A Novel Algorithm for Solving Bi-Objective Fractional Transportation Problems with Fuzzy Numbers

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Abstract. A new method is proposed for finding a set of efficient solutions to bi-objective fractional transportation problems with fuzzy numbers using ranking function. This method is an important tool for the decision makers to obtain efficient solutions and select the preferred optimal solution from the satisfaction level. The procedure allows the user to identify next efficient solution to the problem from the current efficient solution. This new approach enables the decision makers to evaluate the economic activities and make satisfactory managerial decisions when they are handling a variety of logistic problems involving two objectives. An illustrative example is presented to clarify the idea of the proposed approach.

1.   Introduction

Transportation problem nourishes economic and social activity and is cardinal to operations research and management science. In the classical transportation problem of linear programming, the traditional objective is one of minimizing the total cost multi-objective transportation problem and linear fractional programming problem has attracted the attention of many researchers in the past. In general, the real life problems are modeled with multi-objectives which are measured in different scales and at the same time in conflict. In actual classical transportation problems, the multi-objective functions are generally considered, which includes average delivery time of the commodities, reliability of transportation, product deterioration and so on. A number of optimization problems are actually multi-objective optimization problems (MOPs), where the objectives are conflicting. As a result, there is usually no single solution which optimizes all objectives simultaneously. A number of techniques have been developed to find a compromise solution to MOPs. The reader is referred to the recent book by Miettinen [1] about the theory and algorithms for MOPs. Fractional programming problems (FPPs) arise from many applied areas such as portfolio selection, stock cutting, game theory, and numerous decision problems in management science. Many approaches for FPPs have been exploited in considerable details. See, for example, Avriel et al. [2], Schaible[3] and Stancu-Minasian [4]. Multi-objective Linear fractional programming problems useful targets in production and financial planning and return on investment. There are several ways to solving the linear fractional programming (LFP) and multi-objective linear fractional programming (MOLFP) problems [5, 6,7]. Tantawy (2007) proposed a new method for solving linear fractional programming problems [8]. Singh, Sharma and Dangwal Proposed a solution concept to MOLFP problem using the Taylor polynomial series at optimal point of each linear fractional function in feasible region [9]. Sulaiman and Abulrahim Used transformation technique for solving multi-objective linear fractional programming problems to single objective linear fractional programming problem through a new method using mean and median and then solve the problem by

modified simplex method [10, 11]. Bodkhe et al., (2010) used the fuzzy programming technique with hyperbolic membership function to solve a bi-objective TP as vector minimum problem [12]. in 2005, an algorithm was proposed by Omar and Yunes for solving multi objective 1transportation problems using fuzzy factors [13]. In 2011, a new method was presented by Pendian for solving two objective transportation problems [14]. Pandian and Natarajan, (2010) have introduced a new method for finding an optimal solution for transportation problems [15]. Amit and Pushpinder (2010) have introduced ranking of generalized trapezoidal fuzzy numbers based on rank [16].

In this paper, we propose a new method namely, dripping method for finding the set of efficient solutions to bi-objective transportation fractional problem with fuzzy numbers using ranking function and percent of function one solution is introduced for transportation problem and explained with the proposed model. In the proposed method, we can identify next solution to the problem from the current solution which differs from utility function method, goal programming approach, fuzzy programming technique, genetic approach and evolutionary approach. The percentage level of satisfaction of a solution of the bi-objective transportation fractional problem is introduced. The dripping method is illustrated with help of a numerical example. This new approach enables the decision makers to evaluate the economical activities and make self-satisfied managerial decisions when they are handling a variety of logistic problems involving two objectives.

2. The Fractional Transportation Simplex Method [17]

As in the case of a general linear fractional programming problem, the solution process of a linear fractional transportation problem (LFTP) consists of two phases:

1) Finding an initial basic feasible solution (BFS);

2) Improving the current basic feasible solution until the optimality cri-

terion is satisfied.

Since the process of finding initial BFS for LFTP is the same as in the linear problem (LP) case, we will focus mainly on the second stage. Consider the following LFPT problem:

$$\text{(LFTP)} \qquad \text{Maximize} \qquad Q(x) = \frac{P(x)}{D(x)} = \frac{\sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij} \, x_{ij} + p_0}{\sum_{i=1}^{m} \sum_{j=1}^{n} d_{ij} \, x_{ij} + d_0}$$

$$\text{Subject to} \qquad \sum_{j=1}^{n} x_{ij} = a_i \quad for \ i = 1, 2, \cdots, m \qquad (1)$$

$$\sum_{i=1}^{m} x_{ij} = b_j \quad for \ j = 1, 2, \cdots, n \qquad (2)$$

$$x_{ij} \geqslant 0 \ , \ i = 1, 2, \cdots, m \ ; \ j = 1, 2, \cdots, n \qquad (3)$$

Here and in what follows we suppose that $D(x) > 0$, $\forall x = (x_{ij}) \in S$, where S denotes a feasible set defined by constraints (1)-(3). Further, we assume that $a_i > 0$, $b_j > 0$, $i = 1, 2, \cdots, m$, $j = 1, 2, \cdots, n$ and total demand equals to total supply, i.e.

