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Spaceability of the Set of Non Strongly McShane (Product) Integrable Functions

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Abstract. Let X be a unitial Banach algebra, $\mathcal{G}$ be the set of functions $f : [0,1] \to X$ that are non strongly McShane integrable. The aim of this paper is to show that there is an infinite dimensional closed vector space in $\mathcal{G} \cup \{0\}$.

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1. Introduction

In this paper we investigate large linear structures within the set of functions with special integrability properties. The appropriate terminology are lineability, spaceability and algebrability. These terminologies were considered by many authors (see e.g. [1, 2, 3, 4, 6, 7]), whose definitions are as follow:

Definition 1.1. *Suppose that κ is a cardinal number.*

- *Let $\mathcal{L}$ be a vector space and A be a subset of $\mathcal{L}$. We say that A is κ-lineable if $A \cup \{0\}$ contains a κ-dimensional vector space;*

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- *Let $\mathcal{L}$ be a Banach space and A be a subset of $\mathcal{L}$. We say that A is spaceable if $A \cup \{0\}$ contains an infinite dimensional closed vector space;*
- *Let $\mathcal{L}$ be a linear commutative algebra and A be a subset of $\mathcal{L}$. We say that A is κ-algebrable if $A \cup \{0\}$ contains a κ-generated algebra B (i.e., the minimal system of generators of B has cardinality κ);*
- *Let $\mathcal{L}$ be a linear commutative algebra and A be a subset of $\mathcal{L}$. We say that A is strongly κ-algebrable if $A \cup \{0\}$ contains a κ-generated algebra B that is isomorphic with a free algebra (denote by $X = \{x_\alpha : \alpha < \kappa\}$ the set of generators of this free algebra);*
- *The set $X = \{x_\alpha : \alpha < \kappa\}$ is a generating set of some free algebra contained in $A \cup \{0\}$ if and only if the set $\widetilde{X}$ of elements of the form $x_{\alpha_1}^{k_1} x_{\alpha_2}^{k_2} ... x_{\alpha_n}^{k_n}$ is linearly independent and all linear combinations of elements from $\widetilde{X}$ are in $A \cup \{0\}$; equivalently for any $\kappa \in \mathbb{N}$, any nonzero polynomial P in κ variables without a constant term and any distinct $y_1, y_2, ..., y_k \in X$, we have $P(y_1, ..., y_k) \neq 0$ and $P(y_1, ...y_k) \in A$.*

So far a variety of sets of functions have turned out to enjoy some of the above properties. For example the set of continuous nowhere differentiable functions on [0,1] is lineable [9], and also spaceable [8], the set of continuous nowhere Hölder functions is $\mathfrak{c}$-algebrable, the set of differentiable functions on $\mathbb{R}$ which are nowhere monotone is $\mathfrak{c}$-lineable and so on. For a good survey about large algebraic structures see [12].

The present paper deals with strongly McShane (product) integrable functions (see Definitions 1.4 and 1.5) from the point of view of large algebraic structures. In fact we show that the set of non strongly McShane (product) integrable functions is spaceabl as well as strongly $\mathfrak{c}$-algebrable.

We need the following notation.

A *tagged partition* of an interval $[a, b]$ is a collection of point-interval pairs $D = (\xi_i, [t_{i-1}, t_i])_{i=1}^m$, where $a = t_0 \leqslant t_1 \leqslant ... \leqslant t_m = b$ and $\xi_i \in [t_{i-1}, t_i]$ for every $i \in \{1, 2, ..., m\}$.

If we replace $\xi_i \in [t_{i-1}, t_i]$ by $\xi_i \in [a, b]$, then the collection D is called a *free tagged* partition. Given a function $\delta : [a, b] \to \mathbb{R}^+$ (called a gauge on $[a, b]$), a free tagged partition is called *δ-fine* if

$$[t_{i-1}, t_i] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i)), \quad i = \{1, 2, ..., m\}.$$

In the rest of this paper, we assume that X is a real and unital Banach algebra with infinite dimension, and for all $f : [a, b] \to X$, we define $\|f\|_X(x) := \|f(x)\|_X$, for each $x \in [a, b]$.

