Journal of Mathematical Extension
Vol. 14, No. 2, (2020), 121-133
ISSN: 1735-8299
URL: http://www.ijmex.com

Weighted Composition Operators Acting on Dirichlet Type Spaces

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Abstract. In this paper we study the weighted composition operator ψC_ϕ on the Dirichlet type spaces $D^p_\alpha(\mathbb{D})$. We show that if the weighted composition operator ψC_ϕ is bounded on such spaces(for $1 \leqslant p < 2$) then the related measure $\mu_{p,\alpha,\psi}$ is a Carleson measure. Also we show that if the weighted composition operator ψC_ϕ is an isomety on $D^p_\alpha(\mathbb{D})$ then $\psi.\phi$ is a rotation map on $\mathbb{D}$.

AMS Subject Classification: 47B38; 30H05; 47B33
Keywords and Phrases: Weighted composition operator, dirichlet type spaces, boundedness, carleson measure, isometry

1. Introduction

Let $\mathbb{D}$ be the open unit disc in the complex plane $\mathbb{C}$. The Lebesgue area measure on $\mathbb{D}$ is defined by $dA(z) = rdrd\theta = dxdy$. Denote by $H(\mathbb{D})$ the class of all analytic functions on $\mathbb{D}$.

Given $\alpha > -1$ and $p \geqslant 1$, the standard weighted Bergman space $A^p_\alpha(\mathbb{D})$ consists of all analytic functions f on $\mathbb{D}$ for which

$$||f||^p_{A^p_\alpha} = \int_{\mathbb{D}} |f(z)|^p(1 - |z|^2)^\alpha dA(z)$$

Received: July 2018; Accepted: December 2018
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is finite. The space A_0^2, usually denoted by A^2, is called the (unweighted) Bergman space. Note that $A_\alpha^p(\mathbb{D})$ is Banacah space for $p \geqslant 1$ and Hilbert space for $p = 2$ (see [5], [6] and [21] for the theory of these spaces).

The weighted Dirichlet type space $\mathcal{D}_\alpha^p$ is the space of all f in $H(\mathbb{D})$ such that $f' \in A_\alpha^p$, equipped with the norm $||f||_{\mathcal{D}_\alpha^p} = |f(0)| + ||f'||_{A_\alpha^p}$. The space $\mathcal{D} = \mathcal{D}^2$ is the classical Dirichlet space of analytic functions. Clearly $\mathcal{D}^p \subset \mathcal{D}^q$ when $1 \leqslant q < p$.

Given an analytic functions φ and ψ in the unit disc $\mathbb{D}$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$, the weighted composition operator ψC_φ defined by

$$\psi C_\varphi f(z) = \psi(z) f(\varphi(z)),$$

for $f \in H(\mathbb{D})$ and $z \in \mathbb{D}$. The composition operator C_φ is obtained when $\psi = 1$. Clearly these operators are linear. By an easy application of the Closed Graph Theorem, the action of these operators on any classical Banach or Hilbert space of analytic functions in the unit disc implies theirs boundedness on that space.

A linear operator T on a Banach space X is said to be an linear isometry if

$$||Tf||_X = ||f||_X,$$

for any $f \in X$. On a Hilbert space this is equivalent to $T^*T = I$; if the (Hilbert space) isometry is also onto, it is called a unitary operator and is characterized by the property $T^*T = TT^* = I$.

Let μ be a finite positive Borel measure on $\mathbb{D}$. For $|\xi| = 1$ and $0 < \delta \leqslant 2$, $S(\xi, \delta)$ is the Calreson set $\{z \in \mathbb{D} : \ |z - \xi| < \delta\}$. The measure μ is said to be Carleson if there is a constant C such that $\mu(S(\xi, \delta) \leqslant C\delta^{2+\alpha}$ for all ξ and δ. Carleson measures have been useful in the study of composition operators in several settings (see [7], [11], [12], [13], [20]). For $w \in \mathbb{D}$, let $N_2(\phi, w)$ denotes the number of zeroes (counting multiplicities) of the equation $\phi(z) - w$. for $1 \leqslant p < 2$ and $w \in \mathbb{D}$ and analytic map ψ on $\mathbb{D}$, we define modified counting function

$$N_{p,\alpha,\psi}(\phi, w) = \sum \frac{(1 - |z|^2)^\alpha |\psi(z)|^p}{|\phi'(z)|^{2-p}}$$

where the sum extends over the zeros of $\phi - w$, repeated by multiplicity. In particular, $N_{p,\alpha,\psi}(\phi, w) = 0$ for $w \notin \phi(\mathbb{D})$. Clearly with $\psi = 1$, $\alpha = 0$ and $p = 2$ we have $N_2(\phi, w)$.