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

We now show how the simplex method may be adapted to the case when an LFPT problem is to be solved. First, we have to introduce special simplex multipliers u_i', v_j' and u_i'', v_j'' associated with numerator $P(X)$ and denominator $D(x)$, respectively. Elements u_i' and u_i'' , $i = 1, 2, \cdots, m$, correspond to m supply constraints and elements v_j' and v_j'', $j = 1, 2, \cdots, n$, correspond to n demand constraints. We calculate these variables from the following systems of linear equations

$$u_i' + v_j' = p_{ij} \quad and \quad u_i'' + v_j'' = d_{ij} \ , \quad (ij) \in J_B \qquad (4)$$

Then, using these variables u_i', v_j', u_i'' and v_j'' we define the following 'reduced costs' Δ_{ij}' and Δ_{ij}''

$$\left. \begin{array}{l} \Delta_{ij}' = u_i' + v_j' - p_{ij} \\[2mm] \Delta_{ij}'' = u_i'' + v_j'' - d_{ij} \end{array} \right\} \quad for \ \ i = 1, 2, \cdots, m \ , \ j = 1, 2, \cdots, n \qquad (5)$$

It is easy to show that the latter may also be expressed as follows

$$\Delta_{ij}(x) = \Delta'_{ij} - Q(x)\Delta''_{ij} \quad , \quad i = 1, 2, \cdots, m \ , \ j = 1, 2, \cdots, n \quad (6)$$

Theorem 2.1. [18] *Basic feasible solution* $x = (x_{ij})$ *of* LFTP *problem is optimal if*

$$\Delta_{ij}(x) \geqslant 0 \quad , \quad i = 1, 2, \cdots, m \ , \ j = 1, 2, \cdots, n \quad (7)$$

3.   Bi-objective Fractional Transportation Problem

Consider the following Bi-objective Fractional Transportation Problems (BFTP):

$$(\text{BFTP})\text{Maximize} \qquad Q_1 = \frac{P_1(x)}{D_1(x)} = \frac{\sum_{i=1}^{m}\sum_{j=1}^{n} p_{ij}^1 \, x_{ij} + p_0^1}{\sum_{i=1}^{m}\sum_{j=1}^{n} d_{ij}^1 \, x_{ij} + d_0^1}$$

$$\text{Maximize} \qquad Q_2 = \frac{P_2(x)}{D_2(x)} = \frac{\sum_{i=1}^{m}\sum_{j=1}^{n} p_{ij}^2 \, x_{ij} + p_0^2}{\sum_{i=1}^{m}\sum_{j=1}^{n} d_{ij}^2 \, x_{ij} + d_0^2}$$

$$\text{Subject to} \qquad \sum_{j=1}^{n} x_{ij} = a_i \ \ for \ i = 1, 2, \cdots, m \qquad (8)$$

$$\sum_{i=1}^{m} x_{ij} = b_j \ \ for \ j = 1, 2, \cdots, n \qquad (9)$$

$$x_{ij} \geqslant 0 \ , \ i = 1, 2, \cdots, m \ ; \ j = 1, 2, \cdots, n \quad (10)$$

Here and in what follows we suppose that $D_1(x) > 0$, $D_2(x) > 0$, $\forall x = (x_{ij}) \in S$, where S denotes a feasible set defined by constraints (8) to (10). Further, we assume that $a_i > 0$, $b_j > 0$, $i = 1, 2, \cdots, m$, $j = 1, 2, \cdots, n$ and total demand equals to total supply, i.e.

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j.$$

Definition 3.1. [7] *A feasible solution X^0 is said to be an efficient solution to the problem* (BFTP) *if there exists no other feasible X of BFTP such that $Q_1(X) \geqslant Q_1(X^0)$ and $Q_2(X) > Q_2(X^0)$ or $Q_2(X) \geqslant Q_2(X^0)$ and $Q_1(X) > Q_1(X^0)$. Otherwise, it is called non-efficient solution to the problem* (BFTP).

Definition 3.2. [7] *Let $\mathbb{R}$ be real numbers set $\tilde{a}$ fuzzy number is a map with bellow conditions:*

1) $\mu_{\tilde{a}}$ *is continuous.*

2) $\mu_{\tilde{a}}$ *on $[a_1, a_2]$ is ascending and continuous.*

3) $\mu_{\tilde{a}}$ *on $[a_3, a_4]$ is descending and continuous.*

That a_1, a_2, a_3 and a_4 are real numbers and fuzzy numbers is shown as $\tilde{a} = [a_1, a_2, a_3, a_4]$ and it is called trapezoidal fuzzy number.

Definition 3.3. [7] *If $\tilde{a} = (a_1, a_2, a_3, a_4)$ is trapezoid fuzzy number, membership function is as follows:*

$$\mu_{\tilde{a}}(x) = \begin{cases} \dfrac{x - a_1}{a_2 - a_1} & a_1 \leqslant x \leqslant a_2 \\[2mm] 1 & a_2 \leqslant x \leqslant a_3 \\[2mm] \dfrac{x - a_4}{a_3 - a_4} & a_3 \leqslant x \leqslant a_4 \end{cases}$$

Definition 3.4. [15] *Let $\tilde{a} = (a_1, a_2, a_3, a_4)$ and $\tilde{b} = (b_1, b_2, b_3, b_4)$ be two trapezoidal fuzzy numbers then*

i $\tilde{a} \oplus \tilde{b} = (a_1, a_2, a_3, a_4) \oplus (b_1, b_2, b_3, b_4) = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4)$

ii $\tilde{a} \ominus \tilde{b} = (a_1, a_2, a_3, a_4) \ominus (b_1, b_2, b_3, b_4) = (a_1 - b_1, a_2 - b_2, a_3 - b_3, a_4 - b_4)$

iii $k\tilde{a} = k(a_1, a_2, a_3, a_4) = \begin{cases} (ka_1, ka_2, ka_3, ka_4) & for \quad k \geqslant 0 \\ (ka_4, ka_3, ka_2, ka_1) & for \quad k < 0 \end{cases}$

iv $\tilde{a} \otimes \tilde{b} = (a_1, a_2, a_3, a_4) \otimes (b_1, b_2, b_3, b_4) = (t_1, t_2, t_3, t_4)$
$$where \quad t_1 = \min\{a_1 b_1, a_1 b_4, a_4 b_1, a_4 b_4\};$$
$$t_2 = \min\{a_2 b_2, a_2 b_3, a_3 b_2, a_3 b_3\};$$
$$t_3 = \max\{a_2 b_2, a_2 b_3, a_3 b_2, a_3 b_3\} and$$
$$t_4 = \max\{a_1 b_1, a_1 b_4, a_4 b_1, a_4 b_4\}$$

$$\mathbf{v} \quad \frac{\tilde{a}}{\tilde{b}} = \frac{(a_1, a_2, a_3, a_4)}{(b_1, b_2, b_3, b_4)} = \left(\frac{a_1}{b_4}, \frac{a_2}{b_3}, \frac{a_3}{b_2}, \frac{a_4}{b_1} \right)$$

