s)\Lambda_3(|D^\beta(x-y)(s)|) \\
& +a_4(s)\Lambda_4(|\int_0^s h(\xi)(x(\xi)-y(\xi))d\xi|)+a_5(s)\Lambda_5(|\phi x(s)-\phi y(s)|)]ds \\
+ & \frac{1-t}{A_\lambda\Gamma(\alpha-1)}\int_{E\cap[0,\mu]}(\mu-s)^{\alpha-2}[a_1(s)\Lambda_1(|x(s)-y(s)|) \\
& +a_2(s)\Lambda_2(|x'(s)-y'(s)|)+a_3(s)\Lambda_3(|D^\beta(x-y)(s)|) \\
& +a_4(s)\Lambda_4(|\int_0^s h(\xi)(x(\xi)-y(\xi))d\xi|)+a_5(s)\Lambda_5(|\phi x(s)-\phi y(s)|)]ds \\
\leq & \frac{1}{\Gamma(\alpha)}\int_{E\cap[0,t]}(t-s)^{\alpha-1}[a_1(s)\Lambda_1(\|x-y\|)+a_2(s)\Lambda_2(\|x'-y'\|) \\
& +a_3(s)\Lambda_3(\frac{\|x'-y'\|}{\Gamma(2-\beta)})+a_4(s)\Lambda_4(m\|x-y\|) \\
& +a_5(s)\Lambda_5(b_1\|x-y\|+b_2\|x'-y'\|)]ds \\
+ & \frac{1-t}{A_\lambda\Gamma(\alpha)}\int_{E\cap[0,\lambda]}H(s)[a_1(s)\Lambda_1(\|x-y\|)+a_2(s)\Lambda_2(\|x'-y'\|) \\
& +a_3(s)\Lambda_3(\frac{\|x'-y'\|}{\Gamma(2-\beta)})+a_4(s)\Lambda_4(m\|x-y\|) \\
& +a_5(s)\Lambda_5(b_1\|x-y\|+b_2\|x'-y'\|)]ds \\
+ & \frac{\int_0^\lambda |t-\xi|g(\xi)d\xi}{A_\lambda\Gamma(\alpha)}\int_E(1-s)^{\alpha-1}[a_1(s)\Lambda_1(\|x-y\|) \\
& +a_2(s)\Lambda_2(\|x'-y'\|)+a_3(s)\Lambda_3(\frac{\|x'-y'\|}{\Gamma(2-\beta)}) \\
& +a_4(s)\Lambda_4(m\|x-y\|)+a_5(s)\Lambda_5(b_1\|x-y\|+b_2\|x'-y'\|)]ds
\end{aligned}$$

$$\begin{aligned}
& + \frac{1-t}{A_\lambda \Gamma(\alpha-1)} \int_{E \cap [0, \mu]} (\mu-s)^{\alpha-2} [a_1(s) \Lambda_1(\|x-y\|) \\
& + a_2(s) \Lambda_2(\|x'-y'\|) + a_3(s) \Lambda_3(\frac{\|x'-y'\|}{\Gamma(2-\beta)}) \\
& + a_4(s) \Lambda_4(m\|x-y\|) + a_5(s) \Lambda_5(b_1\|x-y\| + b_2\|x'-y'\|)] ds \\
& \leq \frac{1}{\Gamma(\alpha)} \int_E (1-s)^{\alpha-1} [a_1(s) \Lambda_1(\Delta\|x-y\|_*) \\
& + \dots + a_5(s) \Lambda_5(\Delta\|x-y\|_*)] ds \\
& + \frac{1-t}{A_\lambda \Gamma(\alpha)} \int_E \|g\|_{[0, \lambda]} (\lambda-s)^{\alpha-1} [a_1(s) \Lambda_1(\Delta\|x-y\|_*) \\
& + \dots + a_5(s) \Lambda_5(\Delta\|x-y\|_*)] ds \\
& + \frac{1}{\Gamma(\alpha)} \int_E (1-s)^{\alpha-1} [a_1(s) \Lambda_1(\Delta\|x-y\|_*) \\
& + \dots + a_5(s) \Lambda_5(\Delta\|x-y\|_*)] ds \\
& + \frac{1-t}{A_\lambda \Gamma(\alpha-1)} \int_E (1-s)^{\alpha-2} [a_1(s) \Lambda_1(\Delta\|x-y\|_*) \\
& + \dots + a_5(s) \Lambda_5(\Delta\|x-y\|_*)] ds.
\end{aligned}$$

