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A Method for Finding the Anchor Points of BCC Model in DEA Context

A. A. Bani
Science and Research Branch, Islamic Azad University

M. Rostamy-Malkhalifeh*
Science and Research Branch, Islamic Azad University

F. Hosseinzadeh Lotfi
Science and Research Branch, Islamic Azad University

D. Akbarian
Arak branch, Islamic Azad University

Abstract. Anchor points are an important subset of the set of extreme efficient points of the Production Possibility Set (PPS) in the Data Envelopment Analysis (DEA). They delineate the efficient from the inefficient part of the PPS frontier. The aim of this paper is to present a method to find anchor points of the PPS of the BCC model based on a searching weak supporting hyperplane passing through unit under consideration. Then, one theorem is proved to generate a necessary and sufficient condition for identification of these points. Finally, some numerical examples are demonstrated to show the applicability of the proposed method.

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*Corresponding author

1. Introduction

Data Envelopment Analysis introduced by Farrel [15] and Charnes, et al. [8], at first, is a non-parametric technique and a useful tool to evaluate Decision Making Units ($DMUs$) with multiple inputs and multiple outputs [10, 14]. Today, this technique is well-known in many respects. The original model of Data Envelopment Analysis was considered by Charnes, et al. [8] with constant returns to scale (CRS), and then was expanded by Banker, et al. [5] for variable returns to scale (VRS) technologies.

In DEA models, each model has a production boundary. In 1957, Farrel expressed the production boundary approach, as the maximum of production with a certain amount of inputs [15]. In fact, the performance boundary is the situation that every DMU is at the best mode, and knowing it will provide useful information to the decision maker. Due to its important, it has been studied by several researchers. Jahanshahloo, et al. [17, 18, 19, 20], Yu, et al. [29], Olesen and Petersen [23], Wei, et al. [28], Hosseinzadeh Lotfi, et al. [16], Amirteimoori and Kordrostami [3], Davtalab Olyaie, et al. [11, 12] are among the most important of them.

Anchor points are one of the main subsets of the set of extreme efficient points of the Production Possibility Set in DEA. An anchor point is an extreme efficient DMU for which some inputs can be increased and/or outputs decreased without penetrating the interior of the PPS. Therefore, it is an extreme efficient element of the Production Possibility Set lying on the transition between the strong efficient frontier and the free-disposability(unbounded face) part of the boundary. As the DEA context shows, the anchor points play a critical role in DEA theory and its applications. Thanassoulis and Allen [26] applied the concept of these points, at first, to generate, unobserved $DMUs$ and extend the DEA frontier. Rouse [24] used this meaning to identify prices for health care services. These points also play an effective role in [27]in presenting the unobserved $DMUs$. Bougnol and Dula [7] provided another method to identify them based on the geometrical properties of anchor points. They discussed Production Possibility Set with the variable returns to scale. In their method, the PPS of the BCC model is projected on all coordinate hyperplanes; an extreme efficient DMU is an anchor point if and only if it belongs to the boundary of at least one simple projection. Mostafaee and Soleimani-damaneh [21] suggested a manner to find the anchor points using the sensitivity analysis techniques and also presented some conditions for characterizing anchor points in [22]. Soleimani-damaneh and Mostafaee [25] offered an algorithm different with the above-mentioned works to identify the anchor $DMUs$ in (non-convex) FDH models. They introduced the extreme unit and anchor point ideas in non-convex technologies in their study. See also [1, 6] for more details. Note that as the set of anchor points is a subset of the set of extreme efficient points, finding the extreme efficient points is a principle. To find them there are several algorithms to do this. See [9, 13].

In this paper, a new method is developed to identify PPS anchor points of the BCC model using searching weak supporting hyperplane passing through unit under consideration. Our proposed method needs less computational effort than Bougnol´s method [7], because we do not need project frame $DMUs$ on the boundary of simple projections one after another.

The rest of the article is as follows: Section 2 explains the preliminary concepts. Section 3 presents our approach to find anchor points. Sections 4 and 5 contain numerical examples and conclusion, respectively.

