Determining Minor and Major Consolidations in Network Inverse Data Envelopment Analysis

P. Kamyab[1], M. R. Mozaffari[2*], J. Gerami [2]

[1]*Department of Mathematics, Rasht Branch, Islamic Azad University, Rasht, Iran*

[2]*Department of Mathematics, Shiraz Branch, Islamic Azad University, Shiraz, Iran*

[*]*Corresponding Author: mozaffari854@yahoo.com*

Abstract

It is necessary to make use of scientific methods when merging the Decision-Making Units (DMUs) in any organization. Tools such as Data Envelopment Analysis (DEA) and network DEA (NDEA) are quite useful for unit mergers in two-stage network processes. In this paper, a two-stage network inverse DEA (InvDEA) process is proposed for the merger of university and bank branches based on linear programming models. It is generally crucial to prioritize the inputs and outputs and find the intermediate vectors in multi-stage networks. Therefore, a two-stage network inverse DEA model is used for the purposes of this study. Finally, some applications of the proposed model are provided in DMU mergers based on vector prioritization using Shannon's entropy, namely the mergers of 5 universities, 24 insurance companies, and 20 commercial banks.

Keywords: *DEA, Network DEA, Inverse DEA, Consolidation, Merger.*

1. Introduction

Many organizations and financial institutions seek to merge their Decision-Making Units (DMUs) with the aim to save money and improve their performance. Data Envelopment Analysis (DEA) is a tool used to evaluate the performance of organizations through mathematical programming. This method was originally established by Farrell in 1957 and later developed by Charnes, Cooper, and Rhodes in 1978, forming the well-known CCR model. With the introduction of variable returns to scale (VRS) technology by Banker, Charnes, and Cooper in 1984, the BCC model was formulated. DEA is a useful technique to measure the relative efficiency of a DMU as a whole, with no regard to its internal structure. In other words, a given DMU is compared to a black box consuming inputs in order to produce outputs. However, in many real-world situations, there exists an internal structure that affects the final performance of the system under evaluation. Therefore, it is necessary to study the internal components of the DMUs in order to recognize the causes of inefficiency. Structures in which the internal structure of the DMUs are taken into account are called network structures, and the DEA technique used to measure the efficiency of such systems is called network DEA (NDEA). Network DEA was first studied by Charnes et al. in 1986. Thereafter, several studies were carried out with a variety of objectives. Kao and Hwang (2008) proposed a model for measuring the efficiency of 24 insurance companies in Taiwan. Kao and Hwang (2010) studied the effects of information technology (IT) on DMU performances in order to measure the efficiency of network systems. Fukuyama and Weber (2010) proposed the SBM model, which maximizes the mean of slack variables, aiming to evaluate the performance of 869 Japanese banks with undesirable outputs. Kao and Liu (2011) discussed efficiency evaluation in two-stage structures with fuzzy data. Lewis et al. (2013) introduced the iterative method in order to simultaneously minimize the input parameter θ and maximize the output parameter ϕ. Amirteimoori (2013) studied the case of car factories in which some products had defects and required repair. Du et al. (2015) presented an NDEA model with a non-homogeneous parallel networks structure. Mehdiloozad et al. (2015) investigated the global reference set in DEA. Mirdehghan and Fukuyama (2015) suggested a two-stage NDEA method based on the enhanced Russell measure (ERM) that provided information regarding the overall efficiency status of the system, as well as the component-based efficiency measures.

Table 1 provides a list of some studies involving two-stage network DEA stratified by the models used in the network structure.

Table 1. Studies on Two-stage Network Data Envelopment Analysis

Author	**Model used in the network system**
Wang et al. 1997	Independent model
Seiford and Zhu 1999	Independent model
Zhu 2000	Independent model
Sexton and Lewis 2003	Independent model
Chen and Zhu 2004	Process distance measure model
Liang et al. 2008	Game theoretic model
Chen, et al. 2009	Ratio-form process efficiency
Liu and Wang 2009	Ratio-form system efficiency
Liu et al. 2010	Independent model
Wang and Chin 2010	Ratio-form process efficiency
Fukuyama and Weber 2010	Slacks-based measure model
Kao and Liu 2011	Ratio-form process efficiency model

Yang et al. 2011	System distance measure model
Yang and Liu 2012	Ratio-form system efficiency model
Yu 2012	Factor distance measure model
Liu and Lu 2012	Process distance measure model
Chen et al. 2013	Factor distance measure model
Lewis et al. 2013	System distance measure model
Zhou et al. 2013	Game theoretic model

The market usually poses threats that can be overcome by restructuring through consolidation for cooperation. A common form of a merger is when two or more firms combine their activities with the aim of performance improvement, which results in a new merger. A predefined performance target usually accompanies a restructuring decision. There is an abundance of studies in DEA literature that discuss the significance of a firm's consolidation. These studies cover a wide range of applications, such as healthcare (Leleu et al. 2012), banking (Halkos and Tzeremes, 2013), and airlines (Kong et al. 2012). Although it is a useful analytical tool for evaluating various alternatives, one cannot use the conventional DEA approach to determine the levels of production factors in a firm for a given efficiency score. Unlike the conventional approach, which aims to calculate the efficiency score of a given DMU, inverse DEA (InvDEA) considers efficiency as a given parameter and calculates the input and output quantities that are necessary for reaching the pre-specified efficiency level.

The idea of inverse DEA was initially introduced by Zhang and Cui (1999), although Wei et al. (2000) were the first to formally study inverse DEA in order to estimate output (input) levels and increase (decrease) output (input) levels while maintaining a constant efficiency score. Multi-objective linear programming (MOLP) can be used within the framework of inverse DEA to answer questions of the following nature: How much more outputs can a given DMU produce if some of its inputs are increased and its current efficiency level is assumed to remain unchanged among a group of DMUs? Or, how much more inputs should the DMU consume if the output level needs to be increased to a certain level while efficiency remains unchanged? In the context of DMU mergers, if the models corresponding to the DMUs are infeasible, merger will not be possible. Unit mergers can be carried out more appropriately when the priorities of input and output vectors are taken into consideration. For instance, in the merger of university branches, faculty members play a central role, and thus, have the highest priority.

An inverse optimization problem for a given feasible solution of an optimization problem involves identifying the smallest perturbation in the objective function coefficients of the optimization problem so that we can achieve the optimal feasible solution of the perturbed model. Recent works on the subject of inverse optimization include the studies conducted by Chow and Recker (2012) and Wang et al. (2014).

After Wei et al. (2000), inverse DEA studies were further developed by a number of researchers, and the approach was employed in many theoretical and applied contexts. Authors such as Amin and Emrouznejad (2007a), Amin and Emrouznejad (2007b), and Jiong et al. (2011) discussed inverse linear programming in their studies. Gattoufi et al. (2014) extended the concept of inverse DEA and proposed an application in bank mergers. Ghiyasi (2015) introduced an inverse VRS model for a resource allocation problem. Lim (2016) presented an InvDEA model with frontier change in order to set a new production target. Ghiyasi (2017) investigated inverse DEA problems in cases where price information is available. Amin and Al-moharrami (2016) proposed a new inverse DEA method for mergers with negative data. Table 2 contains information on some other studies carried out in the context of inverse DEA. Amin et al. (2017) proposed a novel method for determining minor and major consolidations in the market. When the merger of two or more decision-making units has no effect on the efficiency frontier, as it was defined before consolidation, the merger is minor. Otherwise, the consolidation is defined as a major one. In the present paper, we seek to explore the mergers of a number of university branches, insurance companies, and commercial banks, as DMUs, using inverse DEA within the framework of two-stage network DEA.

In two-stage network DEA, the efficiency measures of the first and second stages are calculated separately, and the overall efficiency is then obtained by considering the constraints of both stages at the same time. The relationship between the efficiency measure of each stage and the overall efficiency is of great significance. Obviously, merging DMUs in two-stage network DEA is quite a crucial matter due to the competitive atmosphere among DMUs, centralized use of resources and assets, and the presence of intermediate data, which play a key role. Recently, inverse DEA has made it possible to merge DMUs with the aim of reducing inputs and increasing outputs; though, this gives rise to problems such as the inefficiency of merged units. Such problems can be overcome through the use of scientific methods, and certain strategies can be adopted in order to make the merged units efficient.

