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J-McCoy Rings Relative To A Monoid

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Abstract. Let R be a ring and M be a monoid. We introduce the notion of J-M-McCoy rings, as a generalization of J-McCoy and weak M-McCoy rings, and investigate their properties. It is proved that for u.p.-monoids M and N if $\frac{R}{J(R)}$ is reversible, then R is J-$M \times N$-McCoy. Also, it is shown that a ring R is J-M-McCoy if and only if $R[[x]]$ is J-M-McCoy if and only if $T_n(R)$ is J-M-McCoy, while the J-M-McCoy property is not Morita invariant.

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1. Introduction

Throughout this paper R and M denote an associative ring with identity and a monoid, respectively. Let R be a ring. The symbols $T_n(R), J(R)$ and $Nil(R)$ denote upper triangular matrix $n \times n$ over R, the Jacobson radical of R, and the set of all nilpotent elements of R, respectively. In 1997, the notion of an Armendariz ring is introduced by

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Rege and Chhawcharian. They called a ring R Armendariz if whenever polynomials $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ and $g(x) = b_0+b_1x+b_2x^2+\cdots+b_mx^m \in R[x]-\{0\}$ satisfy $f(x)g(x) = 0$ implies that for each $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$, $a_ib_j = 0$. A noncommutative ring R is called left McCoy if for $f(x) = \sum_{i=1}^{n} a_ix^i$, $g(x) = \sum_{j=1}^{m} b_jx^j \in R[x]-\{0\}$ satisfy $f(x)g(x) = 0$ there exists a nonzero element $c \in R$ such that $ca_i = 0$ for each i [8]. Right McCoy rings are defined similarly. A ring R is called McCoy if it is both left and right McCoy. Commutative rings are McCoy [6]. A number of papers have been written on McCoy property of rings (see, e.g., [1, 9, 3, 5, 2]). In [4] Liu studied a generalization of Armendariz rings which is called M-Armendariz for a monoid M. A ring is said to be M-Armendariz if for two nonzero elements $\alpha = a_1g_1+a_2g_2+\cdots+a_ng_n$, $\beta = b_1h_1+b_2h_2+\cdots+b_mh_m \in R[M]$ with $\alpha\beta = 0$, implies that $a_ib_j = 0$ for each i, j and $g_i, h_j \in M$. Moreover, a generalization of McCoy rings which is called M-McCoy rings whenever M is a monoid is introduced by E. Hashemi in [3]. A ring is called left M-McCoy if whenever $\alpha = a_1g_1 + a_2g_2 + \cdots + a_ng_n$, $\beta = b_1h_1 + b_2h_2 + \cdots + b_mh_m \in R[M] - \{0\}$ satisfies $\alpha\beta = 0$, then there exists a nonzero element $c \in R$ such that $ca_i = 0$ for each i. Right M-McCoy rings are defined analogously, and if a ring R is both left and right M-McCoy, then it is called M-McCoy. Clearly, M-Armendariz rings are M-McCoy. In 2008 [2], Sh. Ghalandarzadeh et al. introduced another generalization of McCoy rings which is called left weak McCoy if whenever $f(x) = \sum_{i=0}^{n} a_ix^n$, $g(x) = \sum_{j=0}^{m} b_jx^m \in R[x] - \{0\}$ satisfy $f(x)g(x) = 0$ then $ca_i \in Nil(R)$ for some $c \in R - \{0\}$ and each i. They defined right weak McCoy rings similarly and said that a ring R is weak McCoy if it is both right and left weak McCoy. In 2010, Alhevaz et al. in [1] investigated weak M-McCoy rings which are a generalization of weak McCoy rings whenever M is a monoid. They defined that a ring R is called left weak M-McCoy if for two nonzero elements $\alpha = \sum_{i=1}^{n} a_ig_i$, $\beta = \sum_{j=1}^{m} b_jh_j \in R[M]$ with $\alpha\beta = 0$ implies that there exists an element $r \in R - \{0\}$ such that $ra_i \in Nil(R)$. Also they introduced right weak M-McCoy rings similarly. If a ring is both left and right weak M-McCoy then it is named weak M-McCoy. As a generalization of weak McCoy rings in 2016, M. Vahdani et al. in [5] called a ring R, left J-McCoy (when $J(R)$ is the Jacobson radical of R), if whenever two nonzero elements $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$, $g(x) = b_0 + b_1x + b_2x^2 + \cdots + b_mx^m \in R[x]$ satisfy $f(x)g(x) = 0$, then there exists an element $c \in R-\{0\}$ such that $ca_i \in J(R)$ for each i. Right J-McCoy rings are defined similarly. A ring R is called J-McCoy if it is both right and left J-McCoy. They proved that weak McCoy rings are J-McCoy, but in general the converse is not true.