2. Background

Consider a set of n $DMUs$, which is associated with m inputs and s outputs. We apply the notation $(x_j, y_j)(j \in J = \{1, \ldots, n\})$ for the observed $DMUs$ that the first component is vectors of the inputs $(x_j = (x_{1j}, \ldots, x_{mj})^t)$ and the second component is vectors of the outputs $(y_j = (y_{1j}, \ldots, y_{sj})^t)$. Also suppose that for each $DMU_j = (x_j, y_j), j \in J, x_j > 0, y_j > 0$. This assumption is not restrictive, because each extreme efficient DMU with some zero input/output factor is an anchor point [22]. The Production Possibility Set is the set of all technologically possible input-output combinations given by the following [4, 5, 29]:

$$T = \{(x, y) | \text{y can be produced by x}\}.$$

One of the DEA models to evaluate efficiency of set of $DMUs$ is BCC model [5]. Its PPS can be defined follows:

$$T_v = \{(x, y) \,|\, x \geqslant \sum_{j=1}^{n} \lambda_j x_j, \quad y \leqslant \sum_{j=1}^{n} \lambda_j y_j, \quad \sum_{j=1}^{n} \lambda_j = 1, \, \lambda_j \geqslant 0, \quad j \in J\},$$

The BCC input-oriented and output-oriented models pertaining to DMU_p, $p \in J$ can be expressed as follows, respectively:

$$\begin{aligned} min \quad & \theta - \varepsilon(\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+) \\ s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = \theta x_{ip}, \quad i = 1, \ldots, m, \\ & \sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = y_{rp}, \quad r = 1, \ldots, s, \\ & \sum_{j=1}^{n} \lambda_j = 1, \\ & \lambda_j, s_i^-, s_r^+ \geqslant 0, \quad \forall j, \forall i, \forall r. \end{aligned} \tag{1}$$

$$\begin{aligned} max \quad & \varphi + \varepsilon(\sum_{i=1}^{m} t_i^- + \sum_{r=1}^{s} t_r^+) \\ s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij} + t_i^- = x_{ip}, \quad i = 1, \ldots, m, \\ & \sum_{j=1}^{n} \lambda_j y_{rj} - t_r^+ = \varphi y_{rp}, \quad r = 1, \ldots, s, \\ & \sum_{j=1}^{n} \lambda_j = 1, \\ & \lambda_j, t_i^-, t_r^+ \geqslant 0, \quad \forall j, \forall i, \forall r. \end{aligned} \tag{2}$$

Where ε is non-Archimedean small and positive number. The models (1) and (2) are called the envelopment forms of the BCC model and their dual models (without ε i.e. $\varepsilon = 0$), which are called the multiplier forms, are as the following, respectively:

$$\begin{aligned} max \quad & \sum_{r=1}^{s} u_r y_{rp} + u_0 \\ s.t. \quad & \sum_{i=1}^{m} v_i x_{ip} = 1, \\ & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + u_0 \leqslant 0, \quad j \in J \\ & v_i, u_r \geqslant 0, \quad \forall i, \forall r, \\ & u_0 \ is \ free. \end{aligned} \tag{3}$$

$$\begin{aligned} min \quad & \sum_{i=1}^{m} v_i x_{ip} - u_0 \\ s.t. \quad & \sum_{r=1}^{s} u_r y_{rp} = 1, \\ & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + u_0 \leqslant 0, \quad j \in J \\ & v_i, u_r \geqslant 0, \quad \forall i, \forall r, \\ & u_0 \ is \ free. \end{aligned} \tag{4}$$

Several definitions are now available.

Definition 2.1. (Envelopment). *DMU_p is pareto efficient or strong efficient in BCC model if and only if either (i) or (ii) happen:*
(i) For every optimal solution of model (1), $(\theta^*, \lambda_j^*, s_i^{-*}, s_r^{+*})$, $j \in J$, $i =$

$1, \ldots, m$, $r = 1, \ldots, s$, *satisfies* $\theta^* = 1$ *and* $s_i^{-*} = 0, s_r^{+*} = 0, i = 1, \ldots, m$, $r = 1, \ldots, s$,
(*ii*) *For every optimal solution of model* (2) $(\varphi^*, \lambda_j^*, t_i^{-*}, t_r^{+*})$, $j \in J$, $i = 1, \ldots, m$, $r = 1, \ldots, s$, *satisfies* $\varphi^* = 1$ *and* $t_i^{-*} = 0, t_r^{+*} = 0, i = 1, \ldots, m$, $r = 1, \ldots, s$,
otherwise, DMU_p *is* $BCC-$*inefficient.*

