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Axioms of (Extended) d-homology Theory

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Abstract. In this article, singular d-homology, singular extended d-homology, relative singular d-homology and relative singular extended d-homology functors are introduced and some axioms, such as dimension axiom, homotopy axiom, excision axiom and exactness axiom, of homology theory relative to these homologies are investigated and are proved to hold.

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1. Introduction and Preliminaries

d-homology and extended d-homology functors have been defined in more general categories than the abelian categories in [4, 9]; some related results and examples have been given as well. In [6], the homotopy and dimension axioms have been proved. Under certain conditions, exactness axiom has been investigated in [9].

In this section, the definitions of a kernel transformation and (extended) $d-$homology are recalled. In Section 2, first the definitions of singular (extended) d- homology are introduced; then a comparison of the singular homology and singular (extended) d- homology is given. Also the axioms of singular (extended) $d-$homology are investigated and established. Finally in Section 3, relative singular (extended) $d-$homology is defined and a comparison of the relative singular homology and relative

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singular (extended) d- homology is given. Some axioms of relative singular (extended) $d-$homology are investigated and established as well.

Throughout the manuscript we let *Rmod* denote the category of R-modules over a principal ideal domain. To this end, for a pointed category $\mathcal{C}$, following the notation of [4], we recall:

- For $f : A \longrightarrow B$, the maps $K_f \xrightarrow{k_f} A$, $B \xrightarrow{c_f} C_f$ and $P_f \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} A$

are respectively the kernel, the cokernel and the kernel pair of f and the image I_f of f is the coequalizer of the kernel pair of f. f can be factored as $f = m_f e_f$ that e_f is the coequalizer of the kernel pair of f.

- For a pair of maps $A \overset{f}{\underset{g}{\rightrightarrows}} B$, the maps $Equ(f, g) \xrightarrow{equ(f,g)} A$

  and $B \xrightarrow{coe(f,g)} Coe(f, g)$ are respectively the equalizer and the coequalizer of (f, g).

For a category $\mathcal{C}$ with a zero object, pullbacks and pushouts, let $\bar{\mathcal{C}}$ be the arrow category and $\hat{\mathcal{C}}$ be the pair-chain category of $\mathcal{C}$. Let $K : \bar{\mathcal{C}} \longrightarrow \mathcal{C}$ be the kernel functor and $I : \bar{\mathcal{C}} \longrightarrow \mathcal{C}$ be the image functor.

- The natural transformation $j : I \circ pr_1 \to K \circ pr_2 :\ \hat{\mathcal{C}} \longrightarrow \mathcal{C}$ takes the object $(f, g) \in \hat{\mathcal{C}}$ to $j_{fg} : I_f \to K_g$.

- The homology functor H^s that takes the object $(f, g) \in \hat{\mathcal{C}}$ to $H^s_{fg} = Coker(j_{fg})$ is called the standard homology functor. See [1, 10]

- Let S be the squaring functor. A kernel transformation in a category $\mathcal{C}$ is a natural transformation $d : S \circ K \longrightarrow K : \bar{\mathcal{C}} \longrightarrow \mathcal{C}$ such that for all (f, g) in $\hat{\mathcal{C}}$, the pullback, $j^*_{fg} : R_{fg} \longrightarrow K^2_g$, of j_{fg} along d_g and the coequalizer of the pair $j_1 = pr_1 \circ j^*_{fg}$ and $j_2 = pr_2 \circ j^*_{fg}$ exist, where pr_1 and pr_2 are the projection maps.

Any kernel transformation in *Rmod* is of the form $d = rpr_1 + spr_2$, for some $r, s \in R$.

- The d-homology functor $H^d : \hat{\mathcal{C}} \longrightarrow \mathcal{C}$ takes $(f, g) \in \hat{\mathcal{C}}$ to $H^d_{fg} = Coe(j_1, j_2)$.

- Let $m : A \longrightarrow C$ and $j : B \longrightarrow C$ be two maps in $\mathcal{C}$. Define $A +_C B$ also denoted by $A + B$ by the pushout of the pair (α, γ) where is the pullback of (j, m).