Let $\mu_{p,\alpha,\psi}$ be the measure defined on $\mathbb{D}$ by
$d\mu_{p,\alpha,\psi}(w) = N_{p,\alpha,\psi}(\phi, w)dA(w)$, $1 \leqslant p < 2$.

Interest in the spaces of D^p is motivated by the work of R. Roan [17] and B. D. MacCluer [12], who studied composition operators on S^p, the space of functions with derivatives in the Hardy space H^p for $p \geqslant 1$. In [15], G. Mirzakarimi and K. Seddighi characterize compact weighted composition operators on D^2_α. Other related works appears in [13], where MacCluer and J. H. Shapiro studied the boundedness and compactness of composition operators on the weighted Dirichlet space D^p_α in the case $p = 2$. Also the isometric composition operators on analytic function spaces has been studied by many authors. For more information see [1, 2, 3]. In [8] Kolaski gave a characterization of all surjective isometries of a weighted Bergman space A^p_α. The Hilbert Bergman space A^2_α, of course, possesses plenty of isometries. Carswell and Hamond [2] have shown, among other results that the only composition operators that are isometries of the weighted Hilbert Bergman space A^2_α are the rotations. As another well-known analytic function space, Hilbert space H^2 has plenty of isometries. How ever Nordgreen [16] showed that the only isometries of H^2 among the composition operators are the operators induced by inner functions that vanish at the origin. Also Bayart [1] showed that every composition operator on H^2 which is similar to an isometry is induced by an inner function with a fixed point in the disk.

In this article we study the boundedness of the weighted composition operators on Dirichlet type spaces $D^p_\alpha(\mathbb{D})$ in the case $1 \leqslant p < 2$ and $\alpha > -1$. The relations between the boundedness of such operators and a special class of measures on the unit disk, Carleson measure, is shown. Also we study the isometric composition operators on $D^p_\alpha(\mathbb{D})$ and obtain the condition for which the isometric weighted composition operator ψC_ϕ define an isometric composition operator $C_{\psi.\phi}$ on the space $D^p_\alpha(\mathbb{D})$.

2. Weighted Composition Operators on Dirichlet Type Spaces

In this section weighted composition operators on Dirichlet type spaces is examined.

Definition 2.1. *Suppose $1 \leqslant p < 2$, $\alpha > -1$ and μ is a finite positive Borel measure on $\mathbb{D}$. Then μ is a Carleson measure for $A^p_\alpha(\mathbb{D})$, if and only if $A^p_\alpha(\mathbb{D}) \subset L^p(\mu)$ and the identity map*

$$I_\alpha : A^p_\alpha \to L^p(\mu)$$

is a bounded linear operator. It is equivalent with following statement: There exists a constant C such that

$$\int_{\mathbb{D}} |f(z)|^p d\mu(z) \leqslant C||f||^p_{A^p_\alpha(\mathbb{D})},$$

for allf $\in A^p_\alpha(\mathbb{D})$.

The action of composition operators and weighted composition operators on analytic function spaces such as Bergman, Hardy, Dirichlet and Dirichlet type spaces has been studied by many authors (see for example [7], [9], [10], [19] and [20]). The books [4] and [18] are good monographs in this content.

In [20], Zorboska has studied bounded and compact composition operators on weighted Dirichlet spaces. His method involves integral averages of determining function for the operator. In this article we study the boundedness of the weighted composition operators on the Dirichlet type spaces $\mathcal{D}^p_\alpha(\mathbb{D})$ for $1 \leqslant p < 2$.