Motivated by above results, we introduce J-M-McCoy rings as a generalization of J-McCoy and weak M-McCoy rings. In general, we can show that weak M-McCoy rings are J-M-McCoy, but the converse is not always true.

2. Different Conditions on Monoids

We start this section by the following definition:

Definition 2.1. *For a monoid M, a ring R is said to be right J-M-McCoy if whenever elements $\alpha = a_1g_1 + \cdots + a_ng_n$, $\beta = b_1h_1 + \cdots + b_mh_m \in R[M] - \{0\}$ satisfy $\alpha\beta = 0$, then there exists a nonzero element $c \in R$ with $a_ic \in J(R)$. We define left J-M-McCoy rings similarly. If a ring R is both left and right J-M-McCoy, then we say that the ring R is J-M-McCoy.*

Note that, for Artinian rings, weak M-McCoy and J-M-McCoy ring are the same.

Clearly, R is right (resp. left) J-McCoy ring if and only if R is right (resp. left) J-M-McCoy where $M = (\mathbb{N} \cup \{0\}, +)$. Also, weak M-McCoy rings are J-M-McCoy because if whenever two nonzero elements $\alpha = a_1g_1+\cdots+a_ng_n$, $\beta = b_1h_1+\cdots+b_mh_m \in R[M]-\{0\}$ satisfy $\alpha\beta = 0$ then for each $x \in R$ we have $x\alpha\beta = 0$. Since R is weak M-McCoy, there exists nonzero element $c \in R$ such that $xa_ic \in Nil(R)$ and so $1 - xa_ic \in U(R)$ and we have $a_ic \in J(R)$, so it shows the result. But the converse is not always true by the [[5], Example 2.2] for $M = (\mathbb{N} \cup \{0\}, +)$.

For a monoid M and M' a submonoid of M, we have R is J-M'-McCoy, if R is J-M-McCoy. For it, let $\alpha = \sum_{i=1}^{n} a_ig_i$, $\beta = \sum_{j=1}^{m} b_jh_j \in R[M'] -$

$\{0\}$ such that $\alpha\beta = 0$. Since $g_i, h_j \in M' \subseteq M$ for each $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$, then $\alpha, \beta \in R[M]$, hence there exists $r \in R$ such that $a_i r \in J(R)$ for all $1 \leqslant i \leqslant n$, since R is J-M-McCoy. Therefore, R is J-M'-McCoy.

Recall that a ring R is said to be J-semisimple (*semiprimitive*) if $J(R) = 0$. So for J-semisimple rings, if M is a cyclic group of order $m \geqslant 2$, then R is not J-M-McCoy ring by [[3], Lemma 1.11]. Also, if G is a finitely generated abelian group, then G is torsion-free (i.e. $T(G) = \{e\}$) if and only if there exists a right J-G-McCoy ring R such that $|R| \geqslant 2$ by [[3], Theorem 1.14]. Note that, $T(G) = \{g \in G \mid \exists n > 0 : g^n = e\}$ is called the torsion subgroup of the abelian group G.

The following example shows that J-semisimple property is not a superfluous condition in the above discussion.

