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Investigating a Solution of a Multi-Singular Pointwise Defined Fractional Integro-Differential Equation with Caputo Derivative Boundary Condition

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Abstract. We investigate the existence of solutions for a multi-singular pointwise defined fractional integro-differential equation under some boundary conditions.

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1. Introduction

Singular differential equations have been used from a distance past for describing a large number of natural phenomena ([7]). In the recent century, some papers have been published about the existence of solutions for different singular fractional differential equations ([2], [6], [6], [11], [14]-[3]).

In 1996, Delbosco and Rodino studied the problem $D^s u(t) = f(t, u)$ with initial condition $u(a) = b$, where $a > 0$, $b \in \mathbb{R}$, $0 < s < 1$, and D is

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the standard Riemann-Liouville fractional derivative ([6]). In 2005, Bai and Liu investigated the existence of positive solutions for the problem $D_0^\alpha u(t) + f(t, u(t)) = 0$ with boundary conditions $u(0) = u(1) = 0$, where $0 < t < 1$, $0 < \alpha \leqslant 2$, and D_0^α is the Riemann-Liouville standard derivation ([3]). In 2008, Qiu and Bai studied the existence of a positive solution for the nonlinear fractional differential equation $D_{0+}^\alpha u(t) + f(t, u(t)) = 0$ with boundary conditions $u(0) = u'(1) = u''(1) = 0$, where $0 < t < 1$, $2 < \alpha < 3, D_{0+}^\alpha$ is the Caputo derivation and $f : [0, 1] \times [0, \infty) \to [0, \infty)$ is such that $\lim_{t\to 0^+} f(t, .) = \infty$ ([9]). In 2010, Agarwal, O'Regan, and Stanek investigated the singular fractional Dirichlet problem $D^\alpha u(t)+f(t, u(t), D^\mu u(t)) = 0$ with boundary value problem $u(0) = u(1) = 0$, where $1 < \alpha \leqslant 2$, $\mu > 0$, $\alpha - \mu \geqslant 1$, $f \in Car([0, 1]\times\beta)$, f is positive and singular at $x = 0$, and D is the standard Riemann-Liouville derivative [1]. In 2012, Cabada and Wang investigated the existence of positive solutions for the nonlinear fractional differential equation $D^\alpha u(t) + f(t, u(t)) = 0$ with boundary conditions $u(0) = u''(1) = 0$ and $u(1) = \int_0^1 u(s)ds$, where $0 < t < 1$, $2 < \alpha < 3$, $0 < \lambda < 2$, D is the Caputo fractional derivative and $f : [0, 1] \times [0, \infty) \to [0, \infty)$ is a continuous function ([5]). In 2014, Li reviewed the existence of solutions for singular problem $D^q u(t) + f(t, u(t), D^\sigma u(t)) = 0$ with boundary conditions $u(0) = u'(1) = 0$ and $u'(1) = D^\alpha u(t)$, where $0 < t < 1$, $2 < q < 3$, $0 < \sigma < 1$, $f : (0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function that may have singularity at $t = 0$ and D is the standard Caputo derivative ([10]). In 2016, the multi-singular pointwise defined fractional integro-differential equation

$$D^\mu x(t) + f(t, x(t), x'(t), D^\beta x(t), I^p x(t)) = 0$$

with boundary conditions $x'(0) = x(\xi)$, $x(1) = \int_0^\eta x(s)ds$ and $x^{(j)}(0) = 0$ for $j = 2, ..., [\mu] - 1$ was investigated, where $0 < t < 1$, $\mu \in [2, 3)$, $x \in C^1[0, 1]$, $\beta, \xi, \eta \in (0, 1)$, $p > 1$, D is Caputo fractional derivative and $f : [0, 1] \times \mathbb{R}^4 \to \mathbb{R}$ is a function such that $f(t, ., ., ., .)$ is singular at some point $t \in [0, 1]$ ([14]). In fact we say that, $D^\alpha x(t) + f(t) = 0$ is a pointwise defined equation on $[0, 1]$ if there exists set $E \subset [0, 1]$ such that the measure of E^c is zero and the equation holds on E ([15]). In 2017, Shabibi, Rezapour, and Vaezpour investigated the singular fractional integro-differential equation

$$D^{\alpha}x(t) + f(t, x(t), x'(t), D^{\beta}x(t), \int_0^t h(\xi)x(\xi)d\xi) = 0$$

with boundary conditions $x(0) = x'(0)$ and $x(1) = D^{\beta}x(\mu)$, where $0 < t < 1$, $x \in C^1[0,1]$, $\alpha > 2$, $0 < \beta, \gamma, \mu < 1$, $h \in L^1([0,1])$, $||h||_1 = m$, D is the Caputo fractional derivative and $f(t, x_1, x_2, x_3, x_4)$ is singular at some points $t \in [0,1]$ ([16]). Using idea of these papers, we investigate the existence of solutions for the folowing nonlinear pointwise defined fractional integro-differential equation