• Let d be a kernel transformation, $(f,g) \in \hat{\mathcal{C}}$ and Δ be the diagonal map. We have $m_{d_g\Delta_g} : I_{d_g\Delta_g} \longrightarrow K_g$, such that $m_{d_g\Delta_g} e_{d_g\Delta_g} = d_g\Delta_g$. Now in the following diagram since the outer square is a pullback and the inner square is a pushout, there is a unique map $\beta : I_{d_g\Delta_g} + I_f \longrightarrow K_g$ making the following diagram commute.

$$\begin{tikzcd}
P_{jm} \arrow[r, "\alpha"] \arrow[d, "\gamma"'] \arrow[dr, phantom, "\mathrm{po}"] & I_{d_g\Delta_g} \arrow[d, "h"] \arrow[ddr, bend left, "m_{d_g\Delta_g}"] & \\
I_f \arrow[r, "i"'] \arrow[drr, bend right, "j_{fg}"'] & I_{d_g\Delta_g} + I_f \arrow[dr, dotted, "\beta"] & \\
 & & K_g
\end{tikzcd}$$

The extended d-homology functor $\bar{H}^d : \hat{\mathcal{C}} \longrightarrow \mathcal{C}$ takes $(f,g) \in \hat{\mathcal{C}}$ to $\bar{H}^d_{fg} = C_\beta$, the cokernel of β.

2. Axioms of Singular (Extended) d-Homology

Definition 2.1. *Composition of the functors,*

$$Top \xrightarrow{S_{\Delta_*}} Set^{\Delta^{op}} \xrightarrow{F^{\Delta^{op}}} Rmod^{\Delta^{op}} \xrightarrow{C_*} C_*Rmod \xrightarrow{H^d_*} gRmod$$

is called the singular d-homology functor, where Top, $Set^{\Delta^{op}}$, $Rmod^{\Delta^{op}}$, C_*Rmod and $gRmod$ are respectively the categories of topological spaces, simplicial sets, simplicial R-modules, chain complexes of R-modules and graded R-modules and S_{Δ_*}, $F^{\Delta^{op}}$, C_* and H^d_* are respectively the singular functor, the functor generated by the free functor $F : Set \longrightarrow Rmod$, chain generating functor and the d-homology funtor.

Similarly the composition of the functors,

$$Top \xrightarrow{S_{\Delta_*}} Set^{\Delta^{op}} \xrightarrow{F^{\Delta^{op}}} Rmod^{\Delta^{op}} \xrightarrow{C_*} C_*Rmod \xrightarrow{\bar{H}^d_*} gRmod$$

is called the singular extended d-homology functor.

Theorem 2.2. *[6, 7, 9]. Let d be a kernel transformation in $\mathcal{C}$. There are pointwise regular epi natural transformations:*

(a) $p : H^s \longrightarrow H^d : \hat{\mathcal{C}} \longrightarrow \mathcal{C}$;

(b) $p : H^s \to \bar{H}^d : \hat{\mathcal{C}} \longrightarrow \mathcal{C}$.

Proposition 2.3. *[5]. Let $\mathcal{C}$ be an abelian category. For $(f, g) \in \hat{\mathcal{C}}$, $p_{fg} : H^s_{fg} \cong \bar{H}^d_{fg}$ is an isomorphism if and only if $I_{d_g \Delta_g}$ is a subobject of I_f.*

Lemma 2.4. *[9]. Let $\mathcal{C}$ be an abelian category and $d = rpr_1 + spr_2$. If for all A, $r : A \to A$ or $s : A \to A$ is monic, then $H^d \cong_n \bar{H}^d$.*

Theorem 2.5. Dimension Axiom: *Let X be a one point topological space. Then for a kernel transformation $d = rpr_1 + spr_2$ we have:*

$$(a)\ H^d_n(X) = \begin{cases} \frac{R}{(j_1 - j_2)K_0} & for \quad n = 0 \\ 0 & for \quad n \neq 0 \end{cases},$$

in which $K_0 = Ker(d_0)$;

$$(b)\ \bar{H}^d_n(X) = \begin{cases} \frac{R}{(r+s)R} & for \quad n = 0 \\ 0 & for \quad n \neq 0 \end{cases}.$$

Proof. Since $C_*(X)$ is the chain

and $d_0(a, b) = ra + sb$, the result follows, see [5]. $\square$

Lemma 2.6. *[6]. If $f_* \sim g_* : C_* \to D_*$ are homotopic chain maps, then:*

(a) $H^d(f_*) = H^d(g_*)$;

(b) $\bar{H}^d(f_*) = \bar{H}^d(g_*)$.