Now Let $0 < \epsilon < 1$ be arbitry and $\|x-y\|_* < \epsilon$. Since for each $i = 1, \dots, 5$, $\lim_{z \rightarrow 0^+} \frac{\Lambda_i(\Delta z)}{(\Delta z)^{\gamma_i}} = q_i$, hence there exists $\delta > 0$ such that $0 < z < \delta$ implies that $\frac{\Lambda_i(\Delta z)}{(\Delta z)^{\gamma_i}} - q_i < \epsilon$, so $\Lambda_i(\Delta z) < \Delta^{\gamma_i}(q_i + \epsilon)z^{\gamma_i}$. Let $\|x-y\|_* < \min\{\epsilon, \delta\}$, then we have

$$\Lambda_i(\Delta\|x-y\|_*) < \Delta^{\gamma_i}(q_i + \epsilon)\|x-y\|_*^{\gamma_i} < \Delta^{\gamma_i}(q_i + \epsilon)\epsilon^{\gamma_i}$$

for each $i = 1, \dots, 5$. So $\|x-y\|_* < \min\{\epsilon, \delta\}$ implies that

$$\begin{aligned}
|F_2x(t) - F_2y(t)| & \leq \frac{1}{\Gamma(\alpha)} \int_E (1-s)^{\alpha-1} [a_1(s) \Delta^{\gamma_1}(q_1 + \epsilon)\epsilon^{\gamma_1} \\
& + \dots + a_5(s) \Delta^{\gamma_5}(q_5 + \epsilon)\epsilon^{\gamma_5}] ds \\
& + \frac{1-t}{A_\lambda \Gamma(\alpha)} \int_E \|g\|_{[0, \lambda]} (1-s)^{\alpha-1} [a_1(s) \Delta^{\gamma_1}(q_1 + \epsilon)\epsilon^{\gamma_1} \\
& + \dots + a_5(s) \Delta^{\gamma_5}(q_5 + \epsilon)\epsilon^{\gamma_5}] ds
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{\Gamma(\alpha)} \int_E (1-s)^{\alpha-1} [a_1(s) \Delta^{\gamma_1}(q_1 + \epsilon) \epsilon^{\gamma_1} \\
& \quad + \dots + a_5(s) \Delta^{\gamma_5}(q_5 + \epsilon) \epsilon^{\gamma_5}] ds \\
& + \frac{1-t}{A_\lambda \Gamma(\alpha-1)} \int_E (1-s)^{\alpha-2} [a_1(s) \Delta^{\gamma_1}(q_1 + \epsilon) \epsilon^{\gamma_1} \\
& \quad + \dots + a_5(s) \Delta^{\gamma_5}(q_5 + \epsilon) \epsilon^{\gamma_5}] ds.
\end{aligned}$$

Let $\gamma_0 := \min\{\gamma_1, \dots, \gamma_5\}$, then for all $1 \leq i \leq 5$, $\epsilon^{\gamma_i} \leq \epsilon^{\gamma_0}$, so we have

$$\begin{aligned}
\|F_2x - F_2y\| & \leq \left[\frac{1}{\Gamma(\alpha)} \sum_{i=1}^5 \int_E (1-s)^{\alpha-2} a_i(s) \Delta^{\gamma_i}(q_i + \epsilon) ds \right. \\
& + \frac{\|g\|_{[0,\lambda]}}{A_\lambda \Gamma(\alpha)} \sum_{i=1}^5 \int_E (1-s)^{\alpha-2} a_i(s) \Delta^{\gamma_i}(q_i + \epsilon) ds \\
& + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^5 \int_E (1-s)^{\alpha-2} a_i(s) \Delta^{\gamma_i}(q_i + \epsilon) ds \\
& \left. + \frac{1}{A_\lambda \Gamma(\alpha-1)} \sum_{i=1}^5 \int_E (1-s)^{\alpha-2} a_i(s) \Delta^{\gamma_i}(q_i + \epsilon) ds \right] \epsilon^{\gamma_0}. \quad (5)
\end{aligned}$$

Now since $(1-s)^{\alpha-2} a_i(s) \in L^1(E)$, so $(1-s)^{\alpha-2} a_i(s) \Delta^{\gamma_i}(q_i + \epsilon) \in L^1(E)$ and since $\epsilon > 0$ was arbitrary, by (5) we conclude that $\|F_2x - F_2y\| \rightarrow 0$ as $x \rightarrow y$. By the similar way we conclude that

$$\begin{aligned}
& |F_2'x(t) - F_2'y(t)| \\
& \leq \left| \int_E \frac{\partial G}{\partial t}(t, s) [f(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi) x(\xi) d\xi, \phi x(s)) \right. \\
& \quad \left. - f(s, y(s), y'(s), D^\beta y(s), \int_0^s h(\xi) y(\xi) d\xi, \phi y(s))] ds \right| \\
& \leq \frac{1}{\Gamma(\alpha-1)} \int_{E \cap [0,t]} (t-s)^{\alpha-2} |f(s, x(s), x'(s), D^\beta x(s), \\
& \quad \int_0^s h(\xi) x(\xi) d\xi, \phi x(s))
\end{aligned}$$