Definition 1.2. *([14, Theorem 1.3.11]) A function $A : [a, b] \to X$ is called Bochner integrable if there is a sequence of simple functions $A_n : [a, b] \to X$, $n \in \mathbb{N}$ such that*

$$\lim_{n \to \infty} A_n(x) = A(x) \quad a.e., \quad on \quad [a, b],$$

and

$$\lim_{n \to \infty} \int_a^b \|A_n - A\|_X = 0.$$

Definition 1.3. *([13]) A function $A : [a, b] \to X$ is called Henstock-Kurzweil integrable if there exists a vector $S_f \in X$ with the following property: For each $\epsilon > 0$ there is a gauge $\delta : [a, b] \to \mathbb{R}^+$ such that*

$$\left\| \sum_{i=1}^{m} f(\xi_i)(t_i - t_{i-1}) - S_f \right\|_X < \varepsilon, \tag{1}$$

for every δ-fine tagged partition of $[a, b]$. In this case, $S_f \in X$ is called Henstock-Kurzweil integral of f over $[a, b]$, and is denoted by $\int_a^b f(t)dt$. If (1) holds for all δ-fine free tagged partitions of $[a, b]$, then f is called McShane integrable over $[a, b]$.

Definition 1.4. *([13]) A function $A : [a, b] \to X$ is called strongly Henstock-Kurzweil integrable if there is a function $B : [a, b] \to X$ such that for each $\epsilon > 0$ there is a gauge $\delta : [a, b] \to \mathbb{R}^+$ such that*

$$\sum_{i=1}^{m} \|A(\xi_i)(t_i - t_{i-1}) - (B(t_i) - B(t_{i-1}))\|_X < \varepsilon, \tag{2}$$

for every δ-fine tagged partition of $[a, b]$. If (2) holds for all δ-fine free tagged partitions of $[a, b]$, then f is called strongly McShane integrable over $[a, b]$.

Definition 1.5. *([13, definition 3.4]) A function $A : [a, b] \to X$ is called strongly Kurzweil product integrable if there is a function $W : [a, b] \to X$ such that $W(t)^{-1}$ exists for all $t \in [a, b]$, both W and W^{-1} are bounded, and for every $\epsilon > 0$, there is a gauge $\delta : [a, b] \to \mathbb{R}^+$ such that*

$$\sum_{i=1}^{m} \left\| I + A(\xi_i)(t_i - t_{i-1}) - (W(t_i)W(t_{i-1})^{-1}) \right\|_X < \varepsilon, \tag{3}$$

for every δ-fine tagged partition of $[a, b]$. In this case, we define the strongly McShane product integral as $\prod_a^b (I + A(t)dt) = W(b)^{-1}W(a)$. If (3) holds for all δ-fine free tagged partitions of $[a, b]$, then A is called strongly McShane product integrable over $[a, b]$, and is defined as $\prod_a^b (I + A(t)dt) = W(b)^{-1}W(a)$.

2. Spaceability of Sets of Non Strongly McShane (product) Integrable Functions

Let $\mathcal{G}$ be the set of functions $f : [0, 1] \to X$ that are non strongly McShane (product) integrable, such that $\mathcal{G}$ endowed with the supremum norm. Our aim is to show that the set $\mathcal{G}$ is spaceable. In order to do this, let us recall some theorems that will be needed, and use the idea from [10, 11] to produce the assertions.

Theorem 2.1. *([14, Theorem 5.1.4]) A function $f : [a, b] \to X$ is Bochner integrable if and only if f is strongly McShane integrable.*

Theorem 2.2. *([13, Theorem 4.14]) For every function $f : [a, b] \to X$, f is strongly McShane product integrable if and only if f is Bochner integrable.*

Theorem 2.3. *([14, Theorem 1.4.3]) A measurable function $f : [a, b] \to$*

X is Bochner integrable if and only if $\|f\|_X : [a,b] \to [0,\infty)$ is Bochner integrable.