Theorem 2.2. *The composition operator C_ϕ is bounded on $D^p_\alpha(\mathbb{D})$ if and only if $\mu_{p,\alpha,1}$ is a Carleson measure.*

Proof. Suppose C_ϕ is bounded on D^p_α then for $f \in D^p_\alpha(\mathbb{D})$ with $f(0) = 0$ there exists a constant C such that $||C_\phi(f)||_{D^p_\alpha(\mathbb{D})} \leqslant C||f||_{D^p_\alpha(\mathbb{D})}$. Since the evaluation at $\phi(0)$ is bounded operator on $D^p_\alpha(\mathbb{D})$, so

$$\int_{\mathbb{D}} |f'(\phi(z)|^p|\phi'(z)|^p(1-|z|^2)^\alpha dA(z) \leqslant C \int_{\mathbb{D}} |f'(z)|^p(1-|z|^2)^\alpha dA(z). \quad (1)$$

By the usual change of variable formula, if $w = \phi(z)$ then $dA(w) = |\phi'(z)|^2 dA(z)$. So

$$\int_{\mathbb{D}} |f'(\phi(z))|^p |\phi'(z)|^p (1-|z|^2)^\alpha dA(z) = \int_{\mathbb{D}} |f'(w)|^p N_{p,\alpha,1}(\phi, w) dA(w)$$
$$= \int_{\mathbb{D}} |f'(w)|^p d\mu_{p,\alpha,1}(w). \tag{2}$$

Let $g \in A^p_\alpha(\mathbb{D})$ and $f(z) = \int_0^z g(w)dw$. Then $f'(z) = g(z)$ and inequality (1) implies that

$$\int_{\mathbb{D}} |g(z)|^p d\mu_{p,\alpha,1}(z) \leqslant C \int_{\mathbb{D}} |g(z)|^p (1-|z|^2)^\alpha dA(z).$$

Hence by Remark 1, $\mu_{p,\alpha,1}$ is a Carleson measure.

Now suppose that $\mu_{p,\alpha,1}$ is a Carleson measure. Then for $g \in A^p_\alpha(\mathbb{D})$ we have

$$\int_{\mathbb{D}} |g(z)|^p d\mu_{p,\alpha,1}(z) \leqslant C \int_{\mathbb{D}} |g(z)|^p (1-|z|^2)^\alpha dA(z),$$

for a Constant C. But for $f \in D^p_\alpha(\mathbb{D})$ we have $f' \in A^p_\alpha(\mathbb{D})$. By using formula (2) and since eveluation at $\phi(0)$ is a bounded linear operator on D^p_α, we have

$$||fo\phi||_{D^p_\alpha(\mathbb{D})} = \int_{\mathbb{D}} |(fo\phi)'(z)|^p (1-|z|^2)^\alpha dA(z) + |fo\phi(0)|$$
$$= \int_{\mathbb{D}} |f'(z)|^p d\mu_{p,\alpha,1}(z) + |fo\phi(0)|$$
$$\leqslant C_1 \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^\alpha dA(z) = C_1 ||f||_{D^p_\alpha(\mathbb{D})},$$

for constant C_1. So C_ϕ is bounded on $D^p_\alpha(\mathbb{D})$. □

Theorem 2.3. *Suppose that the weighted composition operator ψC_ϕ is bounded on $D^p_\alpha(\mathbb{D})$, then $\mu_{p,\alpha,\psi}$ is a Carleson measure.*

Proof. Suppose that ψC_ϕ is bounded on $D^p_\alpha(\mathbb{D})$ and $f \in D^p_\alpha(\mathbb{D})$ with $f(0) = 0$. Hence there exists a constant C such that

$$||\psi C_\phi(f)||_{D^p_\alpha(\mathbb{D})} \leqslant C ||f||_{D^p_\alpha(\mathbb{D})}.$$