$$D^{\alpha}x(t) + f(t, x(t), D^{\beta}x(t), \int_0^t h(\xi)x(\xi)d\xi, \phi(x(t))) = 0 \tag{1}$$

with boundary conditions $x^{(j)}(0) = 0$ for $j \geqslant 2$, $\int_0^{\mu} x(\xi)d\xi = 0$ and $x'(1) = x(\eta)$, where $\alpha \geqslant 2$, $\eta, \mu, \beta \in (0,1)$, $\phi : X \to X$ is a map such that

$$\|\phi(x) - \phi(y)\| \leqslant b_1\|x - y\| + b_2\|x' - y'\|$$

for some non-negative real numbers b_1 and $b_2 \in [0, \infty)$ and all $x, y \in X = C^1[0,1]$, where D is the Caputo fractional derivative and $f \in L^1$ is singular at some points $[0,1]$.

Here, we use $\|.\|_1$ for the norm of $L^1([0,1])$, $\|.\|$ for the sup norm of $Y = C[0,1]$ and $\|x\|_* = \max\{\|x\|, \|x'\|\}$ for the norm of $X = C^1[0,1]$. The Riemann-Liouville integral of order p with the lower bondary $a \geqslant 0$ for a function $f : (a, \infty) \to \mathbb{R}$ is defined by $I^p_{a^+}f(t) = \frac{1}{\Gamma(p)}\int_a^t (t-s)^{p-1}f(s)ds$, provided that the right-hand side is pointwise defined on (a, ∞) ([8]). We denote $I^p_{0^+}f(t)$ by $I^pf(t)$. The Caputo fractional derivative of order $\alpha > 0$ is defined by ${}^cD^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^n(s)}{(t-s)^{\alpha+1-n}}ds$, where $n = [\alpha] + 1$ and $f : (a, \infty) \to \mathbb{R}$ is a function ([8]). Let Ψ be the family of nondecreasing functions $\psi : [0, \infty) \to [0, \infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$. One can check that $\psi(t) < t$ for all $t > 0$ ([12]). Let $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ be two maps. Then, T is called an α-admissible map whenever $\alpha(x, y) \geqslant 1$ implies $\alpha(Tx, Ty) \geqslant 1$ ([12]). Let (X, d) be a metric space, where $\psi \in \Psi$ and $\alpha : X \times X \to [0, \infty)$ is a map. A self-map

$T : X \to X$ is called an α-ψ-contraction whenever $\alpha(x,y)d(Tx,Ty) \leqslant \psi(d(x,y))$ for all $x,y \in X$ ([12]).

Lemma 1.1. *([12]) Let (X,d) be a complete metric space, $\psi \in \Psi$, $\alpha : X\times X \to [0,\infty)$ a map and $T : X \to X$ an α-admissible α-ψ-contraction. If T is continuous and there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geqslant 1$, then T has a fixed point.*

Lemma 1.2. *([13]) Let $n-1 \leqslant \alpha < n$ and $x \in C(0,1) \cap L^1(0,1)$. Then, we have $I^\alpha D^\alpha x(t) = x(t) + \sum_{i=0}^{n-1} c_i t^i$ for some real constants $c_0, \dots, c_{n-1}$.*

2. Main Results

Now, we are ready to present our first key result.