Example 2.2. Let $T_n(\mathbb{Z}_8)$ be the upper triangular matrix ring over $\mathbb{Z}_8$ which is J-semisimple, and $M = (\mathbb{Z}_2, +)$ be a monoid which is not cyclic and torsion-free, by Corollary 3.6, $T_n(\mathbb{Z}_8)$ is J-M-McCoy for any monoid M.

For a monoid M, a ring R is said to be *J-M-Armendariz* if whenever $\alpha = a_1g_1 + \cdots + a_ng_n$, $\beta = b_1h_1 + \cdots + b_mh_m \in R[M] - \{0\}$ satisfy $\alpha\beta = 0$, then $a_ib_j \in J(R)$ for each i, j. The above example shows that every J-M-McCoy ring is not necessarily J-M-Armendariz, because $T_n(\mathbb{Z}_8)$ is not weak M-Armendariz by [[7], Proposition 2.12] so it is not J-M-Armendariz.

For a monoid M, N is an *ideal* of M, if $N \subseteq M$ and if for each $n \in N$ and $m \in M$ then $nm \in N$. An element a of a monoid M is *left cancellative* if $ax = ay$ implies $x = y$ for all x, y, and is *right cancellative* if $xa = ya$ implies $x = y$ for all x, y. It is cancellative if it is both left and right cancellative. A monoid M is *cancellative* if all of its elements are.

Proposition 2.3. *For a monoid M and an ideal N of M, let R be a right (resp. left) J-N-McCoy ring. Then R is right (resp. left) J-M-McCoy ring, if M is a cancellative monoid.*

Proof. Let $\alpha = \sum_{i=1}^{n} a_ig_i$, $\beta = \sum_{j=1}^{m} b_jh_j$ are two nonzero elements of $R[M]$ such that $\alpha\beta = 0$. For $r \in N$, $rg_i \neq rg_j$ and $h_i r \neq h_j r$, when $i \neq j$, since M is cancellative. Also $rg_1, rg_2, \cdots, rg_n, h_1 r, h_2 r, \cdots, h_m r \in N$. Now from $(\sum_{i=1}^{n} a_i r g_i)(\sum_{j=1}^{m} b_j h_j r) = 0$, it follows that $a_i c \in J(R)$ for each i and some $c \in R - \{0\}$, since R is right J-N-McCoy, and so R is right J-M-McCoy. $\square$

For any $\alpha = \sum_{i=1}^{n} a_i g_i \in R[M]$ define $\bar{\alpha} = \sum_{i=1}^{n}(a_i + J(R))g_i \in \frac{R}{J(R)}[M]$. It is easy to see that the mapping $\psi : R[M] \longrightarrow \frac{R}{J(R)}[M]$ defined by $\psi(\sum_{i=1}^{n} a_i g_i) = \sum_{i=1}^{n} \bar{a}_i g_i$ is a ring homomorphism. Recall that a ring R is said to be *reversible* if $ab = 0$ implies that $ba = 0$ for all $a, b \in R$. Let N and N' be two nonempty finite subset of M. If there exists an element $m \in M$ such that $m = nn'$ where $n \in N$ and $n' \in N'$ and m is unique product in this form, then M is called *u.p.-monoid.*

For an ordered monoid M with $\leqslant$, M is said to be *strictly totally ordered monoid* if for any $g_1, g_2, h \in M, g_1 < g_2$ implies that $g_1 h < g_2 h$ and $hg_1 < hg_2$.