Next, a few differences will be pointed out between the current study and recent studies on inverse DEA. In the present study, we have proposed two separate inverse DEA efficiency models for the first and second stages. Furthermore, an overall inverse DEA network model is presented taking both stages into account. We have separately specified the restrictions placed on input and output vectors for the merger of DMUs in our two-stage network inverse DEA. Moreover, we have merged the priorities of inputs, intermediate measures, and outputs in the two-stage network inverse DEA.

Table 2. Studies on the inverse DEA model

Author	Description
Yan et al. (2002)	Resource allocation problem for guiding an inefficient DMU shifting in the direction of a projected ray from its current location to the frontier
Jahanshahloo et al. (2004)	Input/output estimation in the presence of some undesirable factors
Jahanshahloo et al. (2005)	Sensitivity of efficiency classification in InvDEA models
Hadi-vencheh and Foroughi (2006)	Proposal of GInvDEA for input/output estimation
Liu et al. (2011)	Investigation into the effects of mergers and acquisitions on corporate performances
Frija et al. (2013)	Proposal of a novel data-driven deductive methodology for estimating the impact of increasing water prices in Tunisia
Hadi-vencheh et al. (2015)	Presentation of an InvDEA model in the presence of imprecise data
Zhang and Cui (2016)	Presentation of an extension of the InvDEA model

The papers listed in Table 2 involve the use of inverse DEA models for assisting the managers with their decisions regarding DMU mergers in order to reach pre-defined efficiency goals. The current paper is organized as follows: Section 2 contains certain preliminaries. Section 3 proposes a novel model for estimating the input and output levels in a network structure by combining the concepts of InvDEA and network DEA. Some applications are presented in section 4 to elaborate on the computational method of the proposed model, and significant conclusions are drawn in the final section.

2. Preliminaries

The basic concepts of DEA, Inverse DEA (InvDEA), and Network DEA (NDEA) are briefly explained in this section.

2.1 Basic DEA model

Suppose that *n* DMUs consume *m* inputs to produce *s* outputs. The input and output vectors are presented as $X_j = \left(x_{1j},\ldots,x_{ij}\right)$, i=1,…,m and $Y_j = \left(y_{1j},\ldots,y_{rj}\right)$, r=1,…,s, respectively. For all $j = 1,2,\ldots,n,$ we have $x_{ij} > 0$ and $y_{rj} > 0$. Model (1) is the input-oriented radial envelopment form used to evaluate DMU_o , $o \in \{1,\ldots,n\}$, considering three axioms: inclusion of observation, convexity, and free disposability .

$$
\begin{aligned}
\theta_o = Min \quad & \theta \\
s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij} \le \theta x_{io} \quad , \quad i = 1,\ldots,m \\
& \sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{ro} \quad , \quad r = 1,\ldots,s \qquad (1) \\
& \sum_{j=1}^{n} \lambda_j = 1 \\
& \lambda_j \ge 0 \qquad , \quad j = 1,\ldots,n.
\end{aligned}
$$

Introduced by Banker et al. in 1984, model (1), known as the BCC model, is the first basic DEA model under VRS assumption. By solving model (1) for a given DMU_o, $o = 1,\ldots,n,$ we arrive at the unit's technical efficiency score. Solving model (1) also yields a set of efficient peers for any DMU. These efficient peers are on the efficient frontier and are regarded as reference points for evaluating the efficiency of a given (inefficient) DMU. In this model, θ_o is called the input-oriented efficiency score of DMU_o.

$$
\begin{aligned}
Max \quad & \sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+ \\
s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = \theta x_{io} \quad , \quad i = 1,\ldots,m \\
& \sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = y_{ro} \quad , \quad r = 1,\ldots,s \qquad (2) \\
& \sum_{j=1}^{n} \lambda_j = 1 \\
& s_i^- \ge 0, \ s_r^+ \ge 0, \ \lambda_j \ge 0 \qquad , \quad i = 1,\ldots,m, \ r=1,\ldots,s, \ j = 1,\ldots,n.
\end{aligned}
$$

Note that the values assigned to s_i^- and s_r^+ do not affect the optimal value of θ_o , which is determined from the model (1).

Definition 1. Strong Pareto efficiency: The performance of DMU_O is strongly Pareto efficient if and only if $\theta_o = 1$ in Model (1), and all slacks $s_i^{-*} = s_r^{+*} = 0$ in Model (2).

Definition 2. Weak Pareto efficiency: The performance of DMU_O is weakly Pareto efficient if and only if $\theta_o = 1$ in Model (1), and $s_i^{-*} \ne 0$ and/or $s_r^{+*} \ne 0$ for some i or r .

The output-oriented version of model (1) is as follow:

$$\begin{aligned}
\phi(X_o,Y_o) = Max \quad & \phi \\
s.t. \quad & \sum_{j=1}^{n} \mu_j x_{ij} \le x_{io} \quad , \quad i=1,\dots,m \\
& \sum_{j=1}^{n} \mu_j y_{rj} \ge \phi y_{ro} \quad , \quad r=1,\dots,s \qquad (3) \\
& \sum_{j=1}^{n} \mu_j = 1 \\
& \mu_j \ge 0 \quad , \quad j=1,\dots,n.
\end{aligned}$$

Here, $\phi_o = \phi(X_o,Y_o)$ is called the output-oriented efficiency score of DMU_o.

The following model is used to evaluate DMU_o under VRS assumption by simultaneously reducing the inputs and increasing the outputs.

$$\begin{aligned}
Min \quad & \tilde{\theta} - \tilde{\phi} \\
s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij} \le \tilde{\theta} x_{io}, \quad i=1,\dots,m \\
& \sum_{j=1}^{n} \lambda_j y_{rj} \ge \tilde{\phi} y_{ro}, \quad r=1,\dots,s \qquad (4) \\
& \sum_{j=1}^{n} \lambda_j = 1 \\
& \lambda_j \ge 0, \quad j=1,\dots,n.
\end{aligned}$$

A given DMU under evaluation is called weakly-efficient if and only if $\tilde{\theta}=1$ and $\tilde{\phi}=1$. In Model (4), If we consider the slacks variables related to input and output restrictions as $\left(\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+\right)$, and then maximize it, the DMU under evaluation would become strongly efficient whenever $s_i^- = 0$ and $s_r^+ = 0$ for each i and r.

2.2 Network DEA

The simplest network structure is the basic two-stage structure, in which all external inputs $\left(x_{1j},\ldots,x_{mj}\right)$ are consumed in the first stage to produce intermediate measures $\left(z_{1j},\ldots,z_{dj}\right)$ and the final outputs $\left(y_{1j},\ldots,y_{sj}\right)$ are produced in the second stage. Only the input vectors and the intermediate measures are taken into account in the evaluation of the first stage. Intermediate measures are the outputs of the first stage. In the evaluation of the second stage, $\left(y_{1j},\ldots,y_{sj}\right)$ and $\left(z_{1j},\ldots,z_{dj}\right)$ are taken into consideration.

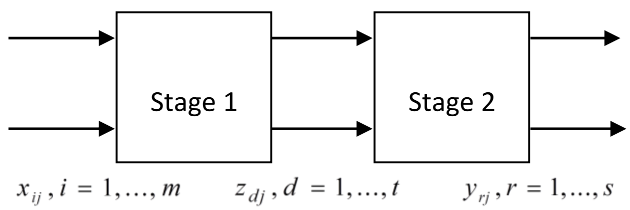


Fig. 1. Two stage structure

2.3. Inverse DEA

The first study in inverse DEA was carried out by Wei et al. (2000). In an InvDEA model, the problem of determining the best output (input) level for a given input (output) level is discussed with the condition that the optimal value of the DEA model remains constant. The following cases are investigated in an InvDEA problem:

1) How much should the inputs of a given DMU_o increase if the input level was to increase while the efficiency measure ϕ_o remains unchanged?