Lemma 2.1. *Let $\alpha \geqslant 2$, $[\alpha] = n-1$, $\mu,\eta \in (0,1)$ and $f \in L^1([0,1])$. The solution of the problem $D^\alpha x(t) + f(t) = 0$ with the boundary conditions $x^{(j)}(0) = 0$ for $j \geqslant 2$, $\int_0^\mu x(\xi)\,d\xi = 0$ and $x'(1) = x(\eta)$ is $x(t) = \int_0^1 G(t,s)f(s)ds$, where $G(t,s)$ is given by $G(t,s) = \frac{-(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\mu(1-\eta+t)-A_{\mu,\eta}}{A_{\mu,\eta}\Gamma(\alpha-1)}(1-s)^{\alpha-2} + \frac{A_{\mu,\eta}+\mu(\eta+t-1)}{A_{\mu,\eta}\Gamma(\alpha)}(\eta-s)^{\alpha-1} + \frac{A_{\mu,\eta}(1-\eta)+t}{A_{\mu,\eta}\Gamma(\alpha+1)}(\mu-s)^\alpha$ whenever $0 \leqslant s \leqslant t \leqslant 1$, $s \leqslant \mu,\eta$, $G(t,s) = \frac{-(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\mu(1-\eta+t)-A_{\mu,\eta}}{A_{\mu,\eta}\Gamma(\alpha-1)}(1-s)^{\alpha-2} + \frac{A_{\mu,\eta}+\mu(\eta+t-1)}{A_{\mu,\eta}\Gamma(\alpha)}(\eta-s)^{\alpha-1}$ whenever $0 \leqslant s \leqslant t \leqslant 1$, $\mu \leqslant s \leqslant \eta$, $G(t,s) = \frac{-(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\mu(1-\eta+t)-A_{\mu,\eta}}{A_{\mu,\eta}\Gamma(\alpha-1)}(1-s)^{\alpha-2} + \frac{A_{\mu,\eta}(1-\eta)+t}{A_{\mu,\eta}\Gamma(\alpha+1)}(\mu-s)^\alpha$ whenever $0 \leqslant s \leqslant t \leqslant 1$, $\eta \leqslant s \leqslant \mu$, $G(t,s) = \frac{-(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\mu(1-\eta+t)-A_{\mu,\eta}}{A_{\mu,\eta}\Gamma(\alpha-1)}(1-s)^{\alpha-2}$ whenever $0 \leqslant s \leqslant t \leqslant 1$, $\mu,\eta \leqslant s$, $G(t,s) = \frac{\mu(1-\eta+t)-A_{\mu,\eta}}{A_{\mu,\eta}\Gamma(\alpha-1)}(1-s)^{\alpha-2} + \frac{A_{\mu,\eta}+\mu(\eta+t-1)}{A_{\mu,\eta}\Gamma(\alpha)}(\eta-s)^{\alpha-1} + \frac{A_{\mu,\eta}(1-\eta)+t}{A_{\mu,\eta}\Gamma(\alpha+1)}(\mu-s)^\alpha$ whenever $0 \leqslant t \leqslant s \leqslant 1$, $s \leqslant \mu,\eta$, $G(t,s) = \frac{\mu(1-\eta+t)-A_{\mu,\eta}}{A_{\mu,\eta}\Gamma(\alpha-1)}(1-s)^{\alpha-2} + \frac{A_{\mu,\eta}+\mu(\eta+t-1)}{A_{\mu,\eta}\Gamma(\alpha)}(\eta-s)^{\alpha-1}$ whenever $0 \leqslant t \leqslant s \leqslant 1$, $\mu \leqslant s \leqslant \eta$, $G(t,s) = \frac{\mu(1-\eta+t)-A_{\mu,\eta}}{A_{\mu,\eta}\Gamma(\alpha-1)}(1-s)^{\alpha-2} + \frac{A_{\mu,\eta}(1-\eta)+t}{A_{\mu,\eta}\Gamma(\alpha+1)}(\mu-s)^\alpha$ whenever $0 \leqslant t \leqslant s \leqslant 1$, $\eta \leqslant s \leqslant \mu$, $G(t,s) = \frac{\mu(1-\eta+t)-A_{\mu,\eta}}{A_{\mu,\eta}\Gamma(\alpha-1)}(1-s)^{\alpha-2}$ whenever $0 \leqslant t \leqslant s \leqslant 1$, $\mu,\eta \leqslant s$ and $A_{\mu,\eta} := \mu(1-\eta) + \frac{\mu^2}{2} > 0$.*

Proof. Using some calculation, one can see that $x(t) = \int_0^1 G(t,s)f(s)ds$ is a solution for the problem $D^\alpha x(t)+f(t) = 0$. Now consider the problem $D^\alpha x(t) + f(t) = 0$. Using Lemma 1.2, it is inferred that

$$x(t) = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)ds + c_0 + c_1 t,$$

where c_0 and c_1 are some real numbers. Hence

$$x'(t) = -\frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2} f(s)ds + c_1.$$

Since $x'(1) = x(\eta)$, we have

$$-\frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2} f(s)ds + c_1 = -\frac{1}{\Gamma(\alpha)}\int_0^\eta (\eta-s)^{\alpha-1} f(s)ds + c_0 + c_1\eta$$

and so

$$c_0 = -\frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2} f(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^\eta (\eta-s)^{\alpha-1} f(s)ds + (1-\eta)c_1.$$

It is easy to see that

$$\int_0^\mu x(\xi)d\xi = -\frac{1}{\Gamma(\alpha+1)}\int_0^\mu (\mu-s)^{\alpha} f(s)ds - \frac{\mu}{\Gamma(\alpha-1)}$$

$$\times\int_0^1 (1-s)^{\alpha-2} f(s)ds + \frac{\mu}{\Gamma(\alpha)}\int_0^\eta (\eta-s)^{\alpha-1} f(s)ds + \mu(1-\eta)c_1 + c_1\frac{\mu^2}{2}.$$