Lemma 2.7. *The functors*

(a) $S_{\Delta_*} : Top \to Set^{\Delta^{op}}$;

(b) $F^{\Delta^{op}} : Set^{\Delta^{op}} \to Rmod^{\Delta^{op}}$;

(c) $C_* : Rmod^{\Delta^{op}} \to C_*Rmod$;

preserve homotopy.

Proof. *(a)* If $H : X \times I \to Y$ is a homotopy from f to g, then $h^n_i : Top(\Delta_n, X) \to Top(\Delta_{n+1}, Y)$ defined by $h^n_i(k) = H \circ (k \times 1) \circ \langle \sigma_i, \pi_i \rangle$ is a homotopy from $S_{\Delta_*}(f)$ to $S_{\Delta_*}(g)$ in which $\sigma_i(x_0, ..., x_{n+1}) =$

$(x_0, ..., x_{i-1}, x_{i+1}, ..., x_{n+1})$ and $\pi_i(x_0, ..., x_{n+1}) = \sum\limits_{t=i}^{n} x_{t+1}$.

(b) If h is a homotopy from f to g in $Set^{\Delta^{op}}$, then $\bar{h}_i^n(\sum\limits_{t=1}^{m} r_t x_t) = \sum\limits_{t=1}^{m} r_t h_i^n(x_t)$ is a homotopy from $F^{\Delta^{op}}(f)$ to $F^{\Delta^{op}}(g)$.

(c) If h is a homotopy from f to g in $Rmod^{\Delta^{op}}$, then $h_n = \sum\limits_{i=0}^{n}(-1)^i h_i^n$ is a homotopy from $C_*(f)$ to $C_*(g)$. $\square$

Theorem 2.8. Homotopy Axiom:[6, 7]. *If are homotopic continuous maps, then:*

(a) $H_*^d(f) = H_*^d(g) : H_*^d(X) \longrightarrow H_*^d(Y)$;

(b) $\bar{H}_*^d(f) = \bar{H}_*^d(g) : \bar{H}_*^d(X) \longrightarrow \bar{H}_*^d(Y)$.

Proof. Follows from 2.6 and 2.7. $\square$

Proposition 2.9. *Regular epi natural transformations* $p : H^s \to H^d$ *and* $p : H^s \to \bar{H}^d$ *induce regular epi natural transformations:*

(a) $p : H_*^s \to H_*^d : Top \to gRmod$;

(b) $p : H_*^s \to \bar{H}_*^d : Top \to gRmod$.

Proof. The proof of both parts follows from composing the natural transformations p with the functor $C_* \circ F^{\Delta^{op}} \circ Set^{\Delta^{op}}$. $\square$

3. Relative Singular (Extended) d-Homology

Proposition 3.1. *The functors*

(a) $S_{\Delta_*} : Top \to Set^{\Delta^{op}}$;

(b) $F^{\Delta^{op}} : Set^{\Delta^{op}} \to Rmod^{\Delta^{op}}$;

(c) $C_* : Rmod^{\Delta^{op}} \to C_*Rmod$;

preserve monomorphisms.

Proof. Using the fact that the monomorphisms in Top are the one to one functions and in $Set^{\Delta^{op}}$, $Rmod^{\Delta^{op}}$ and C_*Rmod are pointwise monomorphisms, the proof follows by straightforward computations. $\square$

Let $Topp$ be the category of topological pairs whose objects are the pairs (X, A), where A is a subspace of the topological space X ; and whose morphisms from (X, A) to (Y, B) are continous maps $f : X \to Y$ for which $f(A) \subseteq B$.

Corollary 3.2. *Let $(X, A) \in Topp$. Then the inclusion map $i : A \hookrightarrow X$ induces the inclusion chain map $i_* =: C_* \circ F^{\Delta^{op}} \circ S_{\Delta_*}(i) : C_*(A) \to C_*(X)$.*

Proof. By 3.1, the composition $C_* \circ F^{\Delta^{op}} \circ S_{\Delta_*}$ preserves monomorphisms. It then follows that i_* is isomorphic to the inclusion. □

By 3.2, $C_*(A)$ can be regarded as a subchain of $C_*(X)$, so we have the quotient chain $\frac{C_*(X)}{C_*(A)}$ and the quotient chain map $q_* : C_*(X) \to \frac{C_*(X)}{C_*(A)}$.