In this regard, assume that the inputs of DMU_o increase from x_{io} , $i=1,\ldots,m$ to $\alpha_{io} = x_{io} + \Delta x_{io}$, $(i=1,\ldots,m)$, where $\Delta x_{io} \geq 0$ and $\Delta x_{io} \neq 0$ (i.e. at least one component increases, and it is also allowed for several or even all components to increase). The objective of the problem is to estimate the output vector $\beta_{ro} = y_{ro} + \Delta y_{ro}$, $(r=1,\ldots,s)$, in a way that the efficiency measure ϕ_o remains unchanged. Wei et al. (2000) proposed the InvDEA model as follows.

$$
\begin{aligned}
& Max\left(\beta_{1o}, \beta_{2o}, \ldots, \beta_{so}\right) \\
& s.t. \quad \sum_{j=1}^{n} \lambda_j x_{ij} \leq \alpha_{io} \quad , \quad i=1,\ldots,m \\
& \qquad \sum_{j=1}^{n} \lambda_j y_{rj} \geq \phi_o \beta_{ro} \ , \quad r=1,\ldots,s \qquad (5) \\
& \qquad \sum_{j=1}^{n} \lambda_j = 1 \\
& \qquad \beta_{ro} \geq y_{ro} \qquad , \quad i=1,\ldots,m \\
& \qquad \lambda_j \geq o \qquad , \quad j=1,\ldots,n.
\end{aligned}
$$

The MOLP model (5) is linearized and solved to determine the maximum possible increase in output production. The variables vector in model (5) is (β_o, λ). All x_{ij}, y_{rj}, and α_{io} are given in the model, and β_{ro} is to be obtained. In this model, ϕ_o is the optimal value of the linear programming problem (3).

Theorem 1. (Wei et al. 2000). Suppose that $\phi_o > 1$, and the inputs for DMU_o are going to increase from x_{io} , $i=1,\ldots,m$ to $\alpha_{io} = x_{io} + \Delta x_{io}$, $(i=1,\ldots,m)$, where $\Delta x_{io} \geq 0$ and $\Delta x_{io} \neq 0$.

(i) Let (λ^*, β_o^*) be a weak Pareto solution of MOLP (5). Then, when the outputs of y_{ro}, $(\mathrm{r}=1,\ldots,\mathrm{s})$ are increased to β^*, we have $\varphi(\alpha_o, \beta_o^*) = \varphi(X_o, Y_o)$.

(ii) Conversely, let (λ^*, β_o^*) be a feasible solution of MOLP (5). If $\phi(\alpha_o, \beta_o) = \phi(X_o, Y_o)$, then (λ^*, β_o^*) is a weak Pareto solution of MOLP (5).

(2) How much should the inputs of a given DMU_o increase if the outputs were to increase while the efficiency measure θ_o in the linear programming problem (1) remains unchanged?

In this case, suppose the outputs of DMU_o increase from y_{ro}, $(\mathrm{r}=1,\ldots,\mathrm{s})$ to $\beta_{ro} = y_{ro} + \Delta y_{ro}$, $(r=1,\ldots,s)$, where $\Delta y_{ro} \geq 0$ and $\Delta y_{ro} \neq 0$ (i.e. at least one component increases, and it is also allowed for several or even all components to increase). In the following InvDEA problem (6), the input vector $\alpha_{io} = x_{io} + \Delta x_{io}$, $(i=1,\ldots,m)$ is estimated in a way that the efficiency measure θ_o remains constant. (Hadi-Vencheh et al. 2008).

$$
\begin{aligned}
& Min\left(\alpha_{1o}, \alpha_{2o}, \ldots, \alpha_{mo}\right) \\
& s.t. \quad \sum_{j=1}^{n} \mu_j x_{ij} \leq \theta_o \alpha_{io}, \quad i = 1, \ldots, m \\
& \qquad \sum_{j=1}^{n} \mu_j y_{rj} \geq \beta_{ro}, \quad r = 1, \ldots, s \qquad (6) \\
& \qquad \sum_{j=1}^{n} \mu_j = 1 \\
& \qquad \alpha_{io} \leq x_{io}, \quad i = 1, \ldots, m \\
& \qquad \mu_j \geq 0, \quad j = 1, \ldots, n.
\end{aligned}
$$

The variables vector in model (6) is (α_o, μ). All x_{ij}, y_{rj}, and β_{ro} are given in the model and α_{io} is to be obtained. In this model, θ_o denotes the optimal value of the linear programming problem (1).

Model (6) is a MOLP model, which can be solved using the weighted sum method, lexicography, or interactive methods. This model was presented by Hadi-Vencheh et al. (2008) for estimating $\left(\alpha_{io}, \beta_{ro}\right)$ with adherence to the axioms of data envelopment analysis.

Theorem 2. Suppose that $\left(\alpha_o^*, \mu^*\right)$ is a weak Pareto solution of model (6). Then, $\theta\left(\alpha_o^*, \mu\right) = \theta\left(X_o, Y_o\right)$ (Hadi-Vencheh et al. 2008).

Definition 3. "Minor and major consolidation": If a merger affects the pre-consolidation efficiency frontier, then it is called a major consolidation. Otherwise, it is known as a minor consolidation. (See Amin et al. 2017).

In order to convert double-objective and triple-objective programming models, we can use methods such as lexicography or the weighted sum method. To employ the weighted sum method, by setting the weights as $W = \left(w_1, \ldots, w_s\right)$, $\sum_{r=1}^{s} w_r = 1$, $w_r > 0$, the objective function of a double-objective programming model can be considered as max $\sum_{r=1}^{s} w_r \beta_r$. Similarly, the objective function of a triple-objective programming model can be formulated as max $\sum_{i=1}^{m} v_i \alpha_i$ by considering the weights $V = \left(v_1, \ldots, v_m\right)$, $\sum_{i=1}^{m} v_i = 1$, $v_i > 0$. In regard to the procedures involved in lexicography see Steuer (1986).

2.4. Shannon's entropy formula

This section introduces Shannon's (1948) entropy formula for determining the degree of relevance (weight) of input, intermediate data, and output vectors. First, we consider the values corresponding to the input, intermediate data, and output vectors in separate matrices. For instance, suppose the matrix corresponding to the input values is as follows.

$$M = \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ \vdots & \vdots & & \vdots \\ r_{m1} & r_{m2} & \cdots & r_{mn} \end{pmatrix}_{m\times n}$$

Now, with regard to the information in Matrix M corresponding to each input, intermediate measure, and output, Shannon's entropy formula consists of the following four steps.

Step 1. Normalization: Set $\bar{M}_{ij} = \dfrac{r_{ij}}{\sum_{i=1}^{m} r_{ij}}, \quad i = 1,\ldots,m \quad , \quad j = 1,\ldots,n$.

Step 2. Calculate the entropy e_k as $e_n = -e_o \sum_{i=1}^{m} \bar{M}_{ij} \cdot \ln \bar{M}_{ij} \quad , \quad \mathrm{j} = 1,\ldots,\mathrm{n}$, where e_o is the entropy constant calculated as $e_o = (\ln n)^{-1}$.

Step 3. Let $f_j = 1 - e_j \;, \quad \mathrm{j} = 1,\ldots,\mathrm{n}$.

Step 4. Let $w_j = \dfrac{f_j}{\sum_{t=1}^{n} f_t}$, where w_j is the corresponding degree of importance.

Shannon's entropy formula is used to calculate the weights of w_j corresponding to the indexes r_{ij}. Weights obtained through Shannon's entropy formula can be used to assign weights to input and output variables in DEA. For further information, refer to soliemani-damaneh and Zarepisheh (2009).