So for a constant C_p we have

$$\int_{\mathbb{D}} |\psi(z)|^p ||f'(\phi(z))||^p |\phi'(z)|^p (1-|z|^2)^\alpha dA(z)$$

$$+\int_{\mathbb{D}} |\psi'(z)|^p ||f(\phi(z))||^p (1-|z|^2)^\alpha dA(z)$$

$$\leqslant C_p \int_{\mathbb{D}} ||f'(z)||^p (1-|z|^2)^\alpha dA(z).$$

Thus

$$\int_{\mathbb{D}} |\psi(z)|^p ||f'(\phi(z))||^p |\phi'(z)|^p (1-|z|^2)^\alpha dA(z)$$

$$\leqslant C_p \int_{\mathbb{D}} ||f'(z)||^p (1-|z|^2)^\alpha dA(z). \tag{3}$$

By the usual change of variable formula, if $w = \phi(z)$, similar to the proof of theorem 2.2,

$$\int_{\mathbb{D}} |\psi(z)|^p ||f'(\phi(z))||^p |\phi'(z)|^p (1-|z|^2)^\alpha dA(z) = \int_{\mathbb{D}} ||f'(w)||^p N_{p,\alpha,\psi} dA(z) \tag{4}$$

$$= \int_{\mathbb{D}} ||f'(w)||^p d\mu_{p,v,\psi}(w).$$

Let $g \in A^p_\alpha(\mathbb{D})$ and $f(z) = \int_0^z g(w)dw$. Then $f'(z) = g(z)$ for $z \in \mathbb{D}$ and according to (3) and (4) we have

$$\int_{\mathbb{D}} ||g(z)||^p d\mu_{p,\alpha,\psi} \leqslant \int_{\mathbb{D}} ||g(z)||^p (1-|z|^2)^\alpha dA(z).$$

Hence by Definition 2.1 $\mu_{p,\alpha,\psi}$ is a Carleson measure. $\square$

Lemma 2.4. *Suppose that the weighted composition operator ψC_ϕ is bounded on $D^p_\alpha(\mathbb{D})$. Then for $1 \leqslant q < p$, $\psi.\phi$ is in $D^q_\alpha(\mathbb{D})$.*

Proof. If ψC_ϕ is bounded on $D^p_\alpha(\mathbb{D})$, then for any $f \in D^p_\alpha(\mathbb{D})$ there exists a constant M such that

$$||\psi C_\phi f||_{D^p_\alpha(\mathbb{D})} \leqslant M ||f||_{D^P_\alpha(\mathbb{D})} < \infty.$$

If we let $f = Identity$, then

$$||\psi.\phi||_{D^p_\alpha(\mathbb{D})} = ||\psi C_\phi(Id)||_{D^p_\alpha(\mathbb{D})} < \infty.$$

So $\psi.\phi$ is in $D^p_\alpha(\mathbb{D})$ and thus for $q < p$, is in $D^q_\alpha(\mathbb{D})$. $\square$

Proposition 2.5. *Suppose that the weighted composition operator ψC_ϕ is bounded on $D^p_\alpha(\mathbb{D})$. Then for $1 \leqslant q < p$, $\mu_{q,\alpha,\psi}$ is a finite measure on $\mathbb{D}$.*

Proof. If ψC_ϕ is bounded on $D^p_\alpha(\mathbb{D})$ then for $f \in D^p_\alpha$ we have

$$||\psi f o\phi||_{D^p_\alpha(\mathbb{D})} < \infty.$$

So if we consider $f = 1$, then

$$\psi C_\phi f(z) = \psi C_\phi 1(z) = \psi(z)1(\phi(z)) = \psi(z).$$

Thus

$$||\psi C_\phi 1||_{D^p_\alpha(\mathbb{D})} = |\psi(0)| + \int_{\mathbb{D}} |\psi'(z)|^p(1 - |z|^2)^\alpha dA(z) < \infty. \tag{5}$$

The triangle inequality gives us

$$|\psi\phi'| \leqslant |\psi'\phi + \psi\phi'| + |\psi'\phi|.$$

Since $Im\ \phi \subset \mathbb{D}$ and $|\phi(z)| \leqslant 1$ for $z \in \mathbb{D}$, so for a constant C_p we have

$$\int_{\mathbb{D}} |\psi(z)|^p|\phi'(z)|^p(1 - |z|^2)^\alpha dA(z)$$

$$\leqslant \int_{\mathbb{D}} [|(\psi\phi')(z) + (\psi'\phi)(z)| + |(\psi'\phi)(z)|]^p(1 - |z|^2)^\alpha dA(z)$$

$$\leqslant C_p \int_{\mathbb{D}} |(\psi\phi)'(z)|^p(1 - |z|^2)^\alpha dA(z) + C_p \int_{\mathbb{D}} |(\psi'\phi)(z)|^p(1 - |z|^2)^\alpha dA(z)$$

$$\leqslant C_p \int_{\mathbb{D}} |(\psi\phi)'(z)|^p(1 - |z|)^2)^\alpha dA(z) + C_p \int_{\mathbb{D}} |\psi'(z)|^p(1 - |z|^2)^\alpha dA(z).$$