$$\begin{aligned}
& -f(s, y(s), y'(s), D^\beta y(s), \int_0^s h(\xi)y(\xi)d\xi, \phi y(s))|ds \\
+ & \frac{1}{A_\lambda \Gamma(\alpha)} \int_{E \cap [0, \lambda]} H(s) |f(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)d\xi, \phi x(s)) \\
& -f(s, y(s), y'(s), D^\beta y(s), \int_0^s h(\xi)y(\xi)d\xi, \phi y(s))|ds \\
+ & \frac{A_\lambda + B_\lambda}{A_\lambda \Gamma(\alpha)} \int_E (1-s)^{\alpha-1} |f(s, x(s), x'(s), D^\beta x(s), \\
& \int_0^s h(\xi)x(\xi)d\xi, \phi x(s)) - f(s, y(s), y'(s), D^\beta y(s), \\
& \int_0^s h(\xi)y(\xi)d\xi, \phi y(s))|ds \\
+ & \frac{1}{A_\lambda \Gamma(\alpha-1)} \int_{E \cap [0, \mu]} (\mu-s)^{\alpha-2} |f(s, x(s), x'(s), D^\beta x(s), \\
& \int_0^s h(\xi)x(\xi)d\xi, \phi x(s)) \\
& -f(s, y(s), y'(s), D^\beta y(s), \int_0^s h(\xi)y(\xi)d\xi, \phi y(s))|ds \\
\leq & \frac{1}{\Gamma(\alpha-1)} \int_{E \cap [0, t]} (t-s)^{\alpha-2} [a_1(s)\Lambda_1(|x(s)-y(s)|) \\
& + a_2(s)\Lambda_2(|x'(s)-y'(s)|) + a_3(s)\Lambda_3(|D^\beta(x-y)(s)|) \\
& + a_4(s)\Lambda_4(|\int_0^s h(\xi)(x(\xi)-y(\xi))d\xi|) + a_5(s)\Lambda_5(|\phi x(s)-\phi y(s)|)]ds \\
+ & \frac{1}{A_\lambda \Gamma(\alpha)} \int_{E \cap [0, \lambda]} H(s) [a_1(s)\Lambda_1(|x(s)-y(s)|) \\
& + a_2(s)\Lambda_2(|x'(s)-y'(s)|) + a_3(s)\Lambda_3(|D^\beta(x-y)(s)|) \\
& + a_4(s)\Lambda_4(|\int_0^s h(\xi)(x(\xi)-y(\xi))d\xi|) + a_5(s)\Lambda_5(|\phi x(s)-\phi y(s)|)]ds \\
+ & \frac{A_\lambda + B_\lambda}{A_\lambda \Gamma(\alpha)} \int_E (1-s)^{\alpha-1} [a_1(s)\Lambda_1(|x(s)-y(s)|) \\
& + a_2(s)\Lambda_2(|x'(s)-y'(s)|) + a_3(s)\Lambda_3(|D^\beta(x-y)(s)|) \\
& + a_4(s)\Lambda_4(|\int_0^s h(\xi)(x(\xi)-y(\xi))d\xi|) + a_5(s)\Lambda_5(|\phi x(s)-\phi y(s)|)]ds
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{A_\lambda \Gamma(\alpha - 1)} \int_{E \cap [0, \mu]} (\mu - s)^{\alpha-2} [a_1(s) \Lambda_1(|x(s) - y(s)|) \\
& + a_2(s) \Lambda_2(|x'(s) - y'(s)|) + a_3(s) \Lambda_3(|D^\beta(x - y)(s)|) \\
& + a_4(s) \Lambda_4(|\int_0^s h(\xi)(x(\xi) - y(\xi)) d\xi|) + a_5(s) \Lambda_5(|\phi x(s) - \phi y(s)|)] ds \\
\leq & \frac{1}{\Gamma(\alpha - 1)} \int_{E \cap [0, t]} (t - s)^{\alpha-2} [a_1(s) \Lambda_1(\|x - y\|) + a_2(s) \Lambda_2(\|x' - y'\|) \\
& + a_3(s) \Lambda_3(\frac{\|x' - y'\|}{\Gamma(2 - \beta)}) + a_4(s) \Lambda_4(m\|x - y\|) \\
& + a_5(s) \Lambda_5(b_1\|x - y\| + b_2\|x' - y'\|)] ds \\
& + \frac{1}{A_\lambda \Gamma(\alpha)} \int_{E \cap [0, \lambda]} H(s) [a_1(s) \Lambda_1(\|x - y\|) + a_2(s) \Lambda_2(\|x' - y'\|) \\
& + a_3(s) \Lambda_3(\frac{\|x' - y'\|}{\Gamma(2 - \beta)}) + a_4(s) \Lambda_4(m\|x - y\|) \\
& + a_5(s) \Lambda_5(b_1\|x - y\| + b_2\|x' - y'\|)] ds \\
& + \frac{\int_0^\lambda g(\xi) d\xi}{A_\lambda \Gamma(\alpha)} \int_E (1 - s)^{\alpha-1} [a_1(s) \Lambda_1(\|x - y\|) + a_2(s) \Lambda_2(\|x' - y'\|) \\
& + a_3(s) \Lambda_3(\frac{\|x' - y'\|}{\Gamma(2 - \beta)}) + a_4(s) \Lambda_4(m\|x - y\|) \\
& + a_5(s) \Lambda_5(b_1\|x - y\| + b_2\|x' - y'\|)] ds \\
& + \frac{1}{A_\lambda \Gamma(\alpha - 1)} \int_{E \cap [0, \mu]} (\mu - s)^{\alpha-2} [a_1(s) \Lambda_1(\|x - y\|) \\
& + a_2(s) \Lambda_2(\|x' - y'\|) + a_3(s) \Lambda_3(\frac{\|x' - y'\|}{\Gamma(2 - \beta)}) + a_4(s) \Lambda_4(m\|x - y\|) \\
& + a_5(s) \Lambda_5(b_1\|x - y\| + b_2\|x' - y'\|)] ds \\
\leq & \frac{1}{\Gamma(\alpha - 1)} \int_E (1 - s)^{\alpha-2} (1 - s)^{\alpha-2} [a_1(s) \Lambda_1(\Delta\|x - y\|_*) \\
& + \dots + a_5(s) \Lambda_5(\Delta\|x - y\|_*)] ds \\
& + \frac{\|g\|_{[0, \lambda]}}{A_\lambda \Gamma(\alpha)} \int_E (1 - s)^{\alpha-2} [a_1(s) \Lambda_1(\Delta\|x - y\|_*) \\
& + \dots + a_5(s) \Lambda_5(\Delta\|x - y\|_*)] ds
\end{aligned}$$