3. Network Inverse DEA (NInvDEA) and suggested models

It is obvious that network DEA-based efficiency evaluation for the purposes of DMU mergers is dependent on the desirable performance of DMUs in separate network stages. Therefore, by focusing on stage 1 and using the InvDEA technique, we would be able to find suitable targets in terms of target efficiency. In a two-stage network structure, suppose that we have n DMUs, where the jth DMU, $j=(1,\ldots,n)$ consumes m inputs $X_j=(x_{1j},\ldots,x_{mj})$ to produce the intermediate measures $Z_j=(z_{1j},\ldots,z_{dj})$ and s outputs $Y_j=(y_{1j},\ldots,y_{sj})$ are produced in the second stage. In this research, mergers occur in stage one, stage two, and the overall stage. In the overall mode, there is no need for the mergers to occur in any specific order. Meanwhile, when we are dealing with DMU mergers, the proposed models can be of use.

Suppose that DMUs k and l are going to merge in order to consolidate their activities. Let $\tilde{M}$ denote the merged DMU and T represent the index set of all DMUs except k and l.

Set $I=\{1,\ldots,m\}=I_1\cup I_2$ and $i\in I=I_1\cup I_2$, $i=1,\ldots m$. Similarly, for the set of output indexes, suppose that $S=\{1,\ldots,s\}=S_1\cup S_2$, and $r\in S=S_1\cup S_2$, $r=1,\ldots,s$, and for indexes relating intermediate measures, set $D=\{1,\ldots,t\}=D_1\cup D_2$ and $d\in D=D_1\cup D_2$, $d=1,\ldots,t$.

Suppose that x_i^M and z_d^M correspond to the inputs and intermediate measures of the merged unit, respectively. Then, the input-oriented NInvDEA model for the consolidation of DMUs in the first stage of a two-stage network structure is proposed as follows.

$$\begin{aligned}
Min\ & \sum_{i\in I_1\cup I_2} p_i\alpha_i - \sum_{d\in D_1\cup D_2} f_d\gamma_d \\
s.t.\ & \sum_{j\in T}\lambda_j^1 x_{ij} \le \theta_o\alpha_i, \quad i\in I_1\cup I_2 \\
& \sum_{j\in T}\lambda_j^1 z_{dj} \ge \gamma_d, \quad d\in D_1\cup D_2 \qquad (7)\\
& \sum_{j\in T}\lambda_j^1 = 1 \\
& 0\le\alpha_i\le x_i^M, \quad i\in I_1\cup I_2 \\
& \gamma_d \ge z_d^M, \quad d\in D_1\cup D_2 \\
& \lambda_j^1\ge 0, \quad j\in T.
\end{aligned}$$

In model (7), the input weights are obtained through Shannon's entropy formula. The target efficiency measure is pre-determined. p_i is a factor specifying the amount of reduction in our inputs. In the first and second constraints, the goal is to reduce the inputs. Furthermore, α_i cannot take any value and is limited to values varying between zero and a given input component. The solution to model (7) represents the maximum possible and required amounts of decrease in the inputs of the merged entity. In this model, θ_o is the optimal value of the linear programming problem (1). In Model (7), by considering λ_j^1, which corresponds to DMU_j in the first stage, we seek to obtain α_{i1} and α_{i2} by adding the restrictions $0 \le \alpha_{i1} \le x_{i1}^M$ and $0 \le \alpha_{i2} \le x_{i2}^M$ under the VRS assumption. The objective in this model is to reduce $\sum_{i \in I_1 \cup I_2} p_i \alpha_i$ and increase $\sum_{d \in D_1 \cup D_2} f_d \gamma_d$.

The objective function of Model (7) is formulated as $\min\ (\alpha_1, \ldots, \alpha_m, \gamma_1, \ldots, \gamma_d)$, in which the weights of p_i and f_d obtained through Shannon's entropy formula for Matrices X and Z, respectively, are used. Although, only in the condition that p_i and f_d take positive values. Therefore, we have: $\min \left(\sum_{i \in I_1 \cup I_2} p_i \alpha_i - \sum_{d \in D_1 \cup D_2} f_d \gamma_d \right)$.

Theorem (2) is specific to Model (6) and cannot be extended to Model (7), as Model (7) is a single-objective linear programming model and does not require the use of MOLP solving procedures like Model (6).

3.1 Network InvDEA model for stage 2

As in the case of the first stage, model (8), presented in the following, can be considered as the NInvDEA model corresponding to the second stage. Only the intermediate products $(z_{1j}, \ldots, z_{dj})$, $d = (1, \ldots, t)$ and the final outputs $(y_{1j}, \ldots, y_{rj})$, $r = (1, \ldots, s)$ are involved in the evaluation. The weights corresponding to the outputs are computed using Shannon's entropy and are denoted by q_r , $\mathrm{r} = (1, \ldots, \mathrm{s})$ in the model.

$$
\begin{aligned}
&Min \quad \sum_{d\in D_1\cup D_2} f_d\gamma_d - \sum_{r\in S_1\cup S_2} q_r\beta_r \\
&s.t. \quad \sum_{j\in T}\lambda_j^2 z_{dj} \le \gamma_d, \qquad \mathrm{d}\in D_1\cup D_2 \\
&\qquad \sum_{j\in T}\lambda_j^2 y_{rj} \ge \phi_o\beta_r, \quad \mathrm{r}\in \mathrm{S}_1\cup \mathrm{S}_2 \qquad (8) \\
&\qquad \sum_{j\in T}\lambda_j^2 = 1 \\
&\qquad \beta_r \ge y_r^M, \qquad \mathrm{r}\in S_1\cup \mathrm{S}_2 \\
&\qquad \gamma_d \le z_d^M, \qquad \mathrm{d}\in \mathrm{D}_1\cup D_2 \\
&\qquad \lambda_j^2 \ge 0, \qquad \mathrm{j}\in \mathrm{T}.
\end{aligned}
$$

The objective function of Model (8) is defined as $\min\ (\gamma_1,\ldots,\gamma_t,-\beta_1,\ldots,-\beta_s)$, in which the weights of f_d and q_r obtained through Shannon's entropy formula for Matrices Z and Y, respectively, are used (in the condition that they take positive values). Therefore, the objective function is converted into $\min\left(\sum_{d\in D_1\cup D_2} f_d\gamma_d - \sum_{r\in S_1\cup S_2} q_r\beta_r\right)$. The linear programming problem (8) aims to reduce $\sum_{d\in D_1\cup D_2} f_d\gamma_d$ and increase $\sum_{r\in S_1\cup S_2} q_r\beta_r$. In this model, the minimum value of γ_d and the maximum value of β_r are calculated for DMU_j under the VRS assumption by considering ϕ_o from Model (3) and λ_j^2 corresponding to the second stage. Note that in Model (8), the minimum value of β_r and the maximum value of γ_d correspond to y_r^M and z_d^M, respectively.

The objective in two-stage network DEA is to calculate the overall efficiency, which requires the efficiency measures of the first and second stages. There are two fundamental questions to be addressed in inverse DEA: ***a***) how should the merger take place for each stage separately? and ***b***) how should the merger take place for the two stages as a whole? Models (7) and (8) deal with question ***a***, while model (9) addresses question ***b***, as will follow. Model (9) is proposed for determining the input, intermediate measure, and output levels required to achieve the pre-defined target efficiency measure.

$$
\begin{aligned}
Min\ & \sum_{i\in I_1\cup I_2} p_i\alpha_i - \sum_{r\in S_1\cup S_2} q_r\beta_r - \sum_{d\in D_1} f_d\gamma_d + \sum_{d\in D_2} \tilde{f}_d\gamma_d \\
s.t.\ & \sum_{j\in T}\lambda_j^1 x_{ij} \le \tilde{\theta}\alpha_i, \quad i\in I_1\cup I_2 \\
& \sum_{j\in T}\lambda_j^1 z_{dj} \ge \gamma_d, \quad d\in D_1 \qquad (9) \\
& \sum_{j\in T}\lambda_j^2 y_{rj} \ge \tilde{\phi}\beta_r, \quad r\in S_1\cup S_2 \\
& \sum_{j\in T}\lambda_j^2 z_{dj} \le \gamma_d, \quad d\in D_2 \\
& \sum_{j\in T}\lambda_j^1 z_{dj} \ge \sum_{j\in T}\lambda_j^2 z_{dj}, \quad d=1,\dots,t \\
& \sum_{j=1}^{n}\lambda_j^1 = 1, \quad \sum_{j=1}^{n}\lambda_j^2 = 1 \\
& 0\le \alpha_i \le x_i^M \quad, i\in I_1\cup I_2 \\
& \gamma_d \ge z_{d1} \quad , \quad d\in D_1 \quad , \quad \gamma_d \le z_{d2} \quad , \quad d\in D_2 \\
& \beta_r \ge y_r^M \quad , \quad r\in S_1\cup S_2 \\
& \lambda_j^1, \lambda_j^2 \ge 0, \quad j\in T.
\end{aligned}
$$