4. Fuzzy Bi-objective Fractional Transportation Problem

Consider the following Fuzzy Bi-objective Fractional Transportation Problem(FBFTP):

$$\text{(FBFTP)} \qquad \tilde{Q}_1(x) = \frac{\tilde{P}_1(x)}{\tilde{D}_1(x)} = \frac{\sum_{i=1}^{m} \sum_{j=1}^{n} \tilde{p}_{ij}^1 \, x_{ij} + \tilde{p}_0^1}{\sum_{i=1}^{m} \sum_{j=1}^{n} \tilde{d}_{ij}^1 \, x_{ij} + \tilde{d}_0^1}$$

$$\tilde{Q}_2(x) = \frac{\tilde{P}_2(x)}{\tilde{D}_2(x)} = \frac{\sum_{i=1}^{m} \sum_{j=1}^{n} \tilde{p}_{ij}^2 \, x_{ij} + \tilde{p}_0^2}{\sum_{i=1}^{m} \sum_{j=1}^{n} \tilde{d}_{ij}^2 \, x_{ij} + \tilde{d}_0^2}$$

$$\text{Subject to} \qquad \sum_{j=1}^{n} x_{ij} = a_i \quad for \ i = 1, 2, \cdots, m \qquad (11)$$

$$\sum_{i=1}^{m} x_{ij} = b_j \quad for \ j = 1, 2, \cdots, n \qquad (12)$$

$$x_{ij} \geqslant 0 \ , \ i = 1, 2, \cdots, m \ ; \ j = 1, 2, \cdots, n \qquad (13)$$

Here and in what follows we suppose that $D_1(x) > 0$, $D_2(x) > 0$, $\forall x = (x_{ij}) \in S$, where S denotes a feasible set defined by constraints (11) to (13). Further, we assume that $a_i > 0$, $b_j > 0$, $i = 1, 2, \cdots, m$, $j = 1, 2, \cdots, n$ and total demand equals to total supply, i.e.

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

Definition 4.1. [16] *The percentage level of satisfaction of the objective of the transportation problem for the solution U to the transportation problem, $L(Z_t, U)$ is defined as follows:*

$$L(Z_t, U) = \begin{cases} \left(\dfrac{Z_t(U)}{Z_t(X_t^0)} \right) \times 100 & \text{if the problem is maximization type} \\[4mm] \left(\dfrac{2Z_t(X_t^0) - Z_t(U)}{Z_t(X_t^0)} \right) & \text{if the problem is minimization type} \end{cases}$$

Where $Z_t(U)$ is the objective value at the solution U and $Z_t(X_t^0)$ is the optimal objective value of the transportation problem.

Definition 4.2. [16] *α-cut set of a fuzzy number $\tilde{a}$ is shown by A_α and are defined as bellow:*

$$A_\alpha = \{x \mid \mu_{\tilde{a}}(x) \geqslant \alpha\} = [a_\alpha^l, a_\alpha^u]$$

Robust ranking technique is one of the most important ways of converting fuzzy numbers into crisp numbers, which can be used to transform bi-objective fractional transportation problems with fuzzy numbers into bi-objective fractional transportation problems with crisp numbers.

Definition 4.3. [15] *Robust ranking technique which satisfy compensation, linearity, and additivity properties and provides results which are consist human intuition. If $\tilde{a}$ is a fuzzy number then the Robust Ranking is defined by*

$$R(\tilde{a}) = \frac{1}{2} \int_0^1 \left(a_\alpha^l + a_\alpha^u\right) d\alpha. \tag{14}$$

Where $[a_\alpha^l, a_\alpha^u]$ is the α-cut of the fuzzy number $\tilde{a}$.
In this paper we use this method for ranking the objective values. The roubast ranking index $R(\tilde{a})$ gives the representative value of fuzzy number $\tilde{a}$.

Definition 4.4. [15] *let $\tilde{a}$ and $\tilde{b}$ are two fuzzy numbers then*

a) *$\tilde{a} \geqslant \tilde{b}$ if and only if $R(\tilde{a}) \geqslant R(\tilde{b})$.*

b) *$\tilde{a} = \tilde{b}$ if and only if $R(\tilde{a}) = R(\tilde{b})$.*

c) *$\tilde{a} \leqslant \tilde{b}$ if and only if $R(\tilde{a}) \leqslant R(\tilde{b})$.*

Using the relation (b), can be easily proved that bi-objective fractional transportation problems with fuzzy numbers is equivalent to bi-objective fractional transportation problems with crisp numbers.

Now, we need the following theorem which is used in the proposed method and this theorem States that the solutions obtained from the proposed method are an efficient solution (pareto optimal solution) or a non-efficient solution, which extracts the appropriate efficient solution from the efficient solutions.