Hence, according to formula (5) and Lemma 4,

$$\int_{\mathbb{D}} |\psi(z)|^p|\phi'(z)|^p(1 - |z|^2)^\alpha dA(z) < \infty.$$

Thus the Holder inequality gives us

$$\mu_{q,\alpha,\psi}(\mathbb{D}) = \int_{\mathbb{D}} N_{q,\alpha,\psi}(\phi, w)dA(w)$$

$$= \int_{\mathbb{D}} |\psi(z)|^q|\phi'(z)|^q(1-|z|^2)^\alpha dA(z)$$

$$\leqslant (\int_{\mathbb{D}} |\psi(z)|^p|\phi'(z)|^p(1-|z|^2)^\alpha dA(z))^{\frac{q}{p}}(\int_{\mathbb{D}} 1(1-|z|^2)^\alpha dA(z))^{\frac{p-q}{p}} \leqslant \infty.$$

So the proposition is proved. □

Theorem 2.6. *Suppose that the weighted composition operator ψC_ϕ is bounded on $D^p_\alpha(\mathbb{D})$ and $1 \leqslant q < p$. Then $\mu_{q,\alpha,\psi}$ is a Carleson measure.*

Proof. Since $q < p$, Holder inequality implies that

$$\mu_{q,\alpha,\psi}(S(\zeta,\delta)) = \int_{\phi^{-1}(S(\zeta,\delta))} |\psi\phi'|^q(1-|z|^2)^\alpha dA(z) \leqslant$$

$$(\int_{\phi^{-1}(S(\zeta,\delta))} |\psi|^p|\phi'|^p(1-|z|^2)^\alpha dA(z))^{\frac{q}{p}}(\int_{\phi^{-1}(S(\zeta,\delta))} 1(1-|z|^2)^\alpha dA(z))^{\frac{p-q}{p}}$$

$$\leqslant \mu_{p,\alpha,\psi}(S(\zeta,\delta))^{\frac{q}{p}} A(\phi^{-1}(S(\zeta,\delta))^{\frac{p-q}{p}}). \tag{6}$$

Since ψC_ϕ is bounded on standard weighted Bergman spaces, then Definition 2.1 implies that the measure $A\phi^{-1}$ is Carleson. Thus inequality (6) yields

$$\mu_{q,\alpha,\psi}(S(\zeta,\delta)) \leqslant (C_1\delta^{2+\alpha})^{\frac{p-q}{p}} \mu_{p,\alpha,\psi}(S(\zeta,\delta))^{\frac{q}{p}}, \tag{7}$$

for a constant C_1. Since ψC_ϕ is bounded on D^p_α so according to Theorem 2.3, there is a constant C_2 such that $\mu_{p,\alpha,\psi} \leqslant C_2\delta^{2+\alpha}$, for all ζ and δ as described above. Thus formula (7) implies that $\mu_{q,\alpha,\psi}(S(\zeta,\delta)) \leqslant C\delta^{2+\alpha}$, for a constant C. So $\mu_{q,\alpha,\psi}$ is a Carleson measure. □

3. Isometric Weighted Composition Operators on Dirichlet Type Spaces

In this section we characterized all isometries among the weighted composition operators on the Dirichlet type spaces by using simple ideas. The following two theorems has been proved in [9]

Theorem 3.1. *Let ϕ be a self map of the unit disk. Then the induced composition operator C_ϕ is a surjective isometry on the weighted Dirichlet type space D^p_α, $1 \leqslant p < \infty$, $-1 < \alpha < \infty$ if and only if ϕ is a rotation map.*

Theorem 3.2. *Let $\phi \in Aut(\mathbb{D})$. Then the induced composition opearator C_ϕ is an isometry on the weighted Dirichlet type space $D^p_\alpha(\mathbb{D})$, $1 \leqslant p < \infty$, and $-1 \leqslant \alpha < \infty$, if and only if ϕ is a rotation map.*

To prove our main results in this section, we use the following lemma that has been proved in [14].