Since $\int_0^\mu x(\xi)d\xi = 0$, we obtain

$$c_1 = \frac{1}{A_{\mu,\eta}\Gamma(\alpha+1)}\int_0^\mu (\mu-s)^{\alpha} f(s)ds + \frac{\mu}{A_{\mu,\eta}\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2} f(s)ds$$

$$-\frac{\mu}{A_{\mu,\eta}\Gamma(\alpha+1)}\int_0^\eta (\eta-s)^{\alpha-1} f(s)ds,$$

where, $A_{\mu,\eta} := \mu(1-\eta) + \frac{\mu^2}{2} > 0$. Thus,

$$c_0 = \frac{-1}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2} f(s)ds +$$

$$\frac{1}{\Gamma(\alpha)}\int_0^\eta (\eta-s)^{\alpha-1}f(s)ds+\frac{1-\eta}{A_{\mu,\eta}\Gamma(\alpha+1)}\int_0^\mu (\mu-s)^\alpha f(s)ds$$

$$+\frac{\mu(1-\eta)}{A_{\mu,\eta}\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}f(s)ds-\frac{\mu(1-\eta)}{A_{\mu,\eta}\Gamma(\alpha)}\int_0^\eta (\eta-s)^{\alpha-1}f(s)ds$$

and so

$$x(t)=-\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)ds-\frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}f(s)ds$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^\eta (\eta-s)^{\alpha-1}f(s)ds$$

$$+\frac{1-\eta}{A_{\mu,\eta}\Gamma(\alpha+1)}\int_0^\mu (\mu-s)^\alpha f(s)ds+\frac{\mu(1-\eta)}{A_{\mu,\eta}\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}f(s)ds$$

$$-\frac{\mu(1-\eta)}{A_{\mu,\eta}\Gamma(\alpha)}\int_0^\eta (\eta-s)^{\alpha-1}f(s)ds+\frac{t}{A_{\mu,\eta}\Gamma(\alpha+1)}\int_0^\mu (\mu-s)^\alpha f(s)ds$$

$$+\frac{\mu t}{A_{\mu,\eta}\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}f(s)ds-\frac{\mu t}{A_{\mu,\eta}\Gamma(\alpha)}\int_0^\eta (\eta-s)^{\alpha-1}f(s)ds.$$

Hence,

$$x(t)=-\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)ds+\frac{\mu(1-\eta+t)-A_{\mu,\eta}}{A_{\mu,\eta}\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}f(s)$$

$$+\frac{A_{\mu,\eta}+\mu(\eta+t-1)}{A_{\mu,\eta}\Gamma(\alpha)}\int_0^\eta (\eta-s)^{\alpha-1}f(s)ds+$$

$$\frac{A_{\mu,\eta}(1-\eta)+t}{A_{\mu,\eta}\Gamma(\alpha+1)}\int_0^\mu (\mu-s)^\alpha f(s)ds.$$

Now, some easy calculations show us that $x(t)=\int_0^1 G(t,s)f(s)ds$. □

Note that, the mappings G and $\frac{\partial G}{\partial t}$ are continuous respect to t. Let f be a map on $[0,1]\times X^2$ such that f is singular at some points of $[0,1]$. Define the function $F: X\to X$ by

$$F_x(t)=-\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,x(s),D^\beta x(s),\int_0^s h(\xi)x(\xi)d\xi,\phi(x(s)))ds$$

$$+\frac{\mu(1-\eta+t)-A_{\mu,\eta}}{A_{\mu,\eta}\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-2}f(s,x(s),D^\beta x(s),\int_0^s h(\xi)x(\xi)d\xi,\phi(x(s)))ds$$
$$+\frac{A_{\mu,\eta}+\mu(\eta+t-1)}{A_{\mu,\eta}\Gamma(\alpha)}\int_0^\eta(\eta-s)^{\alpha-1}f(s,x(s),D^\beta x(s),\int_0^s h(\xi)x(\xi)d\xi,\phi(x(s)))ds$$
$$+\frac{A_{\mu,\eta}(1-\eta)+t}{A_{\mu,\eta}\Gamma(\alpha+1)}\int_0^\mu(\mu-s)^\alpha f(s,x(s),D^\beta x(s),\int_0^s h(\xi)x(\xi)d\xi,\phi(x(s)))ds$$

for all $t\in[0,1]$. Then, we have

$$F'_x(t)=\int_0^1\frac{\partial G}{\partial t}(t,s)f(s,x(s),D^\beta x(s),\int_0^s h(\xi)x(\xi)d\xi,\phi(x(s)))ds$$
$$=-\frac{1}{\Gamma(\alpha-1)}\int_0^t(t-s)^{\alpha-2}f(s,x(s),D^\beta x(s),\int_0^s h(\xi)x(\xi)d\xi,\phi(x(s)))ds$$
$$+\frac{\mu}{A_{\mu,\eta}\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-2}f(s,x(s),D^\beta x(s),\int_0^s h(\xi)x(\xi)d\xi,\phi(x(s)))ds$$
$$+\frac{\mu}{A_{\mu,\eta}\Gamma(\alpha)}\int_0^\eta(\eta-s)^{\alpha-1}f(s,x(s),D^\beta x(s),\int_0^s h(\xi)x(\xi)d\xi,\phi(x(s)))ds$$
$$+\frac{1}{A_{\mu,\eta}\Gamma(\alpha+1)}\int_0^\mu(\mu-s)^\alpha f(s,x(s),D^\beta x(s),\int_0^s h(\xi)x(\xi)d\xi,\phi(x(s))ds.$$

Our key note is that the singular pointwise defined equation (1) has a solution if and only if the map F has a fixed point. Now, we give our main result.