Theorem 4.5. [14] *Let* $X^0 = \{x_{ij}^0 \ , \ i = 1, 2, , m \ ; \ j = 1, 2, , n\}$ *be an optimal solution to* (P_1), *where*

$$(P_1) \quad \text{minimize} \quad Z_1 = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}$$

$$\text{Subject to} \quad \sum_{j=1}^{n} x_{ij} = a_i \quad for \quad i = 1, 2, \cdots, m$$

$$\sum_{i=1}^{m} x_{ij} = b_j \quad for \quad j = 1, 2, \cdots, n$$

$$x_{ij} \geqslant 0 \quad i = 1, 2, \cdots, m \ ; \ j = 1, 2, \cdots, n$$

and $Y^0 = \{y_{ij}^0 \ , \ i = 1, 2, , m \ ; \ j = 1, 2, , n\}$ *be an optimal solution to* (P_2), *where*

$$(P_2) \quad \text{minimize} \quad Z_2 = \sum_{i=1}^{m}\sum_{j=1}^{n} d_{ij}x_{ij}$$

$$\text{Subject to} \quad \sum_{j=1}^{n} x_{ij} = a_i \quad for \quad i = 1, 2, \cdots, m$$

$$\sum_{i=1}^{m} x_{ij} = b_j \quad for \quad j = 1, 2, \cdots, n$$

$$x_{ij} \geqslant 0 \quad i = 1, 2, \cdots, m \ ; \ j = 1, 2, \cdots, n$$

Then, $U' = \{u_{ij}' \ , \ i = 1, 2, \cdots, m \ ; \ j = 1, 2, \cdots, n\}$ *which is obtained from* $X^0 = \{x_{ij}^0 \ , \ i = 1, 2, \cdots, m \ ; \ j = 1, 2, \cdots, n\}$ *or* $Y^0 = \{y_{ij}^0 \ , \ i = 1, 2, \cdots, m \ ; \ j = 1, 2, \cdots, n\}$, *is an efficient/non efficient solution to the problem* (P)

$$(P) \quad \text{minimize} \quad Z_1 = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}$$

$$\text{minimize} \quad Z_2 = \sum_{i=1}^{m}\sum_{j=1}^{n} d_{ij}x_{ij}$$

$$\text{Subject to} \quad \sum_{j=1}^{n} x_{ij} = a_i \quad for \quad i = 1, 2, \cdots, m$$

$$\sum_{i=1}^{m} x_{ij} = b_j \quad for \quad j = 1, 2, \cdots, n$$

$$x_{ij} \geqslant 0 \quad i = 1, 2, \cdots, m \ ; \ j = 1, 2, \cdots, n.$$

5.   Proposed Method

We now propose a new method for finding all the solutions to the bi-objective transportation problem of fractional programming problems with fuzzy number. This method proceeds as follows:

Step 1: Convert fuzzy bi-objective transportation problems into bi-objective transportation problems with crisp number by using Robust Ranking Technique.

Step 2: Construct two individual problems of the given BFTP namely, first objective transportation problem of fractional programming problems (FOFTP) and second objective transportation problem of fractional programming problems (SOFTP).

Step 3: Obtain an optimal solutions to the problems (FOFTP) and (SOFTP) using transportation algorithm of fractional programming problems (in order to solve fuzzy single objective problem, obtain solution with maximum profit method and then optimize it with modified distribution method).

Step 4: Start with an optimal solution of (FOFTP) in the (SOFTP) as a feasible solution which is an efficient solution to BFTP.

Step 5: Select the allocated cell (t, r) with the maximum of difference of profit and penalty in the (SOFTP). Then, construct a rectangular loop/that starts and ends at the allocated cell (t, r) and connect some of the unallocated and allocated cells.

Step 6: Add and subtract λ to and from the transition cells of the loop in such a way that the rim requirements remain satisfied and assign a sequence of values to λ one by one in such a way that the allocated cell remains non-negative. Then, obtain a feasible solution to (SOFTP) for each value of λ which is also an efficient/a non-efficient solution to BFTP by the Theorem 4.5.

Step 7: Check whether the feasible solution to (SOFTP) obtained from the step 5. It is the optimum solution. If not, repeat the Steps 4 and 5 until an optimum solution to (SOFTP) is found. If so, the process can be stopped and movement to the next step can be made.

Step 8: Start with an optimal solution of the (SOFTP) in the (FOFTP) as a feasible solution which is an efficient/non-efficient solution to BFTP.

Step 9: Repeat the steps 4, 5 and 6 for the (FOFTP).

Step 10: Combine all solutions (efficient/non efficient) of BFTP obtained using the optimal solutions of (FOFTP) and (SOFTP). From this, a set of efficient solutions and a set of non-efficient solutions to the BFTP can be obtained.

6.   Numerical Example

The proposed method for solving a FBFTP is illustrated by the following example.

Example 6.1. Assume there are two objectives under consideration: The first objective function is the maximization of the ratio of the total delivery speed to total waste along the shipping route and the second objective function is the maximization of ratio of total profit to total cost. The ratio of the total delivery speed to total waste along the shipping route and the second objective function is the maximization of ratio of total profit to total cost are given in the following tables:

(FOFTP) :

Destination→ source↓	1	2	3	4	supply
1	$(2,3,5,6)$ $(0,1,1,2)$	$(3,5,8,10)$ $(1,2,3,6)$	$(2,3,6,9)$ $(1,2,5,8)$	$(0,1,3,4)$ $(5,5,7,7)$	15
2	$(1,1,2,4)$ $(2,4,5,5)$	$(3,5,6,6)$ $(1,2,3,6)$	$(0,0,2,2)$ $(4,4,7,8)$	$(2,3,5,6)$ $(0,1,3,4)$	25
3	$(1,1,2,4)$ $(3,4,6,7)$	$(0,0,1,3)$ $(1,3,4,4)$	$(2,2,5,7)$ $(2,2,3,5)$	$(2,2,4,4)$ $(1,2,2,3)$	20
demand	14	18	12	16	

(SOFTP) :

Destination→ source↓	1	2	3	4	supply
1	$(9, 10, 10, 11)$ $(12, 14, 17, 17)$	$(13, 13, 14, 16)$ $(11, 11, 13, 13)$	$(6, 7, 9, 10)$ $(14, 14, 17, 19)$	$(10, 10, 13, 15)$ $(5, 7, 9, 11)$	15
2	$(7, 8, 8, 9)$ $(8, 9, 11, 12)$	$(9, 11, 13, 15)$ $(5, 5, 7, 7)$	$(10, 13, 16, 17)$ $(10, 11, 14, 17)$	$(6, 6, 9, 11)$ $(9, 11, 13, 15)$	25
3	$(7, 8, 10, 11)$ $(10, 13, 14, 15)$	$(4, 5, 6, 9)$ $(13, 14, 16, 17)$	$(12, 15, 16, 17)$ $(9, 11, 12, 16)$	$(7, 9, 10, 10)$ $(7, 8, 11, 14)$	20
demand	14	18	12	16	