$$\begin{aligned}
& + \frac{\|g\|_{[0,\lambda]}}{\Gamma(\alpha)} \int_E (1-s)^{\alpha-2} [a_1(s) \Lambda_1(\Delta \|x-y\|_*) \\
& \quad + \dots + a_5(s) \Lambda_5(\Delta \|x-y\|_*)] ds \\
& + \frac{1}{A_\lambda \Gamma(\alpha-1)} \int_E (1-s)^{\alpha-2} [a_1(s) \Lambda_1(\Delta \|x-y\|_*) \\
& \quad + \dots + a_5(s) \Lambda_5(\Delta \|x-y\|_*)] ds \\
& \leq \frac{1}{\Gamma(\alpha-1)} \int_E (1-s)^{\alpha-2} [a_1(s) \Delta^{\gamma_1} (q_1 + \epsilon) \epsilon^{\gamma_1} \\
& \quad + \dots + a_5(s) \Delta^{\gamma_5} (q_5 + \epsilon) \epsilon^{\gamma_5}] ds \\
& + \frac{\|g\|_{[0,\lambda]}}{A_\lambda \Gamma(\alpha)} \int_E (1-s)^{\alpha-2} [a_1(s) \Delta^{\gamma_1} (q_1 + \epsilon) \epsilon^{\gamma_1} \\
& \quad + \dots + a_5(s) \Delta^{\gamma_5} (q_5 + \epsilon) \epsilon^{\gamma_5}] ds \\
& + \frac{\|g\|_{[0,\lambda]}}{\Gamma(\alpha)} \int_E (1-s)^{\alpha-2} [a_1(s) \Delta^{\gamma_1} (q_1 + \epsilon) \epsilon^{\gamma_1} \\
& \quad + \dots + a_5(s) \Delta^{\gamma_5} (q_5 + \epsilon) \epsilon^{\gamma_5}] ds \\
& + \frac{1}{A_\lambda \Gamma(\alpha-1)} \int_E (1-s)^{\alpha-2} [a_1(s) \Delta^{\gamma_1} (q_1 + \epsilon) \epsilon^{\gamma_1} \\
& \quad + \dots + a_5(s) \Delta^{\gamma_5} (q_5 + \epsilon) \epsilon^{\gamma_5}] ds.
\end{aligned}$$

So we conclude that

$$\begin{aligned}
\|F_2x - F_2y\| & \leq \left[\frac{1}{\Gamma(\alpha-1)} \sum_{i=1}^5 \int_E (1-s)^{\alpha-2} a_i(s) \Delta^{\gamma_i} (q_i + \epsilon) ds \right. \\
& + \frac{\|g\|_{[0,\lambda]}}{A_\lambda \Gamma(\alpha)} \sum_{i=1}^5 \int_E (1-s)^{\alpha-2} a_i(s) \Delta^{\gamma_i} (q_i + \epsilon) ds \\
& + \frac{\|g\|_{[0,\lambda]}}{A_\lambda \Gamma(\alpha)} \sum_{i=1}^5 \int_E (1-s)^{\alpha-2} a_i(s) \Delta^{\gamma_i} (q_i + \epsilon) ds \\
& \left. + \frac{1}{A_\lambda \Gamma(\alpha-1)} \sum_{i=1}^5 \int_E (1-s)^{\alpha-2} a_i(s) \Delta^{\gamma_i} (q_i + \epsilon) ds \right] \epsilon^{\gamma_0}.
\end{aligned}$$

Therefore $\|F'_2x - F'_2y\| \rightarrow 0$ as $x \rightarrow y$, hence

$$\|F_2x - F_2y\|_* = \max\{\|F_2x - F_2y\|, \|F'_2x - F'_2y\|\} \rightarrow 0$$

as $x \rightarrow y$, so we conclude that F_2 is continuous in X , hence $F = F_1 + F_2$ is continuous in X . Now we have $\lim_{z \rightarrow \infty} \frac{L(\Delta z, \Delta z, \Delta z, \Delta z, \Delta z)}{\Delta z} = q$, so for each $\epsilon > 0$ there is $r > 0$ such that $z \geq r$ implies that $\frac{L(\Delta z, \dots, \Delta z)}{\Delta z} - q < \epsilon$ hence $L(\Delta z, \dots, \Delta z) < (q + \epsilon)\Delta z$ for $z \geq r$. In other hand $\lim_{z \rightarrow \infty} \frac{K(\Delta z, \Delta z, \Delta z, \Delta z, \Delta z)}{\Delta z} = 0$, so for each $\epsilon > 0$ there is $r' > 0$ such that $z \geq r'$ implies that $\frac{K(\Delta z, \dots, \Delta z)}{\Delta z} < \epsilon$ hence $K(\Delta z, \dots, \Delta z) < \Delta z\epsilon$ for $z \geq r'$. Now since