Proposition 2.4. *Let $\bar{R} = \frac{R}{J(R)}$ a reversible ring. Then R is J-M-McCoy, if M is a u.p.-monoid.*

Proof. Let $\alpha = \sum_{i=1}^{n} a_i g_i$, $\beta = \sum_{j=1}^{m} b_j h_j \in R[M] - \{0\}$ be such that $\alpha\beta = 0$. We have $\bar{\alpha}\bar{\beta} = \bar{\alpha\beta} = \bar{0}$. Since $\frac{R}{J(R)}$ is M-McCoy by [[3],Proposition 1.2], then there exists $\bar{c} \in \frac{R}{J(R)}$ such that $\bar{a}_i \bar{c} = \bar{0}$. Therefore $a_i c \in J(R)$ for each $1 \leqslant i \leqslant n$. Hence R is J-M-McCoy. $\square$

Corollary 2.5. *Let M be a strictly totally ordered monoid and $\bar{R} = \frac{R}{J(R)}$ a reversible ring. Then R is J-M-McCoy.*

Theorem 2.6. *Let M and N be u.p.-monoids and $\frac{R}{J(R)}$ is reversible ring. Then $R[M]$ is J-N-McCoy, and $R[N]$ is J-M-McCoy.*

Proof. We know that by Proposition 2.4, R is J-M-McCoy. We will show that $\frac{R[M]}{J(R[M])}$ is reversible. It is easy to see that $\frac{R[M]}{J(R[M])} \cong \frac{R}{J(R)}[M]$. We claim that $\frac{R}{J(R)}[M]$ is reversible. Let $\bar{\alpha} = \sum_{i=1}^{n}(a_i + J(R))g_i$, $\bar{\beta} = \sum_{j=1}^{m}(b_j + J(R))h_j \in \frac{R}{J(R)}[M] - \{\bar{0}\}$ such that $\bar{\alpha}\bar{\beta} = \bar{0}$, then $\bar{0} = \sum_{i=1}^{n}\sum_{j=1}^{m}(a_i b_j + J(R))g_i h_j$ so for each i, j we have $a_i b_j \in J(R)$, since M is a u.p.-monoid. Hence $(a_i + J(R))(b_j + J(R)) = \bar{0}$, since $\frac{R}{J(R)}$ is a reversible ring, then $b_j a_i \in J(R)$. It follows that $R[M]$ is J-N-McCoy. By the same analogy with the above proof, it follows that $R[N]$ is J-M-McCoy. $\square$

Theorem 2.7. *Let M and N be u.p.-monoids. If $\frac{R}{J(R)}$ is reversible and $J(R[M]) \subseteq J(R)[M]$, then R is J-$M \times N$-McCoy.*

Proof. Suppose $\sum_{i=1}^{s} a_i(m_i, n_i) \in R[M \times N]$. Without loss of generality we assume that $\{n_1, \cdots, n_s\} = \{n_1, \cdots, n_t\}$ with $n_i \neq n_j$ when $1 \leqslant i \neq j \leqslant t$. For any $1 \leqslant p \leqslant t$, denote $A_p = \{i | 1 \leqslant i \leqslant s, n_i = n_p\}$ then $\sum_{p=1}^{k}(\sum_{i\in A_p} a_i m_i) n_p \in R[M][N]$. Note that $m_i \neq m_{i'}$ for any $i, i' \in A_p$ with $i \neq i'$. Now it is easy to see that there exists an isomorphism of rings $R[M \times N] \longrightarrow R[M][N]$ defined by

$$\sum_{i=1}^{s} a_i(m_i, n_i) \longmapsto \sum_{p=1}^{t}(\sum_{i\in A_p} a_i m_i) n_p.$$

Suppose that $(\sum_{i=1}^{s} a_i(m_i, n_i))(\sum_{j=1}^{s'} b_j(m_j', n_j')) = 0$ in $R[M \times N]$. Then from the above isomorphism it follows that

$$(\sum_{p=1}^{t}(\sum_{i\in A_p} a_i m_i) n_p)(\sum_{q=1}^{t'}(\sum_{j\in B_q} b_j m_{j'}) n_p') = 0.$$