Convert fuzzy bi-objective transportation problems into bi-objective transportation problems with classic number by using robust ranking technique obtain a problem with bellow characteristics:

(FOFTP) :

Destination→ source↓	1	2	3	4	supply
1	4 1	6 3	5 4	2 6	15
2	2 4	5 3	1 6	4 2	25
3	2 5	1 3	4 3	3 2	20
demand	14	18	12	16	

(SOFTP) :

Destination→ source↓	1	2	3	4	supply
1	10 15	14 12	8 16	12 8	15
2	8 10	12 6	14 13	8 12	25
3	9 13	6 15	15 12	9 10	20
demand	14	18	12	16	

Obtain fuzzy solution with maximum profit method and then optimize

it with modified distribution method then FOFTP optimal solution is:

$$x_{11} = 14 \ , \ x_{12} = 1 \ , \ x_{21} = 17 \ , \ x_{24} = 8 \ , \ x_{33} = 12 \ , \ x_{34} = 8$$

Obtain fuzzy solution with maximum profit method and then optimize it with modified distribution method then SOFTP optimal solution is:

$$x_{14} = 15 \ , \ x_{21} = 7 \ , \ x_{22} = 18 \ , \ x_{31} = 7 \ , \ x_{33} = 12 \ , \ x_{34} = 1$$

Now, as in Step 3, we consider the optimal solution of the FOFTP in the SOFTP as a feasible solution in the following table:

Destination→ source↓	1		2		3		4		supply
1	10		14		8		12		15
	15	**14**	12	**1**	16		8		
2	8		12		14		8		25
	10		6	**17**	13		12	**8**	
3	9		6		15		9		20
	13		15		12	**12**	10	**8**	
demand	14		18		12		16		

Thus, $\left(\dfrac{251}{136}, \dfrac{704}{543} \right)$ is the bi-objective value of BFTP for the feasible solution

$$x_{11} = 14 \ , \ x_{12} = 1 \ , \ x_{21} = 17 \ , \ x_{24} = 8 \ , \ x_{33} = 12 \ , \ x_{34} = 8$$

According to Step 4, we construct a rectangular loop (2,4) - (2,2) - (1,2) - (1,4) - (2,4). By using the Step 5, we have the following reduced table.

Destination→ source↓	1		2		3		4		supply
1	10		14		8		12		15
	15	**14**	12	$1 - \lambda$	16		8	λ	
2	8		12		14		8		25
	10		6	$17 + \lambda$	13		12	$8 - \lambda$	
3	9		6		15		9		20
	13		15		12	**12**	10	**8**	
demand	14		18		12		16		

Now, the current solution to SOFTP is not the optimum solution. Repetition of Step 5 and 6 results in the following feasible solution which is better than the prior feasible solution of SOFTP.

Thus, by using Steps 5 and 6, we obtain the set of all efficient/non-efficient solution from FOFTP to SOFTP is given bellow:

It	λ	Bi − objective value
1	$\{0,1\}$	$(\dfrac{(94,145+\lambda,245+\lambda,312+2\lambda)}{(153,193-\lambda,274+2\lambda,360+3\lambda)}$, $\dfrac{(561-\lambda,649-3\lambda,757+\lambda,849+3\lambda)}{(481+4\lambda,534+7\lambda,651+3\lambda,746+4\lambda)})$
2	$\{1,\cdots,6\}$	$(\dfrac{(94-2\lambda,146+4\lambda,246+5\lambda,314+4\lambda)}{(153-7\lambda,192-7\lambda,276-8\lambda,363-6\lambda)}$, $\dfrac{(560-2\lambda,646-2\lambda,758-2\lambda,852-2\lambda)}{(410+8\lambda,496+9\lambda,596+10\lambda,690+9\lambda)})$
3	$\{1,\cdots,9\}$	$(\dfrac{(112+2\lambda,170+2\lambda,276+4\lambda,338+6\lambda)}{(153-7\lambda,192-7\lambda,276-8\lambda,363-6\lambda)}$, $\dfrac{(548,634-2\lambda,746-3\lambda,840-3\lambda)}{(458+10\lambda,550+10\lambda,656+10\lambda,744+10\lambda)})$
4	$\{1,\cdots,8\}$	$(\dfrac{(130-2\lambda,188-3\lambda,312-4\lambda,392-5\lambda)}{(66-3\lambda,114-2\lambda,192-4\lambda,300-7\lambda)}$, $\dfrac{(548-7\lambda,616-6\lambda,749-8\lambda,813-7\lambda)}{(548+4\lambda,640+4\lambda,746+6\lambda,834+6\lambda)})$
5	$\{1,\cdots,8\}$	$(\dfrac{(114+3\lambda,164+4\lambda,280+4\lambda,352+\lambda)}{(42+\lambda,98,160-2\lambda,244+\lambda)}$, $\dfrac{(492+6\lambda,588+9\lambda,823+8\lambda,757+5\lambda)}{(580-10\lambda,672-11\lambda,794-11\lambda,882-11\lambda)})$

Similarly, by using Steps 8 and 9, we obtain the set of all solutions S_2 from SOFTP to FOFTP is given bellow:

It	λ	Bi − objective value
1	$\{0,1\}$	$(\dfrac{(138-2\lambda,196-2\lambda,312-4\lambda,360-6\lambda)}{(50+5\lambda,98+4\lambda,144+4\lambda,252+3\lambda)}$, $\dfrac{(540,640+2\lambda,718+2\lambda,797+3\lambda)}{(500-10\lambda,576-10\lambda,693-10\lambda,794-10\lambda)})$
2	$\{1,\cdots,7\}$	$(\dfrac{(136-2\lambda,194-3\lambda,308-6\lambda,354-7\lambda)}{(55+3\lambda,102+3\lambda,148+3\lambda,255+2\lambda)}$, $\dfrac{(556-\lambda,642+\lambda,722+\lambda,800-\lambda)}{(490-\lambda,566-3\lambda,696,784)})$
3	$\{1,\cdots,14\}$	$(\dfrac{(122-3\lambda,173-3\lambda,280-4\lambda,305-2\lambda)}{(76+7\lambda,123+6\lambda,169+3\lambda,269+9\lambda)}$, $\dfrac{(533+\lambda,649-\lambda,699+3\lambda,793+5\lambda)}{(483-4\lambda,545-2\lambda,696-5\lambda,784-5\lambda)})$
4	$\{1,\cdots,7\}$	$(\dfrac{(80+2\lambda,131+2\lambda,224+3\lambda,277+5\lambda)}{(174-3\lambda,207-2\lambda,309-5\lambda,395-5\lambda)}$, $\dfrac{(547+2\lambda,635+\lambda,771-2\lambda,863-2\lambda)}{(427-3\lambda,517-4\lambda,626-5\lambda,714-4\lambda)})$

Now the set of all solutions S of the BTP obtained from FOFTP to SOFTP and from SOFTP to FOFTP is given bellow:

Num	$\tilde{Q}_1(x)$	$\tilde{Q}_2(x)$	$\left(R(\tilde{Q}_1), R(\tilde{Q}_2)\right)$	Level Satisfaction
1	$\left(\dfrac{92}{357}, \dfrac{151}{268}, \dfrac{251}{185}, \dfrac{318}{146}\right)$	$\left(\dfrac{558}{699}, \dfrac{644}{606}, \dfrac{756}{505}, \dfrac{850}{418}\right)$	$(1.0889, 1.3478)$	$(34.21, 97.25)$
2	$\left(\dfrac{90}{351}, \dfrac{156}{268}, \dfrac{256}{178}, \dfrac{322}{139}\right)$	$\left(\dfrac{556}{708}, \dfrac{151}{616}, \dfrac{754}{185}, \dfrac{848}{426}\right)$	$(1.1527, 1.3212)$	$(36.22, 95.34)$
3	$\left(\dfrac{88}{345}, \dfrac{161}{268}, \dfrac{261}{171}, \dfrac{326}{132}\right)$	$\left(\dfrac{554}{717}, \dfrac{151}{626}, \dfrac{752}{185}, \dfrac{846}{434}\right)$	$(1.2224, 1.2955)$	$(38.41, 93.48)$
4	$\left(\dfrac{86}{339}, \dfrac{166}{268}, \dfrac{266}{164}, \dfrac{330}{125}\right)$	$\left(\dfrac{552}{726}, \dfrac{151}{636}, \dfrac{750}{185}, \dfrac{844}{442}\right)$	$(1.2989, 1.2706)$	$(40.81, 91.69)$
5	$\left(\dfrac{84}{333}, \dfrac{171}{268}, \dfrac{271}{157}, \dfrac{334}{118}\right)$	$\left(\dfrac{550}{735}, \dfrac{151}{646}, \dfrac{748}{185}, \dfrac{842}{450}\right)$	$(1.3833, 1.2466)$	$(43.47, 89.95)$
6	$\left(\dfrac{82}{327}, \dfrac{176}{568}, \dfrac{276}{150}, \dfrac{338}{111}\right)$	$\left(\dfrac{548}{744}, \dfrac{151}{656}, \dfrac{746}{185}, \dfrac{840}{458}\right)$	$(1.4769, 1.2241)$	$(46.41, 88.33)$
7	$\left(\dfrac{114}{350}, \dfrac{172}{224}, \dfrac{280}{146}, \dfrac{344}{106}\right)$	$\left(\dfrac{548}{754}, \dfrac{632}{666}, \dfrac{743}{560}, \dfrac{837}{648}\right)$	$(1.5641, 1.1977)$	$(49.15, 86.42)$
8	$\left(\dfrac{116}{353}, \dfrac{174}{220}, \dfrac{284}{142}, \dfrac{350}{101}\right)$	$\left(\dfrac{548}{764}, \dfrac{630}{676}, \dfrac{740}{570}, \dfrac{834}{478}\right)$	$(1.6462, 1.1730)$	$(51.72, 84.64)$
9	$\left(\dfrac{118}{356}, \dfrac{177}{216}, \dfrac{288}{138}, \dfrac{356}{96}\right)$	$\left(\dfrac{548}{774}, \dfrac{628}{686}, \dfrac{737}{580}, \dfrac{831}{488}\right)$	$(1.7353, 1.1492)$	$(54.52, 82.92)$