Definition 3.3. *The quotient chain $\frac{C_*(X)}{C_*(A)}$ is denoted by $C_*(X, A)$ and is called the relative singular chain of (X, A).*

Theorem 3.4. *The mapping $C_* : Topp \to C_*Rmod$ that sends (X, A) to $C_*(X, A)$ is a functor.*

Proof. Follows from straightforward computations. □

The above functor $C_* : Topp \to C_*Rmod$ is called the relative singular chain functor.

Definition 3.5. (*a*) *The relative singular d-homology functor $H^d_* : Topp \to gRmod$ is the composition*

(*b*) *The relative singular extended d-homology functor $\bar{H}^d_* : Topp \to gRmod$ is the composition*

Definition 3.6. *Two maps are homotopic in $Topp$ if there is a homotopy h from f to g in Top and $h(A \times I) \subseteq B$.*

Corollary 3.7. Relative Homotopy Axiom: *If $f \sim g : (X, A) \to (Y, B)$ are homotopic in $Topp$, then:*

(*a*) $H^d_*(f) = H^d_*(g)$;

(*b*) $\bar{H}^d_*(f) = \bar{H}^d_*(g)$.

Proof. Since $H^s(f) = H^s(g)$ (see [2, 3]), the result follows. □

Proposition 3.8. *Regular epi natural transformations $p : H^s \to H^d$*

and $p : H^s \to \bar{H}^d$ *induce regular epi natural transformations:*

(a) $p : H^s_* \to H^d_* : Topp \to gRmod$*;*

(b) $p : H^s_* \to \bar{H}^d_* : Topp \to gRmod$*.*

Proof. The proof of both parts follows from composing the natural transformations p with the functor $C_* \circ F^{\Delta^{op}} \circ Set^{\Delta^{op}}$. $\square$

Lemma 3.9. *In the category Rmod, if* $H^s_{fg} \cong H^s_{f'g'}$*, then:*

(a) $\bar{H}^d_{fg} \cong \bar{H}^d_{f'g'}$*;*

(b) $H^d_{fg} \cong H^d_{f'g'}$*.*

Proof. Let $d = rpr_1 + spr_2$ and $\psi : H^s_{fg} \cong H^s_{f'g'}$. Let $\mu : H^s_{fg} \to H^s_{fg}$ and $\mu' : H^s_{f'g'} \to H^s_{f'g'}$ be the morphisms "multiplication by $r + s$" i.e. $\mu([x]) = (r + s)[x]$. Since $\psi \circ \mu([x]) = (r + s)\psi([x]) = \mu \circ \psi([x])$, $(\psi, \psi) : \mu \to \mu'$ is an isomorphism and hence it induces an isomorphism $\phi : C_\mu \cong C_{\mu'}$. It is easy to see $C_\mu = \frac{K_g/I_f}{(r+s)(K_g/I_f)} \cong \frac{K_g}{I_f+(r+s)K_g} = \bar{H}^d_{fg}$ and $C_{\mu'} \cong \bar{H}^d_{f'g'}$.

Let $\nu : H^s_{fg} \to \bar{H}^d_{fg}$ and $\nu' : H^s_{f'g'} \to \bar{H}^d_{f'g'}$ be the morphism multiplication by r. Again $(\psi, \phi) : \nu \to \nu'$ is an isomorphism and hence it induces an isomorphism $\eta : I_\nu \cong I_{\nu'}$. Since

$H^d_{fg} = coker(j_1 - j_2) = \frac{K_g}{(j_1-j_2)(R_{fg})} = \{[a]|a \in K_g\}$, where $[a] = \{b \in K_g|r(a - b) \in (r + s)K_g + I_f\} = \{b \in K_g|s(a - b) \in (r + s)K_g + I_f\}$, see [4], we have $I_\nu = C_{k_\nu} \cong H^d_{fg}$ and $I_{\nu'} = C_{k_{\nu'}} \cong H^d_{f'g'}$. $\square$