By Theorem 2.6, $R[M]$ is J-N-McCoy. So, there exists $\sum_{k\in C_k} c_k m_k'' \in R[M]$ such that $(\sum_{i\in A_p} a_i m_i)(\sum_{k\in C_k} c_k m_k'') \in J(R[M])$ for any p and l. Hence $a_i c_k \in J(R)$ for all i, k since $J(R[M]) \subseteq J(R)[M]$ for each $1 \leqslant i \leqslant s$ and $1 \leqslant k \leqslant s''$. This means that R is J-$M \times N$-McCoy. $\square$

Let M_i, $i \in I$ be monoids and $\coprod_{i\in I} M_i = \{(g_i)_{i\in I} |$ there exist only finite i's that $g_i \neq e_i$, the identity of M$\}$. Then $\coprod_{i\in I} M_i$ is a monoid with the equation $(g_i)_{i\in I}(g_i')_{i\in I} = (g_i g_i')_{i\in I}$.

Corollary 2.8. *Let M_i, $i \in I$ be u.p.-monoids and $\frac{R}{J(R)}$ is a reversible ring. If R is J-M_{i_0}-McCoy for some $i_0 \in I$, then R is J-$\coprod_{i\in I} M_i$-McCoy.*

Proof. Let $\alpha = \sum_i a_i g_i$, $\beta = \sum_j b_j h_j \in R[\coprod_{i\in I} M_i]$ such that $\alpha\beta = 0$. Then $\alpha, \beta \in R[M_1 \times \cdots \times M_n]$ for some finite subset $\{M_1, \cdots, M_n\} \subseteq$

$\{M_i|i \in I\}$. Thus $\alpha, \beta \in R[M_{i_0} \times M_1 \times \cdots \times M_n]$. The ring R, by Theorem 2.7 and induction is J-$M_{i_0} \times M_1 \times \cdots \times M_n$-McCoy, so there exist $r \in R$ such that $a_i r \in J(R)$ for all i. Hence R is J-$\coprod_{i \in I} M_i$-McCoy. □

For a monoid M we denote by $G(M)$ the largest subgroup of M.

Proposition 2.9. *Let M be a commutative and cancellative monoid with $G(M) = \{e\}$. If R is J-McCoy, $J(R)[M] \subseteq J(R[M])$ and J-M-McCoy then $R[M]$ is J-M-McCoy.*

Proof. Suppose that $(\sum_i \alpha_i x^i)(\sum_j \beta_j x^j) = 0$ where $\alpha_i = \sum a_{ip} g_{ip}$, $\beta_j = \sum b_{jq} h_{jq} \in R[M] - \{0\}$. Set $g = (\prod_i \prod_j g_{ip})(\prod_j \prod_q h_{jq})$. Clearly, for any $r \in R$ and $h \in M$, $(rh)(1g^2) = (1g^2)(rh)$. Thus from $(\sum_i \alpha_i x^i)(\sum_j \beta_j x^j) = 0$ it follows that

$$(\sum_i \alpha_i (1g^2)^i)(\sum_j \beta_j (1g^2)^j) = 0.$$

Then we have

$$(\sum_i \sum_p a_{ip} g_{ip} g^{2i})(\sum_j \sum_q b_{jq} h_{jq} g^{2j}) = 0.$$

Suppose that $g_{i'p'} g^{2i'} = g_{i''p''} g^{2i''}$ for some i' and i'' if $i' = i''$, then $g_{i'p'} = g_{i''p''}$, since M is cancellative and so $p' = p''$. Thus without loss of generality we assume that $i' > i''$. Then $g_{i'p'} g^{2(i'-i'')} = g_{i''p''}$, since M is cancellative. Thus it is easy to see that g_{ip} and h_{jq} are in $G(M)$ for all i, j, p, q. Hence $g_{ip} = h_{jq} = e$ by the hypothesis and then we may assume that $\alpha_i = a_i e$ and $\beta_j = b_j e$ for all i, j. So we have $(\sum (a_i e) x^i)(\sum (b_j e) x^j) = 0$ from which it follows that $(\sum_i a_i x^i)(\sum_j b_j x^j) = 0$. Thus there exists $c_k \in R$ such that $a_i c_k \in J(R)$ for all i, since R is J-McCoy. Hence $(a_i e)(c_k e) \in J(R)[M] \subseteq J(R[M])$ for all i, k. If $h_{j'q'} g^{2j'} = h_{j''q''} g^{2j''}$ for some j' and j'', then by analogy with the above proof, it follows that $(a_i e)(c_k e) \in J(R[M])$ for all i, k. Now suppose that each pair of $g_{ip} g^{2i}$'s is distinct and each pair of $h_{jq} g^{2j}$'s is distinct. Then $a_{ip} c_{kl} \in J(R)$ for all i, p, k, l, since R is J-M-McCoy. Thus $R[M]$ is J-M-McCoy. □