10	$\left(\dfrac{120}{359}, \dfrac{180}{212}, \dfrac{292}{134}, \dfrac{362}{91}\right)$	$\left(\dfrac{548}{784}, \dfrac{626}{696}, \dfrac{734}{590}, \dfrac{828}{498}\right)$	$(1.8327, 1.1262)$	$(57.59, 81.26)$
11	$\left(\dfrac{122}{362}, \dfrac{183}{208}, \dfrac{296}{130}, \dfrac{368}{86}\right)$	$\left(\dfrac{548}{794}, \dfrac{624}{706}, \dfrac{731}{600}, \dfrac{825}{508}\right)$	$(1.9395, 1.1040)$	$(60.94, 79.67)$
12	$\left(\dfrac{124}{365}, \dfrac{186}{204}, \dfrac{300}{126}, \dfrac{374}{81}\right)$	$\left(\dfrac{548}{804}, \dfrac{622}{716}, \dfrac{728}{610}, \dfrac{822}{518}\right)$	$(2.0575, 1.0826)$	$(64.65, 78.12)$
13	$\left(\dfrac{126}{368}, \dfrac{189}{200}, \dfrac{304}{122}, \dfrac{380}{76}\right)$	$\left(\dfrac{548}{814}, \dfrac{620}{726}, \dfrac{725}{620}, \dfrac{819}{528}\right)$	$(2.1885, 1.0619)$	$(68.77, 76.63)$
14	$\left(\dfrac{126}{371}, \dfrac{192}{196}, \dfrac{308}{118}, \dfrac{386}{71}\right)$	$\left(\dfrac{548}{824}, \dfrac{618}{736}, \dfrac{722}{630}, \dfrac{816}{538}\right)$	$(2.3351, 1.0418)$	$(73.37, 75.18)$
15	$\left(\dfrac{130}{374}, \dfrac{195}{192}, \dfrac{312}{114}, \dfrac{392}{66}\right)$	$\left(\dfrac{548}{834}, \dfrac{616}{746}, \dfrac{719}{640}, \dfrac{813}{548}\right)$	$(2.5007, 1.0224)$	$(78.57, 73.78)$
16	$\left(\dfrac{126}{286}, \dfrac{182}{184}, \dfrac{304}{110}, \dfrac{382}{60}\right)$	$\left(\dfrac{534}{846}, \dfrac{604}{758}, \dfrac{765}{648}, \dfrac{799}{556}\right)$	$(2.6399, 1.0114)$	$(82.95, 72.98)$
17	$\left(\dfrac{124}{279}, \dfrac{179}{180}, \dfrac{300}{108}, \dfrac{377}{57}\right)$	$\left(\dfrac{527}{852}, \dfrac{598}{764}, \dfrac{773}{652}, \dfrac{792}{560}\right)$	$(2.7076, 1.0003)$	$(85.08, 72.18)$
18	$\left(\dfrac{122}{272}, \dfrac{176}{176}, \dfrac{296}{106}, \dfrac{372}{54}\right)$	$\left(\dfrac{520}{858}, \dfrac{592}{770}, \dfrac{781}{656}, \dfrac{785}{564}\right)$	$(2.7824, 0.9893)$	$(87.42, 71.38)$
19	$\left(\dfrac{120}{265}, \dfrac{173}{172}, \dfrac{292}{104}, \dfrac{367}{51}\right)$	$\left(\dfrac{513}{864}, \dfrac{586}{776}, \dfrac{789}{660}, \dfrac{778}{568}\right)$	$(2.8656, 0.9785)$	$(90.04, 70.61)$
20	$\left(\dfrac{118}{258}, \dfrac{170}{168}, \dfrac{288}{102}, \dfrac{362}{48}\right)$	$\left(\dfrac{506}{870}, \dfrac{580}{782}, \dfrac{797}{664}, \dfrac{771}{572}\right)$	$(2.9586, 0.9678)$	$(92.96, 69.84)$
21	$\left(\dfrac{114}{244}, \dfrac{164}{160}, \dfrac{280}{98}, \dfrac{352}{42}\right)$	$\left(\dfrac{492}{882}, \dfrac{568}{794}, \dfrac{813}{672}, \dfrac{757}{580}\right)$	$(3.1825, 0.9470)$	$(100, 68.33)$
22	$\left(\dfrac{117}{245}, \dfrac{168}{158}, \dfrac{284}{98}, \dfrac{353}{43}\right)$	$\left(\dfrac{498}{871}, \dfrac{577}{783}, \dfrac{663}{660}, \dfrac{762}{570}\right)$	$(3.1620, 0.9125)$	$(99.35, 65.84)$
23	$\left(\dfrac{120}{246}, \dfrac{172}{156}, \dfrac{288}{98}, \dfrac{354}{44}\right)$	$\left(\dfrac{504}{860}, \dfrac{586}{772}, \dfrac{671}{648}, \dfrac{767}{560}\right)$	$(3.1436, 0.9376)$	$(98.78, 67.65)$