In this model, p_i, $(i=1,\dots,m)$, q_r, $(r=1,\dots,s)$, f_d, $(d=1,\dots,t)$, and $\tilde{f}_d$, $(d=1,\dots,t)$ are priorities assigned to the inputs, outputs and intermediate measures of the merged DMUs, respectively, obtained through Shannon's entropy formula. The first stage of our network structure is completely independent of the second stage. Therefore, the first stage is evaluated independently. The second stage is also evaluated independently based on parameters related to outputs and intermediate measures. In this model, the objective function is defined as $\min\ (\alpha_1,\dots,\alpha_m,-\beta_1,\dots,-\beta_s,-\gamma_1,\dots,-\gamma_d)$. Therefore, using Shannon's entropy formula, the weights of p_i, q_r f_d and $\tilde{f}_d$ are calculated for Matrices X, Y, Z_{d1} and Z_{d2}, respectively (weights must be positive). Thereby, the objective function is converted into $\min\left(\sum_{i=1}^{m} p_i\alpha_i - \sum_{r=1}^{s} q_r\beta_r - \sum_{d=1}^{D} f_d\gamma_d + \sum_{d=1}^{D} \tilde{f}_d\gamma_d\right)$. Each optimal solution of model (9) would be a feasible solution for models (7) and (8).

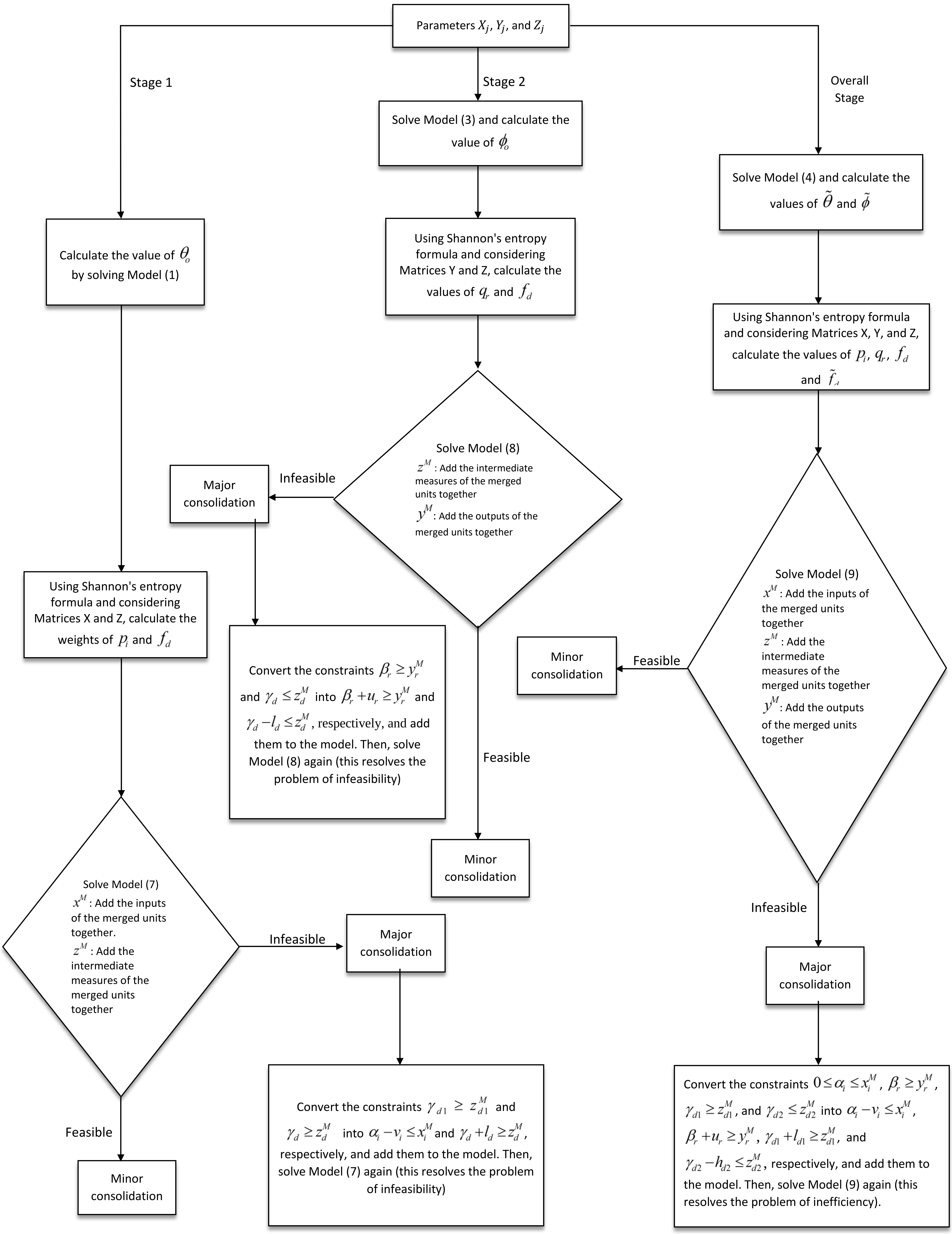
It is interesting to point out that if a DMU was efficient in either of the first or second stages, we can use the proposed models to evaluate the overall efficiency and clearly determine the relationships between overall efficiency and efficiency in any stages of the network structure. The restrictions in Model (8) are determined based on the first stage; however, the model and its restrictions relate to the second stage themselves. The combined restrictions of stages one and two correspond to the overall network in Model (9). Therefore, if a DMU had an efficiency score of 1 in both stages one and two, it will obviously be efficient in the overall evaluation. The main problem in network DEA lies in the relationships between the efficiency of stages one and two and the overall efficiency. Thereby, in the present research, λ_j^1 corresponds to DMU_j in the first stage in Model (7). Similarly, in Model (8), λ_j^2 corresponds to DMU_j in the second stage. In Model (9), which aims to evaluate the overall network, the variables λ_j^1 and λ_j^2 correspond to DMU_j in the first and second stage, respectively. It must be noted that all models adhere to the main axioms of data envelopment analysis, namely "inclusion of observation", "convexity", and "free disposability".

In Model (9), $\tilde{\theta}$ and $\tilde{\phi}$ denote the optimal solutions to Model (4). In this model, DMU_o is projected onto the efficient frontier with the aim to simultaneously reduce the inputs and increase the outputs. Thereby, combining the restrictions $\sum_{j=1}^{n} \lambda_j x_{ij} \leq \tilde{\theta}\alpha_i$ and $\sum_{j=1}^{n} \lambda_j y_{rj} \geq \tilde{\phi}\beta_r$ Model (9) would provide the minimum and maximum values for $\sum_{i=1}^{m} p_i \alpha_i$ and $\sum_{j=1}^{n} \lambda_j y_{rj} \geq \tilde{\phi}\beta_r$ respectively. The link between models (7), (8), and (9) lies in this blending of restrictions in the overall evaluation. As a recommendation for future research, it is particularly important to specify the exact link between the aforementioned models.

The following algorithm is presented in regard to the use of models (7), (8) and (9) for determining minor and major consolidations in network InvDEA.