Definition 2.2. (Multiplier). DMU_p *is pareto efficient or strong efficient in* BCC *model if and only if either* (*iii*) *or* (*iv*) *happen:*
(*iii*) *The optimal objective value of* (3) *is equal to 1 and there exist some optimal solutions such that all decision variables* v_i , u_r , $1 \leqslant i \leqslant m, 1 \leqslant r \leqslant s$ *be strictly positive.*
(*iv*) *The optimal objective value* (4) *is equal to 1 and there exist some optimal solutions such that all decision variables* v_i , u_r , $1 \leqslant i \leqslant m, 1 \leqslant r \leqslant s$ *be strictly positive.*

Definition 2.3. (Envelopment). DMU_p *is weak efficient if and only if there exists an optimal solution such that either* (*v*) *or* (*vi*) *happen:*
(*v*) $\theta^* = 1$ *and* $(s^{+*}, s^{-*}) \neq (0, 0)$ *in* (1)
(*vi*) $\varphi^* = 1$ *and* $(t^{+*}, t^{-*}) \neq (0, 0))$ *in* (2).

Definition 2.4. (Multiplier). DMU_p *is weak efficient if and only if either* (*vii*) *or* (*viii*) *happen:*
(*vii*) *The optimal objective value of* (3) *is equal to 1 and for all its optimal solutions, some decision variables* v_i , u_r , $1 \leqslant i \leqslant m, 1 \leqslant r \leqslant s$ *be zero.*
(*viii*) *The optimal objective value of* (4) *is equal to 1 and for all its optimal solutions some decision variables* v_i , u_r , $1 \leqslant i \leqslant m, 1 \leqslant r \leqslant s$ *be zero.*

Definition 2.5. *The pareto efficient* DMU_p *is extreme efficient if models* (1) *or* (2) *have unique optimal solution with* $\lambda_p^* > 0$ *and* $\lambda_j^* = 0, j \in J - \{p\}$. *Otherwise,* DMU_p *is non-extreme efficient.*

We denote the set of extreme $BCC-$efficient $DMUs$ as E^*. The set E^* is also called the frame of J. The frames are important in DEA because the PPS of the DEA models are constructed by them and the exclusion each of them, alters the shape of the PPS.

The following notes are taken from the above definitions and their proof is straightforward.

Note 2.6. DMU_p is an interior point of PPS and inefficient if and only if in model (1) and model (2) $\theta^* < 1$ and $\phi^* > 1$ respectively.

Note 2.7. DMU_p is an interior point of PPS if and only if objective value in model (3) is smaller than 1 and in model (4) is greater than 1.

The concept of supporting hyperplane in the DEA literature plays a decisive role in identification of anchor points. A hyperplane in a space of $m + s$ dimensions and passing through $DMU_p = (x_p, y_p)$ can be defined as follows:

$$H_p = \{(x, y) \in R^{m+s} | u^t(y - y_p) - v^t(x - x_p) = 0\}, \tag{5}$$

where $u \in R^s$, $v \in R^m$ are normal vectors, and u_0 is defined as:

$$u_0 = v^t x_p - u^t y_p,$$

Therefore, the relation (5) can be expressed as:

$$H_p = \{(x, y) \in R^{m+s} | u^t y - v^t x + u_0 = 0\}.$$

In general, a supporting hyperplane divides space into two half-spaces. If the Production Possibility Set can be located in one of the half-spaces, then H_p is supporting hyperplane on PPS at the point (x_p, y_p). In other words, supporting hyperplane touches the PPS at the DMU_p. Thus, the following definition can be represented.

Definition 2.8. *The hyperplane*

$$H = \{(x, y) \in R^{m+s} | u^t y - v^t x + u_0 = 0, \ (u, v) \geqslant 0, \ (u, v) \neq 0\},$$

is supporting on PPS at a point of boundary, $DMU_p = (x_p, y_p)$, if and only if both (i) and (ii) happen:
(i) $u^t y_p - v^t x_p + u_0 = 0$
(ii) $\forall (x, y) \in PPS \Rightarrow u^t y - v^t x + u_0 \leqslant 0.$
H is called strong supporting if $(u, v) > 0$ and called weak supporting if some of components (u, v) is zero.