3. Different Conditions on Rings

In this section we do the generalization on weak M-McCoy and J-McCoy rings by considering different conditions on rings.

For a monoid M and ring R_k, where $k \in I$, we can easily show that R_k is right (resp. left) J-M-McCoy ring for each $k \in I$ if and only if $R = \prod_{k \in I} R_k$ is right (resp. left) J-M-McCoy.

Proposition 3.1. *For a ring R , a monoid M and an idempotent e element of R, we have:*

1. *If R is a right (resp. left) J-M-McCoy ring, then eRe is a right (resp. left) J-M-McCoy ring;*
2. *If R is an abelian ring (i.e. every idempotent element of R is central), then R is a right (resp. left) J-M-McCoy ring if and only if eRe is a right (resp. left).*

Proof. (1): Let $\alpha = \sum_{i=1}^{n} ea_i eg_i$, $\beta = \sum_{j=1}^{m} eb_j eh_j$ be nonzero elements of $(eRe)[M]$ such that $\alpha\beta = 0$. Since R is a right J-M-McCoy, then there exists $0 \neq s \in R$ such that $(ea_i e)s \in J(R)$. So $(ea_i e)(ese) \in eJ(R)e = J(eRe)$. Therefore, eRe is right J-M-McCoy.

(2): One direction is obvious by (1).

For converse, let $\alpha = \sum_{i=1}^{n} a_i g_i$, $\beta = \sum_{j=1}^{m} b_j h_j$ be two nonzero elements of $R[M]$ such that $\alpha\beta = 0$. Since e is a central idempotent element of R, then $(e\alpha e)(e\beta e) = 0$, where $e\alpha e, e\beta e$ are nonzero elements of $(eRe)[M]$. Therefore, there exists $0 \neq ere \in eRe$ such that $(ea_i e)ere = a_i c \in J(eRe) = J(R) \cap eRe$ where $c = re$, since eRe is right J-M-McCoy and so R is right J-M-McCoy ring, as desired. $\square$

Theorem 3.2. *For a ring R and monoid M, let I be an ideal of R. If $\frac{R}{I}$ is a right (resp. left) J-M-McCoy, then R is right (resp. left) J-M-McCoy, if $I \subseteq J(R)$.*

Proof. Let $\alpha = a_1 g_1 + \cdots + a_n g_n$, $\beta = b_1 h_1 + \cdots + b_m h_m$ be two nonzero elements of $R[M]$ with $\alpha\beta = 0$. Therefore, $(\sum_{i=1}^{n}((a_i + I)g_i))(\sum_{j=1}^{m}((b_j + I)g_j)) = 0$ in $\frac{R}{I}[M]$. Since $\frac{R}{I}$ is a right J-M-McCoy ring, then there exists

$(c+I) \in \frac{R}{I}$ such that $(a_i+I)(c+I) \in J(\frac{R}{I})$, but $I \subseteq J(R)$, so $a_ic \in J(R)$, as desired. $\square$

The converse of the above theorem is not true by the [[5], Example 2.5], where $M = \mathbb{N} \cup \{0\}$.