In Algorithm (1), the value of θ_o is first calculated for the first stage by solving Model (1). Then, the weights of p_i and q_r and f_d are calculated using Shannon's entropy formula, as explained in section two. Next, with consideration to Model (9), the merger would be called a minor consolidation if the model was feasible. If Model (9) was infeasible, we convert the constraints and deem the consolidation a major one.

For the overall network, the value of $\tilde{\theta}$ and $\tilde{\phi}$ is initially calculated by solving Model (4). Then, with the help of Shannon's entropy formula, we calculate the weights of p_i, q_r, f_d, and $\tilde{f}_d$ corresponding to Matrices X, Y, Z_{d1}, and Z_{d2}, respectively. To determine the consolidation type, we take Model (9) into consideration. If the model was feasible, the consolidation would be called a minor one, and otherwise, it would be a major consolidation, in which case, the following modified constraints are considered in the model: $\alpha_i - v_i \leq x_i^M$, $\beta_r + u_r \geq y_r^M$, $\gamma_{d1} + l_{d1} \geq z_{d1}^M$, and $\gamma_{d2} - h_{d2} \leq z_{d2}^M$.


Parameters X_j, Y_j, and Z_j
Stage 1
Stage 2
Overall Stage
Solve Model (3) and calculate the value of ϕ_o
Solve Model (4) and calculate the values of $\tilde{\theta}$ and $\tilde{\phi}$
Calculate the value of θ_o by solving Model (1)
Using Shannon's entropy formula and considering Matrices Y and Z, calculate the values of q_r and f_d
Using Shannon's entropy formula and considering Matrices X, Y, and Z, calculate the values of p_i, q_r, f_d and $\tilde{f}_d$
Solve Model (8)
z^M : Add the intermediate measures of the merged units together
y^M : Add the outputs of the merged units together
Infeasible
Major consolidation
Using Shannon's entropy formula and considering Matrices X and Z, calculate the weights of p_i and f_d
Convert the constraints $\beta_r \geq y_r^M$ and $\gamma_d \leq z_d^M$ into $\beta_r + u_r \geq y_r^M$ and $\gamma_d - l_d \leq z_d^M$, respectively, and add them to the model. Then, solve Model (8) again (this resolves the problem of infeasibility)
Feasible
Minor consolidation
Solve Model (9)
x^M : Add the inputs of the merged units together
z^M : Add the intermediate measures of the merged units together
y^M : Add the outputs of the merged units together
Feasible
Minor consolidation
Infeasible
Major consolidation
Solve Model (7)
x^M : Add the inputs of the merged units together.
z^M : Add the intermediate measures of the merged units together
Infeasible
Major consolidation
Feasible
Minor consolidation
Convert the constraints $\gamma_{d1} \geq z_{d1}^M$ and $\gamma_d \geq z_d^M$ into $\alpha_i - v_i \leq x_i^M$ and $\gamma_d + l_d \geq z_d^M$, respectively, and add them to the model. Then, solve Model (7) again (this resolves the problem of infeasibility)
Convert the constraints $0 \leq \alpha_i \leq x_i^M$, $\beta_r \geq y_r^M$, $\gamma_{d1} \geq z_{d1}^M$, and $\gamma_{d2} \leq z_{d2}^M$ into $\alpha_i - v_i \leq x_i^M$, $\beta_r + u_r \geq y_r^M$, $\gamma_{d1} + l_{d1} \geq z_{d1}^M$, and $\gamma_{d2} - h_{d2} \leq z_{d2}^M$, respectively, and add them to the model. Then, solve Model (9) again (this resolves the problem of inefficiency).

Algorithm 1. Algorithm illustrating the use of models (7), (8) and (9) for determining minor and major consolidations in network InvDEA.

4. An empirical illustration

In this section, some applications of the proposed network inverse DEA (NInvDEA) model are provided in order to demonstrate its utility. In 4.1, the proposed models are applied to a data set extracted from five university branches in Iran (Table 3). In 4.2, 24 Taiwanese non-life insurance companies are evaluated using the proposed models, and the model is employed to evaluate 20 commercial banks in 4.3.

4.1. Case 1. Real- life application: Five university branches in Iran

Consider five university branches, each consuming two inputs and one intermediate measure to produce two final output, as presented in Table 3, in a two-stage network structure. The number of students and the number of staff are the inputs of the first stage, appearing in the second and third columns of Table 3, respectively. In Iran, students are initially admitted into postgraduate programs as education-oriented students, and then have to go through a research-oriented phase after graduating from the education-oriented stage. In the context of our study, education-oriented graduates are considered as the intermediate measures of the two-stage network (fourth column). Tuition fees and research-oriented graduates are the final outputs (fifth and sixth columns, respectively). The efficiency measure is considered equal to 1 for all branches. The purpose of using InvDEA in the case of university branches was to be able to set the efficiency measure equal to 1 for the merged units before analysis.

Table3. Factors used in the numerical example

Factor	Notation	Definition
Stage 1 inputs	X_{ij}^{1}	Number of students
Stage 1 inputs	X_{ij}^{2}	Number of staff
Stage 1 outputs (intermediate products)	Z_{dj}^{1}	Education-oriented graduates
Stage 2 outputs	Y_{rj}^{1}	Tuition fees
Stage 2 outputs	Y_{rj}^{2}	Research-oriented graduates

Table 4. Inputs, Intermediate measures, and outputs for the university branches under study

DMU	X_{ij}^{1}	X_{ij}^{2}	Z_{dj}^{1}	Y_{rj}^{1}	Y_{rj}^{2}
1	3667	98	2088	91675000000	1566
2	18671	378	9839	560130000000	6788
3	10738	224	5669	300664000000	4251
4	1813	44	1162	41699000000	847
5	2973	55	1205	357900000	953

Now, assume that the activities of DMUs 1 and 5 in the first stage are combined. Let $\tilde{M}$ and θ denote the merged unit and the target efficiency score, respectively. We intend to determine the input and intermediate measure levels required for the merged university $\tilde{M}$ in order to achieve the pre-defined target efficiency score. Consider the target efficiency score equal to 1 for the merged university, i.e., $\theta = 1$. Model (7) is infeasible. However, by modifying the constraints $0 \le \alpha_i \le x_i^M$ as $0 \le \alpha_i - v_i \le x_i^M$ and solving the model again, the following results will be obtained.

$$\begin{cases} \left(\alpha_1^*, \alpha_2^*, \gamma^*\right) = (1813, 44, 1162) \\ \left(v_1, v_2, u\right) = (0, 0, 2131). \end{cases}$$

That is to say, the number of students in the merged university $\tilde{M}$ must equal 1813, which can be achieved through a reduction by $4827(= 6640 - 1813)$. Similarly, there should be 44 staff members in $\tilde{M}$, which implies the need for a decrease in the number of staff by $109(= 153 - 44)$ .The number of education-oriented graduates in $\tilde{M}$ should equal 1162, indicating a reduction by $2131(= 3293 - 1162)$.

If model (7) was infeasible, we substitute the restrictions $\alpha_i \le x_i^M$ and $\gamma_d \ge z_d^M$ by the restrictions $0 \le \alpha_i - v_i \le x_i^M$ and $\gamma_d - l_d \ge z_d^M$, and solve the model again. The optimal solution of Model (7) determines our optimal value. To interpret this restriction, we have:

$\alpha_i - v_i \le x_i^M$, $v_i \ge 0$. Therefore, we will have: $\alpha_i - v_i + T_i = x_i^M$. Then, we set $T_i - v_i = \tilde{T}_i$, and we will have: $\alpha_i + \tilde{T}_i = x_i^M$. As can be observed, $\tilde{T}_i$ is free to take any sign, which solves our problem of infeasibility. The constraint $\gamma_d - l_d \ge z_d^M$ has a similar interpretation.

The merger between DMUs 1 and 5 is a major consolidation.