Lemma 3.3. *Let μ be a positive measure on the measure space Ω, $\mathcal{M}$ a subspace of $L^p(\Omega, d\mu)$, $1 \leqslant p < \infty$ and let T be a linear isometry of $\mathcal{M}$. Then*

$$\int_\Omega Tf|Tg|^{p-2}\overline{Tg}d\mu = \int_\Omega f|g|^{p-2}\overline{g}d\mu$$

for all f, g in the subspace $\mathcal{M}$.

Theorem 3.4. *If the weighted composition operator ψC_φ is an isometry of $D^p_\alpha(\mathbb{D})$ then $(\psi.\varphi)(0) = 0$.*

Proof. Consider $\mathcal{M} = D^p_\alpha(\mathbb{D})$ a subspace of $L^p(\mathbb{D}, (1-|z|^2)^\alpha dA)$, respectively. Put $g = 1$ and $Tf = \psi C_\varphi f$ in Lemma 3.3 Use the Mean Value Property to get

$$\int_{\mathbb{D}} fwdA = 2\int_0^1 (\int_0^{2\pi} f(re^{i\theta})dm(\theta))w(r)rdr$$

$$= 2\int_0^1 f(0)w(r)rdr = c_w f(0),$$

for some positive constant c_w. Using the Lemma 3.3, we get

$$\int_{\mathbb{D}} fwdA = \int_{\mathbb{D}} (\psi C_\phi f)|\psi C_\phi 1|^{p-2}(\overline{\psi C_\phi 1})wdA =$$

$$\int_{\mathbb{D}} (\psi C_\varphi f)|\psi|^{p-2}(\overline{\psi})wdA = \int_{\mathbb{D}} |\psi|^p C_\phi fwdA =$$

$$2\int_0^1 (\int_0^{2\pi} (|\psi|^p fo\phi)(re^{i\theta})dm(\theta))w(r)rdr = c_w|\psi|^p(0)(fo\phi)(0).$$

Hence $f(0) = \psi(0)(fo\phi)(0)$. Choosing $f(z) = I$,we get the result. □

Theorem 3.5. *Suppose that the weighted composition operator ψC_ϕ is an isometry of $D^p_\alpha(\mathbb{D})$ and $\psi.\varphi \in Aut(\mathbb{D})$, Then $\psi.\varphi$ is a rotation map.*

Proof. Let $\gamma(z) = (\psi.\phi)(z)$. Then according to theorem 3.4 $\gamma(0) = 0$. But the composition operator ψC_ϕ is an isometry on $D^p_\alpha(\mathbb{D})$, so we have $||\psi C_\phi(f)||_{D^p_\alpha(\mathbb{D})} = ||f||_{D^p_\alpha(\mathbb{D})}$, for $f \in D^p_\alpha(\mathbb{D})$. Consider $f = Identity$, then $||\psi.\phi||_{D^p_\alpha(\mathbb{D})} = ||I||_{D^p_\alpha(\mathbb{D})}$. So $||(\psi.\phi)'||_{A^p_\alpha(\mathbb{D})} = ||1||_{A^p_\alpha(\mathbb{D})}$. Hence

$$0 = ||1||^p_{A^p_\alpha(\mathbb{D})} - ||(\psi\phi)'||^p_{A^p_\alpha(\mathbb{D})} = \int_{\mathbb{D}} |\ |1|^p - |(\psi\phi)'|^p|(1-|z|^2)^\alpha dA(z).$$

This equality is only possible if $|1 - |(\psi.\phi)'|\ | = 0$. So

$$\psi.\phi(z) = |\lambda|z$$

for $|\lambda| = 1$. That is $\psi.\phi$ is a rotation. □

Corollary 3.6. *Suppose that the weighted composition operator ψC_ϕ is an isometry of $D^p_\alpha(\mathbb{D})$ and $\psi.\phi \in Aut(D)$ Then the un-weighted composition operator $C_{\psi.\varphi}$ is an isometry of $D^p_\alpha(\mathbb{D})$.*