Theorem 3.3. *Let R be a ring and M a monoid. R is right (resp. left) J-M-McCoy ring if and only if $R[[x]]$ is right (resp. left) J-M-McCoy ring.*

Proof. Let $R[[x]]$ be a right J-M-McCoy ring. If $\alpha = \sum_{i=1}^{n} a_ig_i$, $\beta = \sum_{j=1}^{m} b_jh_j$ are nonzero elements of $R[M]$ such that $\alpha\beta = 0$. There exists $0 \neq h(x) = \sum_{i=0}^{\infty} d_ix^i$ in $R[[x]]$ such that $a_ih(x) \in J(R[[x]])$, since $R[[x]]$ is right J-M-McCoy ring and $R \subseteq R[[x]]$. Hence $a_id_i \in J(R[[x]]) \cap R \subseteq J(R)$ for all $1 \leqslant i \leqslant n$. Since $h(x) \neq 0$, there exists $d_i \neq 0$ such that $a_id_i \in J(R)$ for $1 \leqslant i \leqslant n$ and so the proof is done. For converse, assume that R is a right J-M-McCoy ring, then by Theorem 3.2, $R[[x]]$ is right J-M-McCoy ring, since $J(R[[x]])$ includes $< x >$ and $R \approx \frac{R[[x]]}{<x>}$. $\square$

Proposition 3.4. *Let N be a monoid and T be the triangular ring $T = \begin{bmatrix} R & M \\ 0 & S \end{bmatrix}$ (where R and S are two rings snd M is an (R,S)-biomodule). Then T is right (resp. left) J-M-McCoy ring if and only if the rings R and S are right (resp. left) J-N-McCoy rings.*

Proof. Let $\alpha_r = \sum_{i=1}^{n} r_ig_i$, $\beta_r = \sum_{j=1}^{m} r_j^{'}h_j \in R[N]$ such that $\alpha_r\beta_r = 0$ and $\alpha_s = \sum_{i=1}^{n} s_ig_i$, $\beta_s = \sum_{j=1}^{m} s_j^{'}h_j \in S[N]$ such that $\alpha_s\beta_s = 0$. Set $\alpha = \sum_{i=1}^{n} \begin{bmatrix} r_i & 0 \\ 0 & s_i \end{bmatrix} g_i$ and $\beta = \sum_{j=1}^{m} \begin{bmatrix} r_j^{'} & 0 \\ 0 & s_j^{'} \end{bmatrix} h_j \in T[N]$. Therefore, $\alpha\beta = 0$. Then there exists $\begin{bmatrix} c & m \\ 0 & d \end{bmatrix} \in T$ such that $\begin{bmatrix} r_i & 0 \\ 0 & s_i \end{bmatrix}\begin{bmatrix} c & m \\ 0 & d \end{bmatrix} \in J(T)$, since T is right J-N-McCoy ring. Note that $J(T) = \begin{bmatrix} J(R) & M \\ 0 & J(S) \end{bmatrix}$ and so $r_ic \in J(R)$ and $s_id \in J(S)$ for all i, j, as desired.

Conversely, assume that R and S are two right J-N-McCoy rings. Take $I = \begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix}$, then $\frac{T}{I} \simeq \begin{bmatrix} R & 0 \\ 0 & S \end{bmatrix}$. Let $\alpha = \begin{bmatrix} r_1 & 0 \\ 0 & s_1 \end{bmatrix} g_1 + \cdots + \begin{bmatrix} r_n & 0 \\ 0 & s_n \end{bmatrix} g_n$