The following theorem states relation between the optimal solution of the BCC multiplier form and the supporting hyperplane on PPS.

Theorem 2.9. *Let DMU_p be a boundary unit of PPS. In evaluation of DMU_p by BCC multiplier form, (u^*, v^*, u_0^*) is an optimal solution if and only if*

$$H^* = \{(x, y) \in R^{m+s} | u^{*t}y - v^{*t}x + u_0^* = 0\}$$

is a supporting hyperplane on the PPS at DMU_p.

Proof. Theorem 5.1 in [10]. □

Definition 2.10. *Suppose (u^*, v^*, u_0^*) is an optimal solution of model (3), then according to the Theorem 2.9 $H^* = \{(x, y) \in R^{m+s} | u^{*t}y - v^{*t}x + u_0^* = 0\}$ is a supporting hyperplane of the T_v. The set $F = \{(x, y) \in R^{m+s} | u^{*t}y - v^{*t}x + u_0^* = 0\} \cap T_v = H^* \cap T_v$ is called a face of T_v.*

Note 2.11. A face of a polyhedral set is the support set of a supporting hyperplane. A facet of a k-dimensional polyhedral set is a $k - 1$ dimensional face.

Note 2.12. The PPS of the BCC model has bounded and unbounded faces. The unbounded faces make up the free-disposability part of the frontier.

The following definition introduces the concept of anchor points in T_v. In fact, this property comes from Result 1 in Bougnol and Dula [7].

Definition 2.13. *$DMU_p = (x_p, y_p) \in E^*$ is called an anchor point if and only if it is located on a supporting hyperplane of T_v, say $H_{(u,v,u_0)}$, such that at least one component of the gradient vector (u, v) is zero.*

Remark 2.14. *The $DMU_p \in E^*$ is an anchor DMU if it belongs to an unbounded face of the PPS of the BCC model.*

3. Identifying the Anchor DMUs of the PPS of the BCC Model

In this section, the anchor $DMUs$ of the PPS of the BCC model are identified as follows.

Corresponding to each $DMU_p = (x_{1p}, \ldots, x_{mp}, y_{1p}, \ldots, y_{sp})$ $(p \in J)$, the following mixed integer linear program is solved:

$$
\begin{aligned}
Max \quad & \sum_{\substack{j=1\\ j\neq p}}^{n} \lambda_j \\
s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij} \leqslant x_{ip}, \qquad i = 1, \ldots, m && (1)\\
& \sum_{j=1}^{n} \lambda_j y_{rj} \geqslant y_{rp}, \qquad r = 1, \ldots, s && (2)\\
& \sum_{j=1}^{n} \lambda_j = 1, && (3)\\
& \sum_{r=1}^{s} u_r y_{rp} - \sum_{i=1}^{m} v_i x_{ip} + u_0 = 0, && (4)\\
& \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + u_0 \leqslant 0, \qquad j \in J - \{p\}, && (5)\\
& \sum_{r=1}^{s} u_r + \sum_{i=1}^{m} v_i = 1, && (6)\\
& \sum_{r=1}^{s} k_r + \sum_{i=1}^{m} z_i \leqslant m + s - 1, && (7)\\
& v_i - M_1 z_i \leqslant 0, \qquad i = 1, \ldots, m, && (8)\\
& u_r - M_2 k_r \leqslant 0, \qquad r = 1, \ldots, s, && (9)\\
& z_i \in \{0, 1\}, \qquad i = 1, \ldots, m,\\
& k_r \in \{0, 1\}, \qquad r = 1, \ldots, s,\\
& v_i \geqslant 0, \qquad i = 1, \ldots, m.\\
& u_r \geqslant 0, \qquad r = 1, \ldots, s,\\
& \lambda_j \geqslant 0, \qquad j = 1, \ldots, n.\\
& u_0 \ is\ free
\end{aligned}
\tag{6}
$$

The values of M_1 and M_2 are very large positive numbers. Finally, the model (6) is presented as follows:

The constraints $(4), (5)$ and (6) guarantee that if $(\lambda^*, u^*, v^*, u_0^*)$ is a feasible solution of model (6), then in accordance with Theorem 2.9, an optimal solution of the BCC form will be (u^*, v^*, u_0^*). Constraint (7) together with constraints (8) and (9) ensure that there is at least one zero component in the vector (u^*, v^*). However, the three constraints $(1), (2)$ and (3) along with the objective function, help to assess the extreme DMU. By Charnes, et al. [9], if divided units of the PPS into six groups, model (6) for $DMUs$ of groups $NE, N\acute{E}, NF$ according to Note (2.6) and Note (2.7) and presence of constraints $(4), (5)$ and(6) will be infeasible. The model is feasible for members F, and the objective function value will be greater than zero due to Definition 2.5. Finally, the preceding program (6) for the $DMUs$ in the sets E and $\acute{E}$ is twofold: or all supporting hyperplanes on the PPS and passing through DMU_p (DMU under consideration) are strong efficient, which in this case will be infeasible with respect to the presence of constraint (7), or there exists at least one weak supporting hyperplane including DMU_p where, the model would be feasible, but the objective function value would be zero for the members of set E and would be greater than zero for the members of set $\acute{E}$ with respect to Definition (2.5). Therefore, in general, the following theorem stating the one necessary and sufficient condition to identify anchor points, can be expressed.

Theorem 3.1. *$DMU_p (p \in J)$ is an anchor point if and only if the model* (6) *is feasible and its optimal objective value is equal to zero.*

Proof. Let DMU_p be an anchor point, in this case, by Definition 2.13, DMU_p is an extreme strong efficient unit; therefore due to Definition 2.2, the BCC multiplier form is feasible, and the objective function value is equal to one, thus constraints $(4), (5)$, and (6) are established. Furthermore, DMU_p is located on a supporting hyperplane of T_v, say $H^* = \{(x, y) \in R^{m+s} | u^{*t} y - v^{*t} x + u_0 = 0\}$, which at least one of the components (u^*, v^*) is zero. So, constraint (7) also holds. Since DMU_p is assumed to be an extreme efficient unit, then according to Definition 2.5, constraints $(1), (2)$ and (3) are established, and the objective function

value of model (6) is zero.
To prove the converse, let program (6) is feasible, and the optimal objective value be zero. Suppose $(\lambda^*, u^*, v^*, u_0^*)$ is the optimal solution for this model. From constraints $(4), (5), (6)$ and Theorem 2.9 is deduced $H^* = \{(x, y) \in R^{m+s} | u^{*t}y - v^{*t}x + u_0 = 0\}$ as a supporting hyperplane of the PPS including DMU_p. Forasmuch as presence constraint (7), at least one of the components (u^*, v^*) is equal to zero. Therefore, the weak hyperplane H^* supports T_v at DMU_p. Thus, since the value of the objective function is zero according to the first three constraints and Definition 2.5, unit under consideration is extreme efficient DMU. Therefore, DMU_p is an extreme efficient DMU that weak supporting hyperplane H^* passes through it. □

4. Numerical Examples

In this section, we deal with explanation of the model suggested in the previous section by presentation of three numerical examples.

4.1 Example($DMUs$ with one input and one output)

Consider a system consisting of 6 $DMUs$ listed in Table (1). According to Figure (1) DMU_1, DMU_2, DMU_3 and DMU_4 are extreme efficient and DMU_6 is weak efficient and DMU_5 is inefficient. Model (6) is infeasible for $\{DMU_3, DMU_4\}$ and $\{DMU_5\}$ due to the presence of constraint $\{(7)\}$ and constraints $\{(4), (5), (6), (7)\}$, respectively. Therefore, they are not anchor points. The mentioned program is feasible for DMU_6 and gives weak supporting hyperplane passing through it ($y = 8$), but, the objective value is greater than zero, therefore it is not anchor. This model is feasible for DMU_1 and DMU_2, the objective value is equal to zero for both of them and the supporting hyperplanes passing through them are $x = 2$ and $y = 8$ respectively. Therefore, by Theorem 3.1 they are anchor points.

Table 1: Example 4.1, Data of input and output of DMUs.