Theorem 2.2. *Let $\alpha\geqslant 2$, $[\alpha]=n-1$, $\mu,\eta\in(0,1)$, $h\in L^1([0,1])$ with $\|h\|_1:=m$, $\phi:X\to\mathbb{R}$ be such that*

$$|\phi(x(t))-\phi(y(t))|\leqslant b_1|x(t)-y(t)|+b_2|x'(t)-y'(t)|$$

for some $b_1,b_2\in[0,\infty)$. Assume that $f:[0,1]\times X^5\to\mathbb{R}$ is a mapping which is singular on some points $[0,1]$ and

$$|f(t,x_1,...,x_5)-f(t,y_1,...,y_5)|\leqslant\sum_{i=1}^{k_0}a_i(t)\Lambda_i(x_1-y_1,...,x_5-y_5)$$

for all $x_1,x_2,y_1,y_2\in X$ and almost all $t\in[0,1]$, where $k_0\in\mathbb{N}$, $a_i:[0,1]\to\mathbb{R}^+$, $\hat{a}_i\in L^1[0,1]$, $\hat{a}_i(s)=(1-s)^{\alpha-2}a_i(s)$, $\Lambda_i:X^5\to\mathbb{R}^+$ is *a nondecreasing mapping respect to all components with* $\frac{\Lambda_i(z,z,z,z,z)}{z^{\gamma_i}} \to q_i$ *as* $z \to 0^+$ *for some* $\gamma_i > 0$, $q_i \in \mathbb{R}^+$ $(1 \leqslant i \leqslant k_0)$. *Suppose that* $|f(t, x_1, ..., x_5)| \leqslant g(t)K(x_1, ..., x_5)$ *for all* $(x_1, ..., x_5) \in X^5$ *and almost all* $t \in [0, 1]$, *where* $a : [0, 1] \to \mathbb{R}^+$, $\hat{g} \in L^1[0, 1]$, $K : X^5 \to \mathbb{R}^+$ *is a nondecreasing mapping respect all their components such that* $\lim_{z\to 0^+} \frac{K(z,z,z,z,z)}{z} \in [0, B)$, *where* $B = (\Delta \|\hat{g}\|_1 L_{\alpha,\mu,\eta})^{-1}$, $A_{\mu,\eta} = \mu(1 - \eta) + \frac{\mu^2}{2}$, $\Delta = \max\{1, \frac{1}{\Gamma(2-\beta)}, m, b_1 + b_2\}$ *and*

$$L_{\alpha,\mu,\eta} = \max\{\frac{1}{\Gamma(\alpha)} + \frac{\mu(2-\eta) - A_{\mu,\eta}}{A_{\mu,\eta}\Gamma(\alpha-1)} + \frac{A_{\mu,\eta} + \mu\eta}{A_{\mu,\eta}\Gamma(\alpha)} + \frac{A_{\mu,\eta}(1-\eta)+1}{A_{\mu,\eta}\Gamma(\alpha+1)},$$
$$\frac{1}{\Gamma(\alpha-1)} + \frac{\mu}{A_{\mu,\eta}\Gamma(\alpha-1)} + \frac{\mu}{A_{\mu,\eta}\Gamma(\alpha)} + \frac{1}{A_{\mu,\eta}\Gamma(\alpha+1)}\}.$$

If $\sum_{i=1}^{k_0} \|\hat{a}_i\|_{[0,1]} q_i \Delta^{\gamma_i} < \frac{1}{L_{\alpha,\mu,\eta}}$, *then the pointwise defined equation* (1) *with boundary conditions* $x^{(j)}(0) = 0$ *for* $j \geqslant 2$, $\int_0^\mu x(\xi)\, d\xi = 0$ *and* $x'(1) = x(\eta)$ *has a solution.*