$$\begin{aligned} & \max\left\{\frac{2}{\Gamma(\alpha)} + \frac{\|g\|_{[0, \lambda]}}{A_\lambda \Gamma(\alpha)} + \frac{1}{A_\lambda \Gamma(\alpha - 1)}, \right. \\ & \left. \frac{1}{\Gamma(\alpha - 1)} + \frac{2\|g\|_{[0, \lambda]}}{A_\lambda \Gamma(\alpha)} + \frac{1}{A_\lambda \Gamma(\alpha - 1)}\right\} q \|\hat{b}\|_{[0, 1]} \in [0, \frac{1}{\Delta}), \end{aligned}$$

there is an ϵ_0 such that

$$\begin{aligned} & \max\left\{\left[\frac{2}{\Gamma(\alpha)} + \frac{\|g\|_{[0, \lambda]}}{A_\lambda \Gamma(\alpha)} + \frac{1}{A_\lambda \Gamma(\alpha - 1)}\right](q + \epsilon_0) \|\hat{b}\|_{[0, 1]} \right. \\ & + \left[\frac{2}{\Gamma(\alpha + 1)} + \frac{1}{A_\lambda \Gamma(\alpha + 1)} + \frac{1}{A_\lambda \Gamma(\alpha)}\right] \epsilon_0, \\ & \left[\frac{1}{\Gamma(\alpha - 1)} + \frac{2\|g\|_{[0, \lambda]}}{A_\lambda \Gamma(\alpha)} + \frac{1}{A_\lambda \Gamma(\alpha - 1)}\right](q + \epsilon_0) \|\hat{b}\|_{[0, 1]} \\ & + \left[\frac{1}{\Gamma(\alpha)} + \frac{2\|g\|_{[0, \lambda]}}{A_\lambda \Gamma(\alpha + 1)} + \frac{1}{A_\lambda \Gamma(\alpha)}\right] \epsilon_0 \left. \right\} \in [0, \frac{1}{\Delta}). \end{aligned}$$

Let $r_1 := r(\epsilon_0)$ and $r_2 := r'(\epsilon_0)$ and put $r_0 := \max\{r_1, r_2\}$ and define $\Omega = \{x \in X : \|x\|_* < r_0\}$. If there exists $y \in \partial\Omega$ such that $y(t) = \lambda_0 F_y(t)$ for some $\lambda_0 \in (0, 1)$ and all $t \in [0, 1]$, then $\|y\|_* = r_0$,

$$y(t) = \lambda_0 \int_0^1 G(t, s) f(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi) x(\xi) d\xi, \phi x(s)) ds$$

and

$$y'(t) = \lambda_0 \int_0^1 \frac{\partial G}{\partial t}(t, s) f(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi) x(\xi) d\xi, \phi x(s)) ds.$$