	DMU_1	DMU_2	DMU_3	DMU_4	DMU_5	DMU_6
x	2	8	3	5	5	10
y	2	8	5	7	2	8

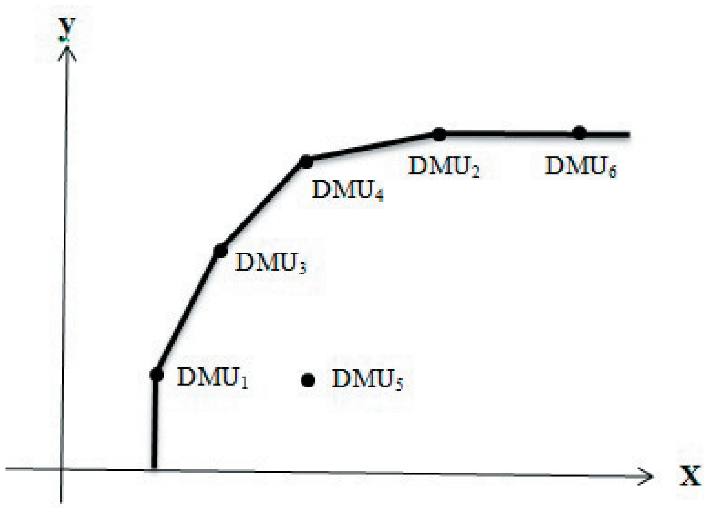

Figure. Example 4.1, DMU_1 and DMU_2 are anchor points.

4.2 Example($DMUs$ with two inputs and one output)

We implement the presented approach for the data listed in Table (2). $DMUs$ have two inputs and one output. For identifying anchor points, it suffices to apply our algorithm for each of the observed $DMUs$. Here, all of them are anchor points, since the mentioned algorithm is feasible, and the objective value is equal to zero for all of them. Moreover, model (6) identifies $y = 10$ as weak supporting hyperplane passing through DMU_3, DMU_4 and $x_1 = 4, x_2 = 4$ pass through DMU_1 and DMU_2, respectively.

Table 2: Example 4.2, Data of inputs and output of DMUs.

	DMU_1	DMU_2	DMU_3	DMU_4
x_1	4	9	8	14
x_2	9	4	14	8
y	7	7	10	10

4.3 Example(Empirical data)

In this subsection, we use a set of data of 20 Iranian banks to evaluate the proficiency of our algorithm (Table 3). The branches are assessed in terms of three inputs and three outputs defined below:

I_1 : staff, I_2 : computer terminals, I_3 : space (m^2),

O_1 : deposits, O_2 : loans, O_3 : charge.

Table 4 in Amirteimoori, et al. [2] presents the input-output data set for these 20 banks. Each of the observed $DMUs$ has been evaluated by the model (6) and the results are summarized in Table 4.

As seen in this table, the model (6) is infeasible for some $DMUs$. These units are either inefficient (internal PPS) or none of the weakly supporting hyperplane.

Therefore, the points are not anchor. On the other hand, this model is feasible for some others (such as $DMU_1, DMU_3, \dots$), but the value of the objective function is larger than zero, resulting in a weak hyperplane passing through the unit under evaluation, but that unit is non-extreme. Therefore, they are not the anchor points.

Problem (6) is feasible for some other units (such as $DMU_4, DMU_8, \dots$) and the value of the objective function is zero.

Thus, according to Theorem 3.1, they are the anchor points.

Table 3: Example (4.3), Data of inputs and outputs of DMUs(extracted from Amirteimoori et al. [2])

Branch	Staff	Computers terminals	Space m^2	Deposits	Loans	Charge
1	0.9503	0.70	0.1550	0.1900	0.5214	0.2926
2	0.7962	0.60	1.0000	0.2266	0.6274	0.4624
3	0.7982	0.75	0.5125	0.2283	0.9703	0.2606
4	0.8651	0.55	0.2100	0.1927	0.6324	1.0000
5	0.8151	0.85	0.2675	0.2333	0.7221	0.2463
6	0.8416	0.65	0.5000	0.2069	0.6025	0.5689
7	0.7189	0.60	0.3500	0.1824	0.9000	0.7158
8	0.7853	0.75	0.1200	0.1250	0.2340	0.2977
9	0.4756	0.60	0.1350	0.0801	0.3643	0.2439
10	0.6782	0.55	0.5100	0.0818	0.1835	0.0486
11	0.7112	1.00	0.3050	0.2117	0.3179	0.4031
12	0.8113	0.65	0.2550	0.1227	0.9225	0.6279
13	0.6586	0.85	0.3400	0.1755	0.6452	0.2605
14	0.9763	0.80	0.5400	0.1443	0.5143	0.2433
15	0.6845	0.95	0.4500	1.0000	0.2617	0.0982
16	0.6127	0.90	0.5250	0.1151	0.4021	0.4641
17	1.0000	0.60	0.2050	0.0900	1.0000	0.1614
18	0.6337	0.65	0.2350	0.0591	0.3492	0.0678
19	0.3715	0.70	0.2375	0.0385	0.1898	0.1112
20	0.5827	0.55	0.5000	0.1101	0.6145	0.7643