Proof. First we show that the map F is continuous. Let $x, y \in X$. Then, we have

$$|F_x(t) - F_y(t)| \leqslant |-\frac{1}{\Gamma(\alpha)}\times$$
$$\int_0^t (t-s)^{\alpha-1} f(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)d\xi, \phi(x(s)))ds$$
$$+\frac{\mu(1-\eta+t) - A_{\mu,\eta}}{A_{\mu,\eta}\Gamma(\alpha-1)}\times$$
$$\int_0^1 (1-s)^{\alpha-2} f(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)d\xi, \phi(x(s)))ds$$
$$+\frac{A_{\mu,\eta} + \mu(\eta+t-1)}{A_{\mu,\eta}\Gamma(\alpha)}\times$$
$$\int_0^\eta (\eta-s)^{\alpha-1} f(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)d\xi, \phi(x(s)))ds$$
$$+\frac{A_{\mu,\eta}(1-\eta)+t}{A_{\mu,\eta}\Gamma(\alpha+1)}\times$$

$$\int_0^{\mu}(\mu-s)^{\alpha}f(s,x(s),x'(s),D^{\beta}x(s),\int_0^{s}h(\xi)x(\xi)d\xi,\phi(x(s)))ds$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^{t}(t-s)^{\alpha-1}f(s,y(s),y'(s),D^{\beta}y(s),\int_0^{s}h(\xi)y(\xi)d\xi,\phi(y(s)))ds$$

$$-\frac{\mu(1-\eta+t)-A_{\mu,\eta}}{A_{\mu,\eta}\Gamma(\alpha-1)}\times$$

$$\int_0^{1}(1-s)^{\alpha-2}f(s,y(s),y'(s),D^{\beta}y(s),\int_0^{s}h(\xi)y(\xi)d\xi,\phi(y(s)))ds$$

$$-\frac{A_{\mu,\eta}+\mu(\eta+t-1)}{A_{\mu,\eta}\Gamma(\alpha)}\times$$

$$\int_0^{\eta}(\eta-s)^{\alpha-1}f(s,y(s),y'(s),D^{\beta}y(s),\int_0^{s}h(\xi)y(\xi)d\xi,\phi(y(s)))ds$$

$$-\frac{A_{\mu,\eta}(1-\eta)+t}{A_{\mu,\eta}\Gamma(\alpha+1)}\times$$

$$\int_0^{\mu}(\mu-s)^{\alpha}f(s,y(s),y'(s),D^{\beta}y(s),\int_0^{s}h(\xi)y(\xi)d\xi,\phi(y(s)))ds|$$

$$\leqslant\frac{1}{\Gamma(\alpha)}\int_0^{t}(t-s)^{\alpha-1}|f(s,x(s),x'(s),D^{\beta}x(s),\int_0^{s}h(\xi)x(\xi)d\xi,\phi(x(s)))$$

$$-f(s,y(s),y'(s),D^{\beta}y(s),\int_0^{s}h(\xi)y(\xi)d\xi,\phi(y(s)))|ds$$

$$+\frac{\mu(1-\eta+t)-A_{\mu,\eta}}{A_{\mu,\eta}\Gamma(\alpha-1)}\times$$

$$\int_0^{1}(1-s)^{\alpha-2}|f(s,x(s),x'(s),D^{\beta}x(s),\int_0^{s}h(\xi)x(\xi)d\xi,\phi(x(s)))$$

$$-f(s,y(s),y'(s),D^{\beta}y(s),\int_0^{s}h(\xi)y(\xi)d\xi,\phi(y(s)))|ds$$

$$+\frac{A_{\mu,\eta}+\mu(\eta+t-1)}{A_{\mu,\eta}\Gamma(\alpha)}\times$$

$$\int_0^{\eta}(\eta-s)^{\alpha-1}|f(s,x(s),D^{\beta}x(s),\int_0^{s}h(\xi)x(\xi)d\xi,\phi(x(s)))$$

$$-f(s,y(s),y'(s),D^\beta y(s),\int_0^s h(\xi)y(\xi)d\xi,\phi(y(s)))|ds$$

$$+\frac{A_{\mu,\eta}(1-\eta)+t}{A_{\mu,\eta}\Gamma(\alpha+1)}\times$$

$$\int_0^\mu(\mu-s)^\alpha|f(s,x(s),x'(s),D^\beta x(s),\int_0^s h(\xi)x(\xi)d\xi,\phi(x(s)))$$

$$-f(s,y(s),y'(s),D^\beta y(s),\int_0^s h(\xi)y(\xi)d\xi,\phi(y(s)))|ds$$

$$\leqslant\frac{1}{\Gamma(\alpha)}\times$$

$$\int_0^t(t-s)^{\alpha-1}\sum_{i=1}^{k_0}a_i(s)[\Lambda_i(x(s)-y(s),x'(s)-y'(s),D^\beta x(s)-D^\beta y(s),$$

$$\int_0^s h(\xi)x(\xi)d\xi-\int_0^s h(\xi)y(\xi)d\xi,\phi(x(s))-\phi(y(s)))]ds$$

$$+\frac{\mu(1-\eta+t)-A_{\mu,\eta}}{A_{\mu,\eta}\Gamma(\alpha-1)}\times$$

$$\int_0^1(1-s)^{\alpha-2}\sum_{i=1}^{k_0}a_i(s)[\Lambda_i(x(s)-y(s),x'(s)-y'(s),$$