Hence

$$\begin{aligned}
& |y(t)| \\
= & \left| \lambda_0 \int_0^1 G(t, s) f(s, y(s), y'(s), D^\beta y(s), \int_0^s h(\xi) y(\xi) d\xi, \phi y(s)) ds \right| \\
\leq & \lambda_0 \left[\int_0^1 |G(t, s)| b(s) L(y(s), y'(s), D^\beta y(s), \int_0^s h(\xi) y(\xi) d\xi, \phi y(s)) ds \right. \\
& \left. + \int_0^1 |G(t, s)| b(s) K(y(s), y'(s), D^\beta y(s), \int_0^s h(\xi) y(\xi) d\xi, \phi y(s)) ds \right] \\
\leq & \lambda_0 \left[\frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} b(s) L(y(s), y'(s), D^\beta y(s), \int_0^s h(\xi) y(\xi) d\xi, \right. \\
& \left. \phi y(s)) ds + \frac{(1-t) \|g\|_{[0, \lambda]}}{A_\lambda \Gamma(\alpha)} \int_0^1 (\lambda-s)^{\alpha-1} b(s) L(y(s), y'(s), D^\beta y(s), \right. \\
& \left. \int_0^s h(\xi) y(\xi) d\xi, \phi y(s)) ds + \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} b(s) L(y(s), y'(s), \right. \\
& \left. D^\beta y(s), \int_0^s h(\xi) y(\xi) d\xi, \phi y(s)) ds \right. \\
& \left. + \frac{1-t}{A_\lambda \Gamma(\alpha)} \int_0^1 (\mu-s)^{\alpha-1} b(s) L(y(s), y'(s), D^\beta y(s), \int_0^s h(\xi) y(\xi) d\xi, \right. \\
& \left. \phi y(s)) ds \right. \\
& \left. + \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} K(y(s), y'(s), D^\beta y(s), \int_0^s h(\xi) y(\xi) d\xi, \phi y(s)) ds \right. \\
& \left. + \frac{(1-t) \|g\|_{[0, \lambda]}}{A_\lambda \Gamma(\alpha)} \int_0^1 (\lambda-s)^{\alpha-1} K(y(s), y'(s), D^\beta y(s), \right. \\
& \left. \int_0^s h(\xi) y(\xi) d\xi, \phi y(s)) ds + \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} K(y(s), y'(s), D^\beta y(s), \right. \\
& \left. \int_0^s h(\xi) y(\xi) d\xi, \phi y(s)) ds + \frac{1-t}{A_\lambda \Gamma(\alpha-1)} \int_0^1 (\mu-s)^{\alpha-2} K(y(s), y'(s), \right. \\
& \left. D^\beta y(s), \int_0^s h(\xi) y(\xi) d\xi, \phi y(s)) ds \right] \\
\leq & \lambda_0 \left[\frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} b(s) L(r, r_0, \frac{r_0}{\Gamma(2-\beta)}, mr_0, (b_1 + b_2)r_0) ds \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{(1-t)\|g\|_{[0,\lambda]}}{A_\lambda \Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} b(s) L(r, r_0, \frac{r_0}{\Gamma(2-\beta)}, \\
& mr_0, (b_1 + b_2)r_0) ds \\
& + \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} b(s) L(r, r_0, \frac{r_0}{\Gamma(2-\beta)}, mr_0, (b_1 + b_2)r_0) ds \\
& + \frac{1-t}{A_\lambda \Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-2} b(s) L(r, r_0, \frac{r_0}{\Gamma(2-\beta)}, mr_0, (b_1 + b_2)r_0) ds \\
& + \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} K(r, r_0, \frac{r_0}{\Gamma(2-\beta)}, mr_0, (b_1 + b_2)r_0) ds \\
& + \frac{(1-t)\|g\|_{[0,\lambda]}}{A_\lambda \Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} K(r, r_0, \frac{r_0}{\Gamma(2-\beta)}, mr_0, (b_1 + b_2)r_0) ds \\
& + \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} K(r, r_0, \frac{r_0}{\Gamma(2-\beta)}, mr_0, (b_1 + b_2)r_0) ds \\
& + \frac{1-t}{A_\lambda \Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} K(r, r_0, \frac{r_0}{\Gamma(2-\beta)}, mr_0, (b_1 + b_2)r_0) ds] \\
\leq & \lambda_0 \left[\frac{1}{\Gamma(\alpha)} L(\Delta r, \dots, \Delta r) \int_0^1 (1-s)^{\alpha-2} b(s) ds \right. \\
& + \frac{(1-t)\|g\|_{[0,\lambda]}}{A_\lambda \Gamma(\alpha)} L(\Delta r, \dots, \Delta r) \int_0^1 (1-s)^{\alpha-2} b(s) ds \\
& + \frac{1}{\Gamma(\alpha)} L(\Delta r, \dots, \Delta r) \int_0^1 (1-s)^{\alpha-2} b(s) ds \\
& + \frac{1-t}{A_\lambda \Gamma(\alpha-1)} L(\Delta r, \dots, \Delta r) \int_0^1 (1-s)^{\alpha-2} b(s) ds \\
& + \frac{1}{\Gamma(\alpha)} \int_0^1 K(\Delta r, \dots, \Delta r) (1-s)^{\alpha-1} ds \\
& + \frac{(1-t)\|g\|_{[0,\lambda]}}{A_\lambda \Gamma(\alpha)} K(\Delta r, \dots, \Delta r) (1-s)^{\alpha-1} ds \\
& + \frac{1}{\Gamma(\alpha)} K(\Delta r, \dots, \Delta r) (1-s)^{\alpha-1} ds \\
& \left. + \frac{1-t}{A_\lambda \Gamma(\alpha-1)} K(\Delta r, \dots, \Delta r) (1-s)^{\alpha-2} ds \right]
\end{aligned}$$

$$\begin{aligned}
 \leq & \lambda_0 \left[\frac{1}{\Gamma(\alpha)} (q + \epsilon_0) \Delta r_0 \|\hat{b}\|_{[0,1]} + \frac{(1-t)\|g\|_{[0,\lambda]}}{A_\lambda \Gamma(\alpha)} (q + \epsilon_0) \Delta r_0 \|\hat{b}\|_{[0,1]} \right. \\
 & + \frac{1}{\Gamma(\alpha)} (q + \epsilon_0) \Delta r_0 \|\hat{b}\|_{[0,1]} + \frac{1-t}{A_\lambda \Gamma(\alpha-1)} (q + \epsilon_0) \Delta r_0 \|\hat{b}\|_{[0,1]} \\
 & + \frac{1}{\Gamma(\alpha+1)} \Delta r_0 \epsilon_0 + \frac{(1-t)\|g\|_{[0,\lambda]}}{A_\lambda \Gamma(\alpha+1)} \Delta r_0 \epsilon_0 \\
 & \left. + \frac{1}{\Gamma(\alpha+1)} \Delta r_0 \epsilon_0 + \frac{1-t}{A_\lambda \Gamma(\alpha)} \Delta r_0 \epsilon_0 \right],
 \end{aligned}$$

so we have

$$\begin{aligned}
 \|y\| \leq & \lambda_0 \left[\left(\frac{2}{\Gamma(\alpha)} + \frac{\|g\|_{[0,\lambda]}}{A_\lambda \Gamma(\alpha)} + \frac{1}{A_\lambda \Gamma(\alpha-1)} \right) (q + \epsilon_0) \|\hat{b}\|_{[0,1]} \right. \\
 & \left. + \left(\frac{2}{\Gamma(\alpha+1)} + \frac{1}{A_\lambda \Gamma(\alpha+1)} + \frac{1}{A_\lambda \Gamma(\alpha)} \right) \epsilon_0 \right] \Delta r_0 < r_0.
 \end{aligned}$$