Corollary 3.10 *For* $d = rpr_1 + spr_2$ *we have:*

(a) $\bar{H}^d \cong \frac{H^s}{(r+s)H^s}$*;*

(b) $H^d \cong \frac{H^s}{Ker(\nu)}$*.*

Proof. Follows from 3.9. $\square$

Theorem 3.11. Excision Axiom: *Let* $(X, A) \in Topp$ *and* $U \subseteq X$ *be open such that* $cl(U) \subseteq int(A)$*. Then the excision map* $i : (X - U, A - U) \hookrightarrow (X, A)$ *induces the following isomorphisms.*

(a) $H^d_*(X - U, A - U) \cong H^d_*(X, A)$*;*

(b) $\bar{H}^d_*(X - U, A - U) \cong \bar{H}^d_*(X, A)$*.*

Proof. Since $H_*^s(X-U, A-U) \cong H_*^s(X,A)$, the result follows from 3.10. $\square$

Let $\mathcal{C}$ be an abelian category and d be a kernel transformation. For a $\mathcal{C}$-chain:

$$C_* : \ldots \longrightarrow C_{n+1} \xrightarrow{\delta_{n+1}} C_n \xrightarrow{\delta_n} C_{n-1} \longrightarrow \ldots$$

we establish the following notation.

$$k_n = k_{\delta_n} : K_n \longrightarrow C_n \ , \ I_n = I_{\delta_{n+1}}, \ \ \Delta_n : K_n \longrightarrow K_n^2$$

is the diagonal map.

Lemma 3.12. *[9] Let $\mathcal{C}$ be an abelian category, d be a kernel transformation and the sequence*

$$0 \longrightarrow C'_* \xrightarrow{\nu_*} C_* \xrightarrow{\pi_*} C''_* \longrightarrow 0$$

of $\mathcal{C}$-chains be short exact. Then:

(a) With $f_n = K(\nu_n, \nu_{n-1})$, there is a unique map $l_n I'_n + I_{d'_n\Delta'_n} I_n + I_{d_n\Delta_n}$ such that the following diagram commutes.

$$\begin{tikzcd} I'_n + I_{d'_n\Delta'_n} \arrow[r, "\beta'_n"] \arrow[d, "l_n"'] & K'_n \arrow[d, "f_n"] \\ I_n + I_{d_n\Delta_n} \arrow[r, "\beta_n"'] & K_n \end{tikzcd}$$

(b) With $f'_n = K(\pi_n, \pi_{n-1})$, there is a unique map $l'_n I_n + I_{d_n\Delta_n} I''_n + I_{d''_n\Delta''_n}$ such that the following diagram commutes.

$$\begin{tikzcd} I_n + I_{d_n\Delta_n} \arrow[r, "\beta_n"] \arrow[d, "l'_n"'] & K_n \arrow[d, "f'_n"] \\ I''_n + I_{d''_n\Delta''_n} \arrow[r, "\beta''_n"'] & K''_n \end{tikzcd}$$

Theorem 3.13. *[9] Let $\mathcal{C}$ be an abelian category, d be a kernel transformation and the sequence*

$$0 \longrightarrow C'_* \xrightarrow{\nu_*} C_* \xrightarrow{\pi_*} C''_* \longrightarrow 0$$

of $\mathcal{C}$-chains be short exact. If l'_n is epic, then for each n, we have the *following exact sequence.*

$$\bar{H}'^{d}_{n} \longrightarrow \bar{H}^{d}_{n} \longrightarrow \bar{H}''^{d}_{n} \longrightarrow H'^{s}_{n-1} \longrightarrow H^{s}_{n-1} \longrightarrow H''^{s}_{n-1}$$

Corollary 3.14. *[8, 9] Let $\mathcal{C}$ be an abelian category and $d = rpr_1 + spr_2$, $r, s \in \mathbb{Z}$. If for all $A \in \mathcal{C}$, $r : A \longrightarrow A or for all A$, $s : A \longrightarrow A$ is monic, then for each short exact sequence of $\mathcal{C}$-chains:*

$$0 \longrightarrow C'_* \xrightarrow{\nu_*} C_* \xrightarrow{\pi_*} C''_* \longrightarrow 0$$

for which l'_n is epic, there is the following exact sequence.