$$D^\beta x(s)-D^\beta y(s),\int_0^s h(\xi)x(\xi)d\xi$$

$$-\int_0^s h(\xi)y(\xi)d\xi,\phi(x(s))-\phi(y(s)))]ds$$

$$+\frac{A_{\mu,\eta}+\mu(\eta+t-1)}{A_{\mu,\eta}\Gamma(\alpha)}\times$$

$$\int_0^\eta(\eta-s)^{\alpha-1}\sum_{i=1}^{k_0}a_i(s)[\Lambda_i(x(s)-y(s),x'(s)-y'(s),$$

$$D^\beta x(s)-D^\beta y(s),\int_0^s h(\xi)x(\xi)d\xi$$

$$-\int_0^s h(\xi)y(\xi)d\xi,\phi(x(s))-\phi(y(s)))]ds$$

$$+\frac{A_{\mu,\eta}(1-\eta)+t}{A_{\mu,\eta}\Gamma(\alpha+1)}\times$$

$$\int_0^\mu (\mu-s)^\alpha \sum_{i=1}^{k_0} a_i(s)[\Lambda_i(x(s)-y(s), x'(s)-y'(s),$$

$$D^\beta x(s)-D^\beta y(s), \int_0^s h(\xi)x(\xi)d\xi$$

$$-\int_0^s h(\xi)y(\xi)d\xi, \phi(x(s))-\phi(y(s)))]ds$$

$$\leqslant \frac{1}{\Gamma(\alpha)}\times$$

$$\sum_{i=1}^{k_0}\int_0^t (t-s)^{\alpha-1}a_i(s)[\Lambda_i(|x(s)-y(s)|, |x'(s)-y'(s)|, |D^\beta(x(s)-y(s))|,$$

$$|\int_0^s h(\xi)(x(\xi)-y(\xi))d\xi|, |\phi(x(s))-\phi(y(s))|)]ds$$

$$+\frac{\mu(1-\eta+t)-A_{\mu,\eta}}{A_{\mu,\eta}\Gamma(\alpha-1)}\times$$

$$\sum_{i=1}^{k_0}\int_0^1 (1-s)^{\alpha-2}a_i(s)[\Lambda_i(|x(s)-y(s)|, |x'(s)-y'(s)|,$$

$$|D^\beta(x(s)-y(s))|, |\int_0^s h(\xi)(x(\xi)-y(\xi))d\xi|, |\phi(x(s))-\phi(y(s))|)]ds$$

$$+\frac{A_{\mu,\eta}+\mu(\eta+t-1)}{A_{\mu,\eta}\Gamma(\alpha)}\times$$

$$\sum_{i=1}^{k_0}\int_0^\eta (\eta-s)^{\alpha-1}a_i(s)[\Lambda_i(|x(s)-y(s)|, |x'(s)-y'(s)|,$$

$$|D^\beta(x(s)-y(s))|, |\int_0^s h(\xi)(x(\xi)-y(\xi))d\xi|, |\phi(x(s))-\phi(y(s))|)]ds$$

$$+\frac{A_{\mu,\eta}(1-\eta)+t}{A_{\mu,\eta}\Gamma(\alpha+1)}\times$$

$$\sum_{i=1}^{k_0}\int_0^{\mu}(\mu-s)^{\alpha}a_i(s)[\Lambda_i(|x(s)-y(s)|,|x'(s)-y'(s)|,$$

$$|D^{\beta}(x(s)-y(s))|,|\int_0^s h(\xi)(x(\xi)-y(\xi))d\xi|,|\phi(x(s))-\phi(y(s))|)]ds.$$

Since $D^{\beta}x(t)=\frac{1}{\Gamma(1-\beta)}\int_0^t x'(s)/(t-s)^{\beta}ds$ for $0<\beta<1$, we have

$$|D^{\beta}x(t)|\leqslant\frac{1}{\Gamma(1-\beta)}\int_0^t|x'(s)|/(t-s)^{\beta}ds$$

$$\leqslant\frac{\|x'\|}{\Gamma(1-\beta)}\int_0^1 1/(t-s)^{\beta}ds=\frac{\|x'\|}{\Gamma(2-\beta)}$$

and so

$$|D^{\beta}x(t)-D^{\beta}y(t)|=|D^{\beta}(x(t)-y(t))|\leqslant\frac{\|x'-y'\|}{\Gamma(2-\beta)}.$$