By same manner we conclude that

$$\begin{aligned}
 \|y'\| \leq & \lambda_0 \left[\left(\frac{1}{\Gamma(\alpha-1)} + \frac{2\|g\|_{[0,\lambda]}}{A_\lambda \Gamma(\alpha)} + \frac{1}{A_\lambda \Gamma(\alpha-1)} \right) (q + \epsilon_0) \|\hat{b}\|_{[0,1]} \right. \\
 & \left. + \left(\frac{1}{\Gamma(\alpha)} + \frac{2\|g\|_{[0,\lambda]}}{A_\lambda \Gamma(\alpha+1)} + \frac{1}{A_\lambda \Gamma(\alpha)} \right) \epsilon_0 \right] \Delta r_0 < r_0,
 \end{aligned}$$

hence $\|y\|_* = \max\{\|y\|, \|y'\|\} < r_0$, so $y \notin \partial\Omega$ that it's a contradiction, so by lemma ([14]) we conclude that F has a fixed point in $\bar{\Omega}$ which is a solution for the problem. $\square$

Example 2.5. Consider the problem

$$D^{\frac{7}{2}}x(t) + f(t, x(t), x'(t), D^{\frac{1}{2}}x(t), \int_0^t \xi^2 x(\xi) d\xi, I^{\frac{1}{3}}x(t)) = 0, \quad (6)$$

with boundary conditions $x'(\frac{1}{2}) = x'(0) + \int_0^{\frac{1}{2}} sx(s)ds$ and $x(1) = x^{(j)}(0) = 0$ for $2 \leq j \leq 4$, where $f(t, x_1, \dots, x_5) = 1 + \sin t + g(t, x_1, \dots, x_5)$, $g(t, x_1, \dots, x_5) = \frac{0.1p(t)\sum_{i=1}^5 |x_i|}{1 + |x_i|}$ for $t \in E := [0.2, 0.6)$ and $g(t, x_1, \dots, x_5) = 0.1t\sum_{i=1}^5 |x_i| + \sum_{i=1}^5 \frac{|x_i|}{1 + |x_i|}$ for $t \in E^c := [0, 0.2) \cup [0.6, 1]$, $p(t) = 0$ whenever $t \in E \cap \mathcal{Q}$ and $p(t) = 1$ whenever $t \in E \cap \mathcal{Q}^c$. Put $\alpha = \frac{7}{2}$,

$\beta = \frac{1}{2}$, $\Lambda_i(x_1, \dots, x_5) = L(x_1, \dots, x_5) = 0.1 \sum_{i=1}^5 |x_i|$, $b(t) = \frac{1}{p(t)}$, $g(t) = t$,
 $h(t) = t^2$, $\lambda = \frac{1}{2}$, $\mu = \frac{1}{2}$, $m = \frac{1}{3}$, $\phi x(t) = I^{\frac{1}{3}}x(t)$, $b_1 = \frac{1}{\Gamma(\frac{1}{2})}$, $b_2 = 0$,
 $q = 0.5$, $K(x_1, \dots, x_5) = 2 + \frac{|x_i|}{1+|x_i|}$. Note that for $t \in E$

$$|f(t, x_1, \dots, x_5) - f(t, y_1, \dots, y_5)| = \frac{0.1}{p(t)} |\sum_{i=1}^5 |x_i| - |y_i|| \leq \frac{0.1}{p(t)} \sum_{i=1}^5 |x_i - y_i|,$$

and for $t \in E$ we f is continuous, $\lim_{z \rightarrow \infty} \frac{\Lambda_i(z, z, z, z, z)}{z} = 0.5 = q$, for all
 $1 \leq i \leq 5$, $\lim_{z \rightarrow \infty} \frac{L(z, z, z, z, z)}{z} = 0$,

$$\begin{aligned} |f(t, x_1, \dots, x_5)| &\leq \frac{0.1}{p(t)} \sum_{i=1}^5 |x_i - y_i| + 1 + \sin t + \frac{|x_i|}{1 + |x_i|} \\ &\leq b(t)L(x_1, \dots, x_5) + K(x_1, \dots, x_5), \end{aligned}$$

for almost all $t \in [0, 1]$, $(1 - t)^{\alpha-2}b(t) \in L^1(E)$, $\|\hat{b}\|_E = 0.28$,
 $\|g\|_{[0, \lambda]} = \frac{1}{18}$,