$$H'^{d}_{n} \longrightarrow H^{d}_{n} \longrightarrow H''^{d}_{n} \longrightarrow H'^{s}_{n-1} \longrightarrow H^{s}_{n-1} \longrightarrow H''^{s}_{n-1}$$

Proposition 3.15. *[2, 3] Let $(X, A) \in Topp$. Then the sequence*

$$0 \longrightarrow C_*(A) \xrightarrow{i_*} C_*(X) \xrightarrow{q_*} C_*(X, A) \longrightarrow 0$$

is a short exact sequence of chain complexes.

Proof. Since q_* and i_* are respectively quotient and inclusion chain maps and $Ker(q_*) \cong C_*(A) \cong Im(i_*)$, the result is obtained as desired. $\square$

If we denote the category of short exact sequences of chain complexes by $SesC_*Rmod$, then we have:

Proposition 3.16 *The functor $C_* : Topp \to C_*Rmod$ induces a functor $c : Topp \to SesC_*Rmod$.*

Theorem 3.17. Exactness Axiom: *Let $(X, A) \in Topp$ and $d = rpr_1 + spr_2$. If for each n, $l'_n : I_n + I_{d_n \Delta_n} \to I''_n + I_{d''_n \Delta''_n}$ is epic, then:*

(a) there is the following exact sequence.

$$\begin{array}{ccccccc}\bar{H}^{d}_{n}(A) \longrightarrow \bar{H}^{d}_{n}(X) \longrightarrow \bar{H}^{d}_{n}(X, A) \xrightarrow{\chi_n} H^{s}_{n-1}(A) \\ \swarrow \\ H^{s}_{n-1}(X) \longrightarrow H^{s}_{n-1}(X, A)\end{array}$$

(b) If for all $M \in Rmod$, $r : M \to M$ or $s : M \to M$ is monic, then there is the following exact sequence.

$$\begin{array}{c}H^{d}_{n}(A) \longrightarrow H^{d}_{n}(X) \longrightarrow H^{d}_{n}(X, A) \longrightarrow H^{s}_{n-1}(A) \\ \swarrow \\ H^{s}_{n-1}(X) \longrightarrow H^{s}_{n-1}(X, A)\end{array}$$

Proof. Part (a) follows from the application of 3.15 in 3.13 and part (b) follows from the application of 3.15 in 3.14. $\square$

Proposition 3.18. *Let $(X, A) \in Topp$ and $d = rpr_1 + spr_2$. If for each n, $l'_n : I_n + I_{d_n\Delta_n} \to I''_n + I_{d''_n\Delta''_n}$ is epic, then there is the following chain that is exact at $\bar{H}^d_n(X)$ for each n.*

$$
\begin{tikzcd}
\cdots \arrow[r] & \bar{H}^d_n(A) \arrow[r] & \bar{H}^d_n(X) \arrow[r] & \bar{H}^d_n(X,A) \arrow[r, "\bar{\chi}_n"] & \bar{H}^d_{n-1}(A) \arrow[dl] \\
 & & & \bar{H}^d_{n-1}(X) \arrow[r] & \cdots
\end{tikzcd}
$$

Proof. The sequence is obtained by setting $\bar{\chi}_n = p_{\delta_n\delta_{n-1}}\chi_n$. The exactness at $\bar{H}^d_n(X)$ corner follows from 3.17 (a). $\square$

Corollary 3.19. *Let $(X, A) \in Topp$ and $d = rpr_1 + pr_2$ or $d = pr_1 + rpr_2$. Suppose that for each n, $l'_n : I_n + I_{d_n\Delta_n} \to I''_n + I_{d''_n\Delta''_n}$ is epic, then there is the following chain that is exact at $H^d_n(X)$ for each n.*

$$
\begin{tikzcd}
\cdots \arrow[r] & H^d_n(A) \arrow[r] & H^d_n(X) \arrow[r] & H^d_n(X,A) \arrow[r, "\bar{\chi}_n"] & H^d_{n-1}(A) \arrow[dl] \\
 & & & H^d_{n-1}(X) \arrow[r] & \cdots
\end{tikzcd}
$$