and $\beta = \begin{bmatrix} r_1^{'} & 0 \\ 0 & s_1^{'} \end{bmatrix} h_1 + \cdots + \begin{bmatrix} r_m^{'} & 0 \\ 0 & s_m^{'} \end{bmatrix} h_m \in \frac{T}{I}[N]$ such that $\alpha\beta = 0$. Define $\alpha_r = r_1 g_1 + \cdots + r_n g_n$, $\beta_r = r_1^{'} h_1 + \cdots + r_m^{'} h_m \in R[N]$ and $\alpha_s = s_1 g_1 + \cdots + s_n g_n$, $\beta_s = s_1^{'} h_1 + \cdots + s_m^{'} h_m \in S[N]$. From $\alpha\beta = 0$ we have $\alpha_r \beta_r = \alpha_s \beta_s = 0$. Then there exists $c \in R$ and $d \in S$ such that $r_i c \in J(R)$ and $s_j d \in J(S)$ for any $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$, since R and S are two right J-N-McCoy rings. Hence if we put $I = \begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix}$, then $\frac{T}{I}$ is right J-N-McCoy ring and so T is right J-N-McCoy ring by Theorem 3.2. $\square$

Recall that a *regular element* in a ring R is any non-zero-devisor, i.e., any element $x \in R$ such that $r.ann_R(x) = 0$ and $l.ann_R(x) = 0$. Let R be a ring and $X \subseteq R$ a multiplicative set of central regular elements in R. A right ring of fractions (or right quotient ring) for R with respect to X is any overring $S \supseteq R$ such that every element of X is invertible in S and every element of S can be expressed in the form ax^{-1} for some $a \in R$ and $x \in X$. The right ring of fractions for R is denoted by RX^{-1}. Left ring of fractions are defined analogously, using fractions of the form $x^{-1}a$. Of course, if a ring of fractions is commutative, the adjectives "right" and "left" are not needed.

Theorem 3.5. *Let R be a ring and M a monoid. If R is right (resp. left) J-M-McCoy ring, then the right ring of fractions of R (RX^{-1}) is right (resp. left) J-M-McCoy ring.*

Proof. Let R be a right J-M-McCoy ring. If $\alpha = \sum_{i=1}^{n} a_i c_i^{-1} g_i$, $\beta = \sum_{j=1}^{m} b_j d_j^{-1} h_j$ are two nonzero elements of $RX^{-1}[M]$ such that $\alpha\beta = 0$. Suppose that $a_i c_i^{-1} = c^{-1} a_i^{'}$ and $b_j d_j^{-1} = d^{-1} b_j^{'}$ with c, d in X. Then $\alpha^{'} \beta^{'} = 0$ such that $\alpha^{'} = \sum_{i=1}^{n} a_i^{'} g_i$ and $\beta^{'} = \sum_{j=1}^{m} b_j^{'} h_j$ are nonzero elements of $R[M]$, since $\alpha\beta = 0$. Hence there exists a nonzero element $r \in R$ such that $a_i^{'} r \in J(R)$ for each $1 \leqslant i \leqslant n$, since R is right J-M-McCoy. Equivalently, for each $t \in R$ we have $1 - t a_i^{'} r$ is left invertible in R. So $c^{-1} w^{-1} (1 - t w^{-1} a_i c_i^{-1} r c w) = c^{-1} w^{-1} - t w^{-1} a_i c^{-1} r$ is left invertible in RX^{-1} for each $t w^{-1} \in RX^{-1}$ and $a_i c_i^{-1} r c w \in J(RX^{-1})$. Therefore, RX^{-1} is right J-M-McCoy. $\square$

Consider a skew polynomial ring $R = A[x;\alpha]$, where α is an automorphism of the ring A, set $X = \{1, x, x^2, \cdots\}$. The skew-Laurant ring $A[x^{\pm 1};\alpha]$ is both a right and a left ring of fractions for R with respect to X. Therefore, if R is right (resp. left) J-M-McCoy, then $A[x^{\pm 1};\alpha]$ is J-M-McCoy, too.

Corollary 3.6. *For a ring R and a monoid M the followings are equivalent.*
1. A ring R is J-M-McCoy;
2. $T_n(R)$ is J-M-McCoy for any $n \geqslant 2$;
3. $\frac{R[x]}{\langle x^n \rangle}$ is J-M-McCoy where $\langle x^n \rangle$ is the ideal generated by x^n in R.

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