Proof. The corollary follows by using theorem 3.5 and Theorem 3.2. □

4. Conclusion

In this paper we study the relation between the boundedness of the weighted composition operator ψC_ϕ and the Carleson measure $\mu_{p,\alpha,\psi}$. Also we study the isometric weighted composition operators on Dirichlet type spaces $D^p_\alpha(\mathbb{D})$. In Section 2, firs we show that if the weighted composition operator ψC_ϕ is bounded on $D^p_\alpha(\mathbb{D})$, then the measure $\mu_{p,\alpha,\psi}$ is a Carleson. Next, we show that the boundedness of ψC_ϕ on $D^p_\alpha(\mathbb{D})$ implies that for $1 \leqslant q < p$, $\mu_{q,\alpha,\psi}$ is also a Carleson measure.

In Section 3, we show that if the weighted composition operator ψC_ϕ is an isometry on Dirichlet type spaces, then $\psi.\phi$ will be a rotation map.

References

[1] F. Bayart, Similarity to an isometry of a composition operator, *Proc. Amer. Math. Soc.*, 131 (6) (2003), 1789-1791.

[2] B. J. Carswell and C. Hammond, Composition operators with maximal norm on weighted Bergman spaces, *Proc. Amer. Math. Soc.*, 134 (2006), 2599-2605.

[3] B. A. Cload, Composition operators: hyperinvariant subspaces, quasinormals and isometries, *Proc. Amer. Math. Soc.*, 127 (6) (1999), 1697-1703.

[4] C. Cowen and B. MacCluer, *Composition Operators on Spaces of Analytic Functions*, Studies in Advanced Mathematics. Boca Raton: CRC Pres, 1995.

[5] P. Duren and A. Shuster, *Bergman Spaces*, Providence Rhode Island: Amer. Math. Soc., 2003.

[6] H. Hedenmalm, B. Korenblum, and K. Zhu, *Theory of Bergman Spaces*, New York: Springer, 2000.

[7] M. Jovovic and B. D. MacCluer, Composition operator on Dirichlet spaces, *Acta. Sci. Math.,* 63 (1997), 229-247.

[8] C. J. Kolaski, Isometries of weighted Bergman spaces, *Canad. J. Math.*, 34 (4) (1982), 910-915.

[9] L-G. Geng, ZH. Zhou, and ZT. Dong, Isometric composition operators on wighted Dirichlet type spaces, *J. Inequal. App.,* 23 (2012), 1029-1036.

[10] U. Gul, Essential spectra of composition operators on the space of bounded analytic functions, *Turk. J. Math.,* 32 (2008), 475-480.

[11] B. D. MacCluer, Compact composition operator on $H^p(B_N)$, *Mich. Math. J.,* 32 (1985), 237-248.

[12] B. D. MacCluer, Composition operators on S^p, *Houst. J. Math.,* 13 (1987), 245-254.

[13] B. D. Maccluer and J. H. Shapiro, Angular derivatives and compact composition operators on the Hardy and Bergman spaces, *Canad. J. Math.,* 38 (1986), 878-906.

[14] M. J. Martin and D. Vukotic, Isometries of some classical function spaces among the composition operators, *Contemp. Math.*, 393 (2006). 133-138.

[15] G. Mirzakarimi and K. Seddighi, Weighted composition operators on Bergman and Dirichlet spaces, *Georgian Math. J.,* 4 (1997), 373-383.

[16] E. A. Nordgren, Composition operators, *Canad. J. Math.*, 20 (1968), 442-449.

[17] R. Roan, Composition operators on the space of functions, *Duke Math. J.,* 4 (1978), 423-238.

[18] J. H. Shapiro, *Composition Operators and Classical Function Theory.* Berlin: Springer-Verlag, 1993.

[19] E. Wolf, Weighted composition operators between weighted Bergman spaces. *Rev. R. Acad. Cien, Series A. Math.,* 103 (1) (2009), 11-15.

[20] N. Zorboska, Composition operators on weighted Dirichlet spaces, *Proc. Amer. Math. Soc.,* 126 (1998), 2013-2023.

[21] K. Zhu, *Operator Theory in Function Spaces*, Marcel Dekker: New York, 1990.

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