Proof. Follows from 3.18 and the fact that under these conditions, $\bar{H}^d = H^d$. $\square$

Let $Topt$ be the category of triples of topological spaces with objects (X, A, B) such that $B \subseteq A \subseteq X$ and morphisms $f : (X, A, B) \to (Y, C, D)$ in which $f : X \to Y$ is a continuous map, $f(A) \subseteq C$ and $f(B) \subseteq D$. Then the inclusions $B \hookrightarrow A \hookrightarrow X$ induce the inclusion $j_* : C_*(A, B) \to C_*(X, B)$. The cokernel of j_* up to isomorphism is $q_* : C_*(X, B) \to C_*(X, A)$.

Proposition 3.20. *Let $(X, A, B) \in Topt$. Then the sequence*

$$
0 \longrightarrow C_*(A,B) \xrightarrow{j_*} C_*(X,B) \xrightarrow{q_*} C_*(X,A) \longrightarrow 0
$$

is a short exact sequence of chain complexes.

Proof. By the Third Isomorphism Theorem $\frac{C_*(X,B)}{C_*(A,B)} = \frac{C_*(X)/C_*(B)}{C_*(A)/C_*(B)} \cong \frac{C_*(X)}{C_*(A)} = C_*(X, A)$. $\square$

So we have:

Proposition 3.21. *The functor $C_* : Top \to C_*Rmod$ induces a functor $c : Topt \to SesC_*Rmod$.*

Proof. For a morphism $f : (X, A, B) \to (Y, C, D)$ in $Topt$ we have chain maps $f_* : C_*(X) \to C_*(Y)$, $f_*|A : C_*(A) \to C_*(C)$, $f_*|B : C_*(B) \to C_*(D)$ which inducing chain maps $\hat{f}_{*1} : C_*(X, A) \to C_*(Y, C)$, $\hat{f}_{*2} : C_*(X, B) \to C_*(Y, D)$ and $\hat{f}_{*3} : C_*(A, B) \to C_*(C, D)$. Since $q_*^X = coker(j_*^X)$, $q_*^Y = coker(j_*^Y)$ and $\hat{f}_{*2}|A = \hat{f}_{*3}$ and the left square of the following diagram commutes, the chain map $\hat{f}_{*1}$ is that unique map in which the right square commutes.

$$\begin{tikzcd}
0 \arrow[r] & C_*(A,B) \arrow[r, "j_*"] \arrow[d, "\hat{f}_{*3}"'] & C_*(X,B) \arrow[r, "q_*"] \arrow[d, "\hat{f}_{*2}"'] & C_*(X,A) \arrow[r] \arrow[d, "\hat{f}_{*1}"'] & 0 \\
0 \arrow[r] & C_*(C,D) \arrow[r, "j_*"] & C_*(Y,D) \arrow[r, "q_*"] & C_*(Y,C) \arrow[r] & 0
\end{tikzcd}$$ $\square$

Theorem 3.22. Exactness Axiom for Triples: *Let $(X, A, B) \in Topt$ and $d = rpr_1 + spr_2$. If for each n, $l'_n : I_n + I_{d_n\Delta_n} \to I''_n + I_{d''_n\Delta''_n}$ is epic, then:*

1) there is the following exact sequence.

$$\begin{tikzcd}
\bar{H}^d_n(A,B) \arrow[r] & \bar{H}^d_n(X,B) \arrow[r] & \bar{H}^d_n(X,A) \arrow[r, "\chi_n"] & H^s_{n-1}(A,B) \arrow[dl] \\
 & & H^s_{n-1}(X,B) \arrow[r] & H^s_{n-1}(X,A)
\end{tikzcd}$$

2) If for all $M \in Rmod$, $r : M \to M$ defined by $r(m) = rm$ is monic, then there is the following exact sequence.

$$\begin{tikzcd}
H^d_n(A,B) \arrow[r] & H^d_n(X,B) \arrow[r] & H^d_n(X,A) \arrow[r] & H^s_{n-1}(A,B) \arrow[dl] \\
 & & H^s_{n-1}(X,B) \arrow[r] & H^s_{n-1}(X,A)
\end{tikzcd}$$

Proof. Part (1) follows from the application of 3.20 in 3.13 and part (2) follows from the application of 3.20 in 3.14. □

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