Now, consider the consolidation of the activities of DMUs 1 and 5 in the second stage. By solving model (8), the following optimal solution is obtained:

$$\begin{cases} \left(\beta_1^*, \beta_2^*, \gamma^*\right) = \left(5.6\times10^{11}, 6788, 9839\right) \\ \left(u_1, u_2, l\right) = \left(0, 0, 6546\right). \end{cases}$$

As can be seen, the required tuition fee for $\tilde{M}$ is 5.6×10^{11}, which means that the tuition fee should be increased. The number of education-oriented graduates must be 9839, achieved through an increase by $6546\left(=9839-3293\right)$. The required number of research-oriented graduates is 6788, implying the need for an increase by $4269\left(=6788-2519\right)$. Similar to the case of the first stage, the merger in the second stage is a major consolidation. By solving model (9) for $\tilde{M}$, we arrive at the optimal solution as follows:

$$\begin{cases} \left(\alpha_1^*, \alpha_2^*\right) = \left(18671, 378\right) \\ \left(\beta_1^*, \beta_2^*, \gamma^*\right) = \left(5.06\times10^{11}, 6788, 9839\right) \\ \left(v_1, v_2, u_1, u_2, l\right) = \left(12031, 225, 0, 0, 6546\right). \end{cases}$$

One can observe that the required number of students for $DMU_{\tilde{M}}$ is 18671, obtained by adding 12031 students. The number of staff must equal 378, achieved through an increase by 225. The number of education-oriented graduates should be 9839, indicating the need for an increase by 6546. The tuition fee is 5.06×10^{11}, meaning that the tuition fee should be decreased. Similar to the first and second stages, the overall merger is a major consolidation as well.

4.2. Case 2. Real-life application: Taiwanese non-life insurance companies

The data set of 24 Taiwanese non-life insurance companies are provided in Table 6. Operating expenses and insurance expenses are the two inputs of the first stage. Underwriting profit and investment profit are the two intermediate measures; that is to say, they are the outputs of the first stage used as inputs in the second stage. Direct written premiums and reinsurance premiums are the two outputs of the second stage. Table 5 presents the factors related to the insurance companies under study, as follows.

Table 5. Factors used in non-life insurance companies in Taiwan

Factor	**Notation**	**Definition**

Stage 1 inputs	X_{ij}^{1}	Operating expenses: salaries of employees and various types of costs incurred in daily operation
Stage 1 inputs	X_{ij}^{2}	Insurance expenses: expenses paid to agencies, brokers, and solicitors; and other expenses associated with marketing the insurance service
Stage 1 outputs (intermediate products)	Z_{dj}^{1}	Underwriting profit: profits earned from the insurance business
Stage 1 outputs (intermediate products)	Z_{dj}^{2}	Investment profit: profits earned from the investment portfolio
Stage 2 outputs	Y_{rj}^{1}	Direct written premium: premiums received from insured clients
Stage 2 outputs	Y_{rj}^{2}	Reinsurance premium: premiums received from ceding companies

The data set provided in Table 6 is extracted from Kao and Hwang (2008).

Table 6. Data set for 24 Taiwanese non-life insurance companies

DMU	**Company**	X_{ij}^{1}	X_{ij}^{2}	Z_{dj}^{1}	Z_{dj}^{2}	Y_{rj}^{1}	Y_{rj}^{2}
1	Taiwan Fire	1178744	673512	7451757	856735	984143	681687
2	Chung Kuo	1381822	1352755	10020274	1812894	1228502	834704
3	Tai Ping	1177494	592790	4776548	560244	293613	658428
4	China Mariners	601320	594259	3174851	371863	248709	177331
5	Fubon	6699063	3531614	37392862	1753794	7851229	3925272
6	Zurich	2627707	668363	9747908	952326	1713598	415058
7	Taian	1942833	1443100	10685457	643412	2239593	439039
8	Ming Tai	3789001	1873530	17267266	1134600	3899530	622868
9	Central	1567746	950432	11473162	546337	1043778	264098

10	The First	1303249	1298470	8210389	504528	1697941	554806
11	Kuo Hua	1962448	672414	7222378	643178	1486014	18259
12	Union	2592790	650952	9434406	1118489	1574191	909295
13	Shingkong	2609941	1368802	13921464	811343	3609236	223047
14	South China	1396002	988888	7396396	465509	1401200	332283
15	Cathay Century	2184944	651063	10422297	749893	3355197	555482
16	Allianz President	1211716	415071	5606013	402881	854054	197947
17	Newa	1453797	1085019	7695461	342489	3144484	371984
18	AIU	757515	547997	3631484	995620	692731	163927
19	North American	159422	182338	1141951	48329	51950	46857
20	Federal	145442	53518	316829	131920	355624	26537
21	Royal & Sunallianc e	84171	26224	225888	40542	51950	6491
22	Asia	15993	10502	52063	14574	82141	4181
23	AXA	54693	28408	245910	49864	0.1	18980
24	Mitsui Sumitomo	163297	235094	476419	644816	142370	16976

Suppose that DMUs 19 and 24 combine the totality of their activities (i.e., considering neither stage alone). Let $\tilde{M}$ and θ denote the merged unit and the target efficiency measure, respectively. Suppose that $\tilde{M}$ is efficient, i.e., $\theta = 1$. The inputs, intermediate measures, and outputs of the merged unit are the sums of inputs, intermediate measures, and outputs in DMUs 19 and 24, respectively. Since a merger is meant to bring about an improvement in the performance of the merging units, the target efficiency measure must lie within a range between the maximum efficiency measure of the merging entities and 1. To achieve a target efficiency score of 1, we determine the minimum input and intermediate measure levels and the maximum output level based on model (9). The optimal solution for model (9) is:

$$\begin{cases}\left(\alpha_1^*,\alpha_2^*\right)=\left(6699063,3531614\right)\\ \left(\beta_1^*,\beta_2^*\right)=\left(83141,4182\right)\\ \left(\gamma_1^*,\gamma_2^*\right)=\left(373928621,17537941\right).\end{cases}$$

Thereby, according to $\left(\alpha_1^*,\alpha_2^*\right)=\left(6699063,3531614\right)$, the merged unit $\tilde{M}$ must reduce its operating expenses and insurance expenses down to 99542 and 198949, respectively. Furthermore, underwriting profit should be reduced from 661491 to 592589 based on $\left(\beta_1^*,\beta_2^*\right)=\left(83141,4182\right)$, and investment profit should be maintained without any change; moreover, direct written premium should remain unchanged, and reinsurance premium should be reduced down to 835307. As another example, assume a merger between DMUs 18 and 19, resulting in the merged unit $\tilde{M}\cdot$ We consider $\theta=1$. Next, model (9) is used to determine the levels of inputs, intermediate measures, and outputs required for $\tilde{M}$ to reach the target efficiency measure.

Considering the optimal solution to (9), i.e., $\left(\alpha_1^*,\alpha_2^*\right)=\left(891948,730335\right)$, $\left(\beta_1^*,\beta_2^*\right)=\left(12111852,210784\right)$, and $\left(\gamma_1^*,\gamma_2^*\right)=\left(4773451,1467197\right)$, $\tilde{M}$ should reduce operating expenses to 891948, increase underwriting profit to 10899390, reduce reinsurance premium to 11714, and keep insurance expenses, investment profit, and written premium constant. By solving model (7), the following optimal solution is obtained regarding a merger between DMUs 19 and 24.

$$\begin{cases}\left(\alpha_1^*,\alpha_2^*\right)=\left(6699063,3531614\right)\\ \left(\gamma_1^*,\gamma_2^*\right)=\left(37392862,1753794\right).\end{cases}$$

Suppose that $\tilde{M}$ is efficient. That implies that the operating expenses for the merged company $\tilde{M}$ should be increased to 6699063, requiring an increase by 6376344. Insurance expenses should also be increased by 3114182 in order to reach the goal of 3531614. Moreover, underwriting profit and investment profit should reach 37392862 and 1753794, respectively, which again requires an increase in the respective levels. The merger between these two insurance companies is a major consolidation, or in other words, model (7) is infeasible.