24	$\left(\dfrac{123}{247}, \dfrac{176}{154}, \dfrac{292}{98}, \dfrac{355}{45}\right)$	$\left(\dfrac{510}{849}, \dfrac{595}{761}, \dfrac{679}{636}, \dfrac{772}{550}\right)$	$(3.1273, 0.9635)$	$(98.26, 69.52)$
25	$\left(\dfrac{126}{248}, \dfrac{180}{152}, \dfrac{296}{98}, \dfrac{356}{46}\right)$	$\left(\dfrac{516}{838}, \dfrac{604}{750}, \dfrac{687}{624}, \dfrac{777}{540}\right)$	$(3.1129, 0.9902)$	$(97.81, 71.45)$
26	$\left(\dfrac{129}{249}, \dfrac{184}{150}, \dfrac{300}{98}, \dfrac{357}{47}\right)$	$\left(\dfrac{522}{827}, \dfrac{613}{739}, \dfrac{695}{612}, \dfrac{782}{530}\right)$	$(3.1004, 1.0179)$	$(97.42, 73.45)$
27	$\left(\dfrac{132}{250}, \dfrac{188}{148}, \dfrac{304}{98}, \dfrac{358}{48}\right)$	$\left(\dfrac{528}{816}, \dfrac{622}{728}, \dfrac{703}{600}, \dfrac{787}{520}\right)$	$(3.0896, 1.0418)$	$(97.08, 75.17)$
28	$\left(\dfrac{135}{251}, \dfrac{192}{146}, \dfrac{308}{98}, \dfrac{359}{49}\right)$	$\left(\dfrac{534}{805}, \dfrac{631}{717}, \dfrac{711}{588}, \dfrac{792}{510}\right)$	$(3.0807, 1.0764)$	$(96.81, 77.67)$
29	$\left(\dfrac{138}{252}, \dfrac{196}{144}, \dfrac{312}{98}, \dfrac{360}{50}\right)$	$\left(\dfrac{540}{794}, \dfrac{640}{706}, \dfrac{719}{576}, \dfrac{797}{500}\right)$	$(3.0731, 1.1072)$	$(96.56, 79.90)$
30	$\left(\dfrac{126}{265}, \dfrac{179}{163}, \dfrac{278}{117}, \dfrac{319}{70}\right)$	$\left(\dfrac{550}{784}, \dfrac{647}{696}, \dfrac{727}{551}, \dfrac{795}{485}\right)$	$(2.1267, 1.1477)$	$(66.82, 82.82)$
31	$\left(\dfrac{124}{267}, \dfrac{176}{169}, \dfrac{272}{120}, \dfrac{312}{73}\right)$	$\left(\dfrac{549}{784}, \dfrac{648}{696}, \dfrac{728}{548}, \dfrac{794}{484}\right)$	$(2.0116, 1.1504)$	$(63.21, 83.01)$
32	$\left(\dfrac{122}{269}, \dfrac{173}{172}, \dfrac{266}{123}, \dfrac{305}{76}\right)$	$\left(\dfrac{548}{784}, \dfrac{649}{696}, \dfrac{729}{545}, \dfrac{793}{483}\right)$	$(1.9088, 1.1530)$	$(59.97, 83.20)$
33	$\left(\dfrac{110}{305}, \dfrac{161}{181}, \dfrac{264}{147}, \dfrac{297}{104}\right)$	$\left(\dfrac{537}{764}, \dfrac{645}{676}, \dfrac{711}{537}, \dfrac{813}{467}\right)$	$(1.4755, 1.1805)$	$(46.36, 85.19)$
34	$\left(\dfrac{107}{314}, \dfrac{158}{184}, \dfrac{260}{153}, \dfrac{295}{111}\right)$	$\left(\dfrac{538}{759}, \dfrac{644}{671}, \dfrac{714}{535}, \dfrac{818}{463}\right)$	$(1.3891, 1.1925)$	$(43.65, 86.05)$
35	$\left(\dfrac{104}{323}, \dfrac{155}{187}, \dfrac{256}{159}, \dfrac{293}{118}\right)$	$\left(\dfrac{539}{754}, \dfrac{643}{666}, \dfrac{717}{533}, \dfrac{823}{459}\right)$	$(1.3109, 1.2046)$	$(41.19, 86.92)$
36	$\left(\dfrac{101}{332}, \dfrac{152}{190}, \dfrac{252}{165}, \dfrac{291}{125}\right)$	$\left(\dfrac{540}{749}, \dfrac{642}{661}, \dfrac{720}{531}, \dfrac{828}{455}\right)$	$(1.2399, 1.2170)$	$(38.96, 87.82)$
37	$\left(\dfrac{98}{341}, \dfrac{149}{193}, \dfrac{248}{171}, \dfrac{289}{132}\right)$	$\left(\dfrac{541}{744}, \dfrac{641}{656}, \dfrac{723}{529}, \dfrac{833}{451}\right)$	$(1.1691, 1.2295)$	$(36.73, 88.72)$
38	$\left(\dfrac{95}{350}, \dfrac{146}{196}, \dfrac{244}{177}, \dfrac{287}{139}\right)$	$\left(\dfrac{542}{739}, \dfrac{640}{651}, \dfrac{726}{527}, \dfrac{838}{447}\right)$	$(1.1149, 1.2422)$	$(35.03, 89.64)$
39	$\left(\dfrac{92}{359}, \dfrac{143}{199}, \dfrac{240}{183}, \dfrac{285}{146}\right)$	$\left(\dfrac{543}{734}, \dfrac{639}{646}, \dfrac{729}{525}, \dfrac{843}{443}\right)$	$(1.0596, 1.2551)$	$(33.29, 90.59)$
40	$\left(\dfrac{86}{377}, \dfrac{137}{205}, \dfrac{232}{195}, \dfrac{281}{160}\right)$	$\left(\dfrac{545}{724}, \dfrac{637}{363}, \dfrac{735}{521}, \dfrac{853}{435}\right)$	$(0.9606, 1.2815)$	$(30.18, 92.47)$
41	$\left(\dfrac{80}{395}, \dfrac{131}{211}, \dfrac{224}{207}, \dfrac{277}{174}\right)$	$\left(\dfrac{547}{714}, \dfrac{635}{626}, \dfrac{741}{517}, \dfrac{863}{427}\right)$	$(0.8744, 1.3187)$	$(27.47, 95.16)$
42	$\left(\dfrac{86}{380}, \dfrac{137}{294}, \dfrac{233}{201}, \dfrac{292}{165}\right)$	$\left(\dfrac{553}{702}, \dfrac{638}{611}, \dfrac{765}{505}, \dfrac{857}{418}\right)$	$(0.8553, 1.3493)$	$(26.87, 97.36)$
43	$\left(\dfrac{92}{365}, \dfrac{143}{279}, \dfrac{242}{195}, \dfrac{307}{156}\right)$	$\left(\dfrac{559}{690}, \dfrac{641}{596}, \dfrac{759}{493}, \dfrac{851}{409}\right)$	$(0.7434, 1.3765)$	$(23.35, 99.32)$
44	$\left(\dfrac{94}{360}, \dfrac{145}{274}, \dfrac{245}{193}, \dfrac{312}{153}\right)$	$\left(\dfrac{561}{686}, \dfrac{642}{591}, \dfrac{757}{489}, \dfrac{849}{406}\right)$	$(0.7247, 1.3858)$	$(22.77, 100)$

The above satisfaction level table is very much useful for the decision makers to select the appropriate efficient solution to fuzzy bi-objective

fractional transportation problems according to their level of satisfaction of objectives.

7.   Conclusion

In this paper, the proposed method provides the set of efficient solutions for bi-objective fractional transportation problems with fuzzy numbers using ranking function and percent of function one solution is introduced for transportation problem This method is new method that have been used for solving multi-objective fractional transportation problems with fuzzy numbers that decision maker can determine the preferred solution from efficient solution using it. Here the two objectives are inherently taken care of at each iteration and the pairs recorded at any step identify the next efficient pair, thus providing a direction of movement without making use of any utility function. This method enables the decision makers to select an appropriate solution, depending on their financial position and also, their level of satisfaction of objectives.

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