Table 4: Example (4.3), Results from model (6)

Branch	Feasible	Objective Value	Anchor	Weak Supporting Hyperplane
1	Yes	>0	No	$0.0345y_1+0.0978y_2-0.8676x_3+0.0769=0$
2	No	-	No	-
3	Yes	>0	No	$0.4063y_1+0.4547y_2-0.1388x_3-0.4629=0$
4	Yes	=0	Yes	$0.1135y_3-0.8864x_3+0.0725=0$
5	No	-	No	-
6	No	-	No	-
7	Yes	>0	No	$0.2016y_1+0.1491y_2+0.0658y_3-0.2185x_1-0.3648x_3+0.0666=0$
8	Yes	=0	Yes	$-x_3+0.12=0$
9	Yes	=0	Yes	$0.1y_2-0.0014x_1-0.8984x_3+0.0855=0$
10	Yes	=0	Yes	$-x_2+0.55=0$
11	No	-	No	-
12	Yes	>0	No	$0.1637y_2+0.0599y_3-0.0985x_1-0.6777x_3+0.0641=0$
13	No	-	No	-
14	No	-	No	-
15	Yes	>0	No	$0.2738y_1-0.7261x_3+0.0529=0$
16	No	-	No	-
17	Yes	=0	Yes	$0.0998y_2-0.9001x_3+0.0846=0$
18	No	-	No	-
19	Yes	=0	Yes	$0.2455y_1-0.7544x_1+0.2708=0$
20	Yes	=0	Yes	$0.1910y_1+0.2052y_3-0.4179x_1-0.1857x_3+0.1584=0$

5. Conclusions

Anchor $DMUs$ were a new collection in the general classification of $DMUs$ in DEA. An anchor point in DEA was an extreme efficient unit for which some inputs could be increased and/or outputs decreased without penetrating the interior of the Production Possibility Set. Their identification has several interesting DEA applications such as the construction of unobserved $DMUs$ to capture prior value judgments in DEA and to identify $DMUs$ that are efficient for multiple constituencies.

In this paper, we developed a novel method to identify the PPS anchor points of the BCC model by searching weak supporting hyperplane passing through unit under consideration. Our proposed idea needs less computational effort than Bougnol´s method [7], because we do not need project frame $DMUs$ on the boundary of simple projections one after another. Furthermore, in most of the studies, it is first necessary to identify the extreme efficient points of the PPS by models then by running the algorithm for each of the methods, the anchor points are determined from the extreme efficient points. In other words, to achieve the goal, at least two or three or more models need to be solved, but in our approach, we will only rely on solving one model to reach the ultimate goal as, the advantage of our method. Diminution in the computational commitments of the procedures given for recognition of the anchor points can be worth studying in the future.

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Ali Akbar Bani
Graduated Ph.D of Mathematics
Department of Mathematics
Science and Research Branch, Islamic Azad University
Tehran, Iran
E-mail: A.bani@gorganiau.ac.ir ; bani_ali@yahoo.com

Mohsen Rostamy-Malkhalifeh
Associate Professor of Mathematics
Department of Mathematics
Science and Research Branch, Islamic Azad University
Tehran, Iran
E-mail: mohsen_rostamy@yahoo.com

Farhad Hosseinzadeh Lotfi
Professor of Mathematics
Department of Mathematics
Science and Research Branch, Islamic Azad University
Tehran, Iran
E-mail: farhad@hosseinzadeh.ir

Dariush Akbarian
Assistant Professor of Mathematics
Department of Mathematics
Arak Branch, Islamic Azad University
Arak, Iran
E-mail: d-akbarian@iau-arak.ac.ir
E-mail: d_akbarian@yahoo.com