4.3. Case 3. Real-life application: 20 commercial banks

In this section, we provide another application of the proposed method. Table 7 contains the factors considered in our example of 20 commercial banks, and the data set corresponding to the banks are presented in Table 8.

Table 7. Factors related to 20 commercial banks

Factor	Notation	Definition
Stage 1 inputs	X_{ij}^{1}	Interest expenses
Stage 1 inputs	X_{ij}^{2}	Non-interest expenses
Stage 1 outputs (intermediate products)	Z_{dj}^{1}	Total assets
Stage 2 outputs	Y_{rj}^{1}	Interest incomes
Stage 2 outputs	Y_{rj}^{2}	Non-interest incomes

Table 8. Data set related to 20 commercial banks

Bank	X_{ij}^{1}	X_{ij}^{2}	Z_{dj}^{1}	Y_{rj}^{1}	Y_{rj}^{2}
Bank 01	3956.796	1894.426	368189.74	9001.004	8701.497
Bank 02	481.239	319.976	2736416.08	974.854	597.726
Bank 03	305.2	138.6	2573126.00	479.8	252.2
Bank 04	4710.680	3996.259	2492463.15	12920.337	6060.768
Bank 05	1.081	1.282	2106796.00	3.054	0.377
Bank 06	954.437	1208.703	2610582.12	1991.004	7278.097
Bank 07	3.965	5.082	837537.23	13.359	3.003
Bank 08	14.630	16.863	4810226.26	44.659	14.938
Bank 09	11.771	6.579	184253.00	22.952	15.134
Bank 10	364.920	244.750	1687155.00	923.51	1942.935
Bank 11	4897.442	2787.181	2634139.00	11294.607	9363.232
Bank 12	14.665	8.973	2382397.54	28.124	10.971
Bank 13	6.077	14.249	2139275.49	26.994	10.207
Bank 14	397.627	371.535	2521552.18	894.845	1902.878
Bank 15	661.120	830.166	2118418.10	2325.128	1748.531
Bank 16	12.125	7.346	856301.00	33.573	19.530
Bank 17	1222.026	1049.479	1536766.99	2959.509	2651.546

Bank 18	931.172	838.346	2073668.69	2460.798	2765.485
Bank 19	4070.351	2845.498	1024399.25	8377.368	7726.906
Bank 20	3721.233	858.463	1639225585	6953.701	2779.716

Suppose that Bank 19 and Bank 20 combine the entirety of their activities. Consider a target efficiency measure equaling 1 for the merged bank $\tilde{M}$. Model (9) yields the following optimal solution.

$$\left(\alpha_1^*, \alpha_2^*, \beta_1^*, \beta_2^*, \gamma^*\right) = \left(15, 17, 23, 15, 4810226\right) .$$

This means that interest expenses must equal 15, implying the need for a reduction by $7776.584 \left(= 7791.584 - 15\right)$.

Non-interest expenses require a value of 17 in the merged bank $\tilde{M}$, indicating the need for a reduction by 3686.961. Interest incomes must equal 23, achieved through a decrease by 15308. Non-interest incomes should be equal to 15, requiring a reduction by 10491.622. Furthermore, the intermediate measure must have a value of 4810226, which requires a reduction in the respective level. This merger is a major consolidation, indicating the infeasibility of model (9).

Now, consider the consolidation of the activities of Bank 19 and Bank 20 in the first stage, and suppose that the target for the merged bank $\tilde{M}$ is efficient. The post-consolidation parameter values are obtained by model (7) as follows.

$$\begin{cases} \left(\alpha_1^*, \alpha_2^*, \gamma^*\right) = \left(15, 17, 4810226\right) \\ \left(v_1, v_2, l\right) = \left(0, 0, 12606424\right). \end{cases}$$

This implies that the values for interest expenses, non-interest expenses, and the intermediate measure should equal 15, 17, and, 4810226, requiring reductions by 7776.584, 3686.961, and 12606424, respectively. The values obtained by model (8) for the consolidation of Banks 19 and 20 in the second stage are as follows.

$$\begin{cases} \left(\beta_1^*, \beta_2^*, \gamma^*\right) = \left(23, 15, 184253\right) \\ \left(u_1, u_2, l\right) = \left(15308, 10491, 0\right). \end{cases}$$

This indicates that the required values for interest incomes and non-interest incomes are 23 and 15, achieved through reductions by 77768.584 and 3688.961, respectively.

5. Conclusions

Financial institutions are nowadays faced with issues such as branch mergers or the presence of an excessive number of branches. The extension of financial institutions or mergers between them can, to some extent, be an indication of their turnover. However, a larger number of branches or mergers cannot necessarily lead to the success of financial institutions affiliated with a given organization. Therefore, in this paper, we consider a given financial institution as a two-stage network. Tools such as DEA and InvDEA are quite helpful in DMU mergers occurring in two separate stages with consideration to intermediate measures. Generally speaking, two-stage network InvDEA makes it possible to merge DMUs with a desirable level of efficiency, and therefore, the use of two-stage network DEA is crucial in many organizations such as universities or banks. This is because the graduation of students in postgraduate education happens in a two-stage process (education-oriented and research-oriented), and in a banking system, the two-stage process of loans investment and contracts with foreign currency services affect the branch evaluations.

As can be observed in Table 7, results from our analysis of 20 commercial banks showed that before consolidation, Bank 19 and Bank 20 had a 15% and a 14% share in the primary inputs of the decision-making units, respectively. Figure 2 illustrates the shares of these two DMUs in regard to other factors. As can be seen in Figure 3, $\tilde{M}$ has a 92% share in the primary inputs (X_{ij}^{1}) of DMUs 19 and 20 after consolidation, as well as having a 3% share in the secondary inputs (X_{ij}^{2}). Furthermore, $\tilde{M}$ has a 4% and a 1% share in the intermediate measures (Z_{dj}^{1}) and the secondary final outputs (Y_{rj}^{2}), respectively, while having no share in the primary final outputs (Y_{rj}^{1}). For future research, we recommend considering an assumption of returns to scale in network inverse DEA, as well as modelling unit expansion patterns in NInvDEA.

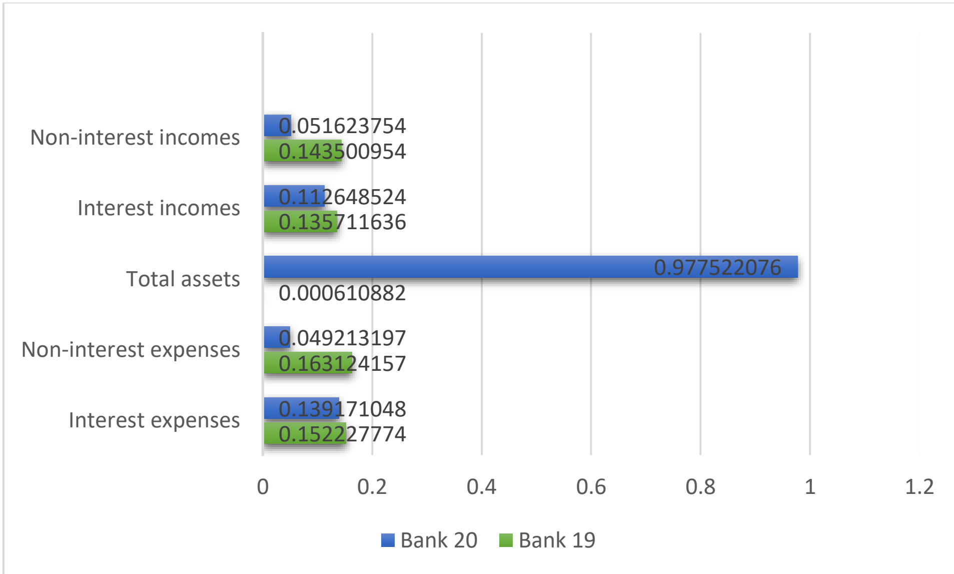


Fig. 2. Shares of DMUs 19 and 20 in measures related to each DMU before consolidation

Fig. 3. Shares of each DMU in measures related to DMU $\widetilde{M}$ after consolidation

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