$$\begin{aligned} |I^{\frac{1}{3}}x(t) - I^{\frac{1}{3}}y(t)| &\leq \frac{1}{\Gamma(\frac{1}{3})} \int_0^t (t-s)^{\frac{1}{3}-1} |x(s) - y(s)| ds \\ &\leq \frac{\|x - y\|}{\Gamma(\frac{1}{3})} \int_0^t \frac{ds}{(t-s)^{\frac{2}{3}}} \leq \frac{\|x - y\|}{\Gamma(\frac{1}{2})} = b_1 \|x - y\|, \end{aligned}$$

$$\int_0^1 h(t) dt = \frac{1}{3} = m,$$

$$\Delta = \max\{1, \frac{1}{\Gamma(2-\beta)}, m, b_1 + b_2\} = \max\{1, \frac{1}{\Gamma(\frac{3}{2})}, \frac{1}{3}, \frac{1}{\Gamma(\frac{1}{2})}\} = \frac{2}{\sqrt{\pi}},$$

$$A_\lambda = \int_0^\lambda (1-t)g(t)dt = \int_0^{\frac{1}{2}} (1-t)t dt = \frac{1}{12}$$

and

$$\begin{aligned}
 & \max\left\{\frac{2}{\Gamma(\alpha)} + \frac{\|g\|_{[0,\lambda]}}{A_\lambda\Gamma(\alpha)} + \frac{1}{A_\lambda\Gamma(\alpha-1)}, \right. \\
 & \left. \frac{1}{\Gamma(\alpha-1)} + \frac{2\|g\|_{[0,\lambda]}}{A_\lambda\Gamma(\alpha)} + \frac{1}{A_\lambda\Gamma(\alpha-1)}\right\}q\|\hat{b}\|_{[0,1]} \\
 = & \max\left\{\frac{2}{\Gamma(\frac{7}{2})} + \frac{\frac{1}{18}}{\frac{1}{2}\Gamma(\frac{7}{2})} + \frac{1}{\frac{1}{2}\Gamma(\frac{5}{2})}, \right. \\
 & \left. \frac{1}{\Gamma(\frac{5}{2})} + \frac{2\cdot\frac{1}{18}}{\frac{1}{2}\Gamma(\frac{7}{2})} + \frac{1}{\frac{1}{2}\Gamma(\frac{5}{2})}\right\} \times 0.5 \times 0.28 \in [0, \frac{\sqrt{\pi}}{2}).
 \end{aligned}$$

Now using Theorem (2.4), the problem (6) has a solution.

References

- [1] O. P. Agrawal, D. O'Regan and S. Stanek, Positive solutions for Dirichlet problems of singular nonlinear fractional differential equations, *J. Math. Anal. Appl.* 371 (2010), 57-68.
- [2] J. Sabatier, O. P. Agrawal and J. A. T. Machado, *Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering* Springer, Dordrecht (2007)
- [3] S. Stanek, The existence of positive solutions of singular fractional boundary value problems, *Computers and Mathematics with Applications*, 62 (2011), 13791388.
- [4] Z. Bai, H. L, Positive solutions for boundary value problem of nonlinear fractional differential equation, *J. Math. Anal. Appl.* 311 (2005), 495505.
- [5] D. Baleanu, Kh. Ghafarnezhad, Sh. Rezapour and M. Shabibi, On the existence of solution of a three steps crisis integro-differential equation, *Advances in difference equations*, 135 (2018), 20 p.
- [6] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional integral and derivative: theory and applications*, Gordon and Breach (1993).

- [7] Y. Liu and P. J. Y. Wong, Global existence of solutions for a system of singular fractional differential equations with impulse effects, *J. Appl. Math. Inform.* 33 (2015), 327-342.
- [8] M. Shabibi, M. Postolache, Sh. Rezapour and S. M. Vaezpour, Investigation of a multi-singular pointwise defined fractional integro-differential equation, *J. Mathl Anal.* 7 (2016), 61-77.
- [9] M. Shabibi, M. Postolache, Sh. Rezapour, Positive solutions for a singular sum fractional differential systems, *International Journal of Analysis and Applications* 13 (2017) 108-118.
- [10] S. Stanek, The existence of positive solutions of singular fractional boundary value problems, *Comput. Math. with Appl.*, 62 (2011), 1379-1388.
- [11] B. Samet, C. Vetro and P. Vetro, Fixed point theorems for α - ψ -contractive type mappings, *Nonlin. Anal.*, 75 (2012) 2154-2165.
- [12] I. Podlubny, *Fractional differential equations*, Academic Press (1999)
- [13] S. Abbas, M. Benchohara, A. Alsaedi and Y. Zhou, Stability results for partial fractional differential equations with noninstantaneous impulses, *Advances in Difference Equations*, 75 (2017),
- [14] E. Zeidler, *Nonlinear Functional Analysis and Its Applications-I: Fixed-point Theorems*, Springer, New York, NY, USA, (1986)

Ali Mansouri

Assistant Professor of Mathematics

Department of Mathematics, Islamic Azad University, Arak Branch

Arak, Iran

E-mail: *arakmath2@yahoo.com*

Mehdi Shabibi

Assistant Professor of Mathematics

Department of Mathematics, Mehran Branch, Islamic Azad University

Mehran, Iran

E-mail: *mehdi_math1983@yahoo.com*