The next proposition is a usefull technique of proving strong McShane integrability of functions. Assume that $\{z_k\}_{k\in\mathbb{N}}$ is a sequence in X and define the function $f : [0,1] \to X$ as follows:

$$f := \sum_{k=1}^{\infty} \chi_{(\frac{1}{2^k}, \frac{1}{2^{k-1}}]} z_k, \quad f(0) = 0. \tag{4}$$

Proposition 2.4. *([14, Proposition 5.4.1]) If $\left\|\frac{1}{2^k} z_k\right\|_X < B$ for all $k \in \mathbb{N}$ and $B > 1$, then the function $f : [0,1] \to X$ given by (4) is strongly McShane integrable if and only if the series $\sum\limits_{k=1}^{\infty} \frac{1}{2^k} z_k$ is absolutely convergent.*

We also need the following.

Assume that $\{x_n\}$ is a sequence in the Banach spase E. Then $\{x_n\}$ is said to be a *basic sequence* whenevere for each vector $x \in \overline{span}\{x_n : n \in \mathbb{N}\}$, we can find a unique sequence $\{a_n\}$ of scalars such that $\sum\limits_{n=1}^{\infty} a_n x_n = x$. The last equality means that $\left\|\sum\limits_{n=1}^{N} a_n x_n - x\right\|_X \to 0$ as $N \to \infty$. In other words, $\{x_n\}$ is the *Schauder basis* of the subspace $\overline{span}\{x_n : n \in \mathbb{N}\}$.

Lemma 2.5. *([5, Lemma 2.1]) Let E be a Banach space and $\{x_n\} \subseteq E\backslash\{0\}$. The following properties are equivalent:*

1. *$\{x_n\}$ is a basic sequence.*
2. *There is a constant $0 < C < \infty$ such that for every pair $r, s \in \mathbb{N}$ with $s \geqslant r$ and every finite sequence of scalars $a_1, ..., a_s$, one has $\left\|\sum\limits_{n=1}^{r} a_n x_n\right\| \leqslant C \left\|\sum\limits_{n=1}^{s} a_n x_n\right\|$.*

Assume that $\{z_k\}_{k\in\mathbb{N}}$ is a sequence in X and $f : [0,1] \to X$ is the function given by (4) with the following property: If $\left\|\frac{1}{2^k} z_k\right\|_X < B$ for all $k \in \mathbb{N}$ and $B > 1$, then the series $\sum\limits_{k=1}^{\infty} \frac{1}{2^k} z_k$ is non absolutely convergent.

Thus due to Theorem 2.4 the function f is non strongly McShane integrable and hence by Theorem 2.1 and Theorem 2.2, f is non strongly McShane (product) integrable. Now suppose that for each $j \in \mathbb{N}$, $(n_k^j)_k$ is a strictly increasing subsequence of $\mathbb{N}$ such that for all natural numbers i, j with $i \neq j$, we have $n_k^i \neq n_m^j$, for all $k, m \in \mathbb{N}$, and that the series $\sum_{k=1}^{\infty} \frac{1}{2^{n_k^j}} z_{n_k^j}$ is not absolutely convergent. Now for each $k \in \mathbb{N}$ put $B_k = (\frac{1}{2^k}, \frac{1}{2^{k-1}}]$. Then for all $j \neq k$, $m(B_i \cap B_k) = 0$, and $m(\cup_k B_k) = 1$, where m denotes the Lebesgue measure. For each j assume $f_j := \sum_{k=1}^{\infty} \chi_{B_{n_k^j}} z_{n_k^j}$, thus $f_j \in \mathcal{G}$, moreover by the disjointness of the supports, $\{f_j\}_j$ is linearly independent in X.

Theorem 2.6. $\overline{span}(\{f_j\}_j) \subset \mathcal{G} \cup \{0\}$. *In particular, $\mathcal{G}$ is spaceable.*

Proof. Let $m, n \in \mathbb{N}$ such that $m < n$, and $a_1, \dots, a_n \in \mathbb{R}$, so $\left\| \sum_{j=1}^{m} a_j f_j \right\|_X \leqslant \left\| \sum_{j=1}^{n} a_j f_j \right\|_X$, and hence $(f_j)_j$ is a basic sequence, and therefore a Schauder basis for $\overline{span}(\{f_j\}_j)$. Noticing that the $\left\{ B_{n_k^j} \right\}_k$ are pairwise disjoint, we get:

$$\begin{aligned}
\left\| \sum_{j=1}^{m} a_j f_j \right\|_X &= \sup \left\{ \left\| \sum_{j=1}^{m} a_j f_j(x) \right\|_X : x \in [0,1] \right\} \\
&= \sup \left\{ \left\| \sum_{k=1}^{\infty} \sum_{j=1}^{m} a_j z_{n_k^j} \chi_{B_{n_k^j}}(x) \right\|_X : x \in [0,1] \right\} \\
&\leqslant \sup \left\{ \left\| \sum_{k=1}^{\infty} \sum_{j=1}^{n} a_j z_{n_k^j} \chi_{B_{n_k^j}}(x) \right\|_X : x \in [0,1] \right\} \\
&= \sup \left\{ \left\| \sum_{j=1}^{n} a_j f_j(x) \right\|_X : x \in [0,1] \right\} = \left\| \sum_{j=1}^{n} a_j f_j \right\|_X .
\end{aligned}$$

It follows by Lemma 2.6 that, for each $g \in \overline{span}(\{f_j\}_j)$ with $g \neq 0$, there is a nonzero sequence $(\alpha_j)_j$ of real numbers satisfying $g = \sum_{j=1}^{\infty} \alpha_j f_j$. The proof is now complete. □

Note that by the definition of lineablity and spaceability, we can conclude the lineability of $\mathcal{G}$.

References

[1] R. Aron, V. I. Gurariy, and J. B. Seoane-Sepúlveda, Lineability and spaceability of sets of functions on R, *Proc. Amer. Math. Soc.,* 133 (2005), 795-803 (electronic).

[2] R. Aron and J. B. Seoane-Sepúlveda, Algebrability of the set of everywhere surjective functions on $\mathbb{C}$, *Bull. Belg. Math. Soc. Simon Stevin,* 14 (2007), 25-31.

[3] A. Bartoszewicz and S. Glab, Strong algebrability of sets of sequences and functions, *Proc. Amer. Math. Soc.,* 141 (2013), 827-835.

[4] A. Bartoszewicz, S. Glab, and T. Poreda, On algebrability of nonabsolutely convergent series, *Linear algebra and its app.,* 435 (2011), 1025-1028.

[5] L. Bernal-González and M. Ordóñez cebrera, Spaceability of strict order integrability, *J. Math. Anal. Appl.,* 385 (2012), 303-309.

[6] A. Farokhinia, Lineability of space of quasi-everywhere surjective functions, *J. Math. Ext.,* 6 (2013), 45-51.

[7] A. Farokhinia, Algebrability of space of quasi-everywhere surjective functions, *B. Math. Anal. Appl.,* 6 (2014), 38-43.

[8] V. P. Fonf, V. I. Gurariy, and M. I. Kadets, An infinite dimensional subspaces of C[0,1] consisting of nowhere differentiable functions, *C. R. Acad. Bulgar Sci.,* 52 (1999), 13-16.

[9] V. I. Gurarii, Subspaces and bases in spaces of continuous functions, *Dokl. Akad. Nauk SSSR,* 167 (1966), 971-973 (Russian).

[10] S. Glab, P. L. Kaufmann, and L. pellegrini, Large structures made of nowhere L^p functions, *Studia Math.,* 221 (2014), 13-34.

[11] S. Glab, P. L. Kaufmann, and L. pellegrini, Spaceability and algebrability of sets of nowhere integrable functions, *Proc. Amer. Math. Soc.,* 141 (2013), 2025-2037.

[12] L. Bernal-González, D. Pellegrino, and J. B. Seoane-Sepúlveda, Linear subsets of nonlinear sets in topological vector spaces, *Amer. Math. Soc.,* 51 (1) (2014), 71-130.

[13] A. Slavík, Kurzweil and McShane product integration in Banach algebras, *J. Math. Anal. App.,* 424 (2015), 748-773.

[14] Š. Schwabic, *Topics in Banach Space Integration*, Series in Real Analysis. Volume 10, World Scientific (2005)

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