Thus

$$|F_x(t)-F_y(t)|\leqslant$$

$$\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k_0}\int_0^t(t-s)^{\alpha-1}[\Lambda_i(\|x-y\|,\|x'-y'\|,\frac{\|x'-y'\|}{\Gamma(2-\beta)},$$

$$m\|x-y\|,b_1\|x-y\|+b_2\|x'-y'\|)]ds$$

$$+\frac{\mu(1-\eta+t)-A_{\mu,\eta}}{A_{\mu,\eta}\Gamma(\alpha-1)}\times$$

$$\sum_{i=1}^{k_0}\int_0^1(1-s)^{\alpha-2}a_i(s)[\Lambda_i(\|x-y\|,\|x'-y'\|,\frac{\|x'-y'\|}{\Gamma(2-\beta)},$$

$$m\|x-y\|,b_1\|x-y\|+b_2\|x'-y'\|)]ds$$

$$+\frac{A_{\mu,\eta}+\mu(\eta+t-1)}{A_{\mu,\eta}\Gamma(\alpha)}\times$$

$$\sum_{i=1}^{k_0}\int_0^{\eta}(\eta-s)^{\alpha-1}a_i(s)[\Lambda_i(\|x-y\|,\|x'-y'\|,\frac{\|x'-y'\|}{\Gamma(2-\beta)},$$

$$m\|x-y\|, b_1\|x-y\|+b_2\|x'-y'\|)]ds$$

$$+\frac{A_{\mu,\eta}(1-\eta)+t}{A_{\mu,\eta}\Gamma(\alpha+1)}\times$$

$$\sum_{i=1}^{k_0}\int_0^{\mu}(\mu-s)^{\alpha}a_i(s)[\Lambda_i(\|x-y\|,\|x'-y'\|,\frac{\|x'-y'\|}{\Gamma(2-\beta)},$$

$$m\|x-y\|, b_1\|x-y\|+b_2\|x'-y'\|)]ds$$

$$\leqslant\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k_0}\Lambda_i(\|x-y\|_*,\|x-y\|_*,\frac{\|x-y\|_*}{\Gamma(2-\beta)},$$

$$m\|x-y\|_*, b_1\|x-y\|_*+b_2\|x-y\|_*)\int_0^t(t-s)^{\alpha-1}a_i(s)ds$$

$$+\frac{\mu(1-\eta+t)-A_{\mu,\eta}}{A_{\mu,\eta}\Gamma(\alpha-1)}\sum_{i=1}^{k_0}\Lambda_i(\|x-y\|_*,\|x-y\|_*,\frac{\|x-y\|_*}{\Gamma(2-\beta)},$$

$$m\|x-y\|_*, b_1\|x-y\|_*+b_2\|x-y\|_*)\int_0^1(1-s)^{\alpha-2}a_i(s)ds$$

$$+\frac{A_{\mu,\eta}+\mu(\eta+t-1)}{A_{\mu,\eta}\Gamma(\alpha)}\sum_{i=1}^{k_0}\Lambda_i(\|x-y\|_*,\|x-y\|_*,\frac{\|x-y\|_*}{\Gamma(2-\beta)},$$

$$m\|x-y\|_*, b_1\|x-y\|_*+b_2\|x-y\|_*)\int_0^{\eta}(\eta-s)^{\alpha-1}a_i(s)ds$$

$$+\frac{A_{\mu,\eta}(1-\eta)+t}{A_{\mu,\eta}\Gamma(\alpha+1)}\sum_{i=1}^{k_0}\Lambda_i(\|x-y\|_*,\|x-y\|_*,\frac{\|x-y\|_*}{\Gamma(2-\beta)},$$

$$m\|x-y\|_*, b_1\|x-y\|_*+b_2\|x-y\|_*)\int_0^{\mu}(\mu-s)^{\alpha}a_i(s)ds$$

$$\leqslant\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k_0}\Lambda_i(\Delta\|x-y\|_*,\Delta\|x-y\|_*,\Delta\|x-y\|_*,$$

$$\Delta\|x-y\|_*,\Delta\|x-y\|_*)\int_0^1(1-s)^{\alpha-2}a_i(s)ds$$

$$+\frac{\mu(1-\eta+t)-A_{\mu,\eta}}{A_{\mu,\eta}\Gamma(\alpha-1)}\sum_{i=1}^{k_0}\Lambda_i(\Delta\|x-y\|_*,\Delta\|x-y\|_*,\Delta\|x-y\|_*,$$

$$\Delta\|x-y\|_*,\Delta\|x-y\|_*)\int_0^1(1-s)^{\alpha-2}a_i(s)ds$$

$$+\frac{A_{\mu,\eta}+\mu(\eta+t-1)}{A_{\mu,\eta}\Gamma(\alpha)}\sum_{i=1}^{k_0}\Lambda_i(\Delta\|x-y\|_*,\Delta\|x-y\|_*,\Delta\|x-y\|_*,$$

$$\Delta\|x-y\|_*,\Delta\|x-y\|_*)\int_0^1(1-s)^{\alpha-2}a_i(s)ds$$

$$+\frac{A_{\mu,\eta}(1-\eta)+t}{A_{\mu,\eta}\Gamma(\alpha+1)}\sum_{i=1}^{k_0}\Lambda_i(\Delta\|x-y\|_*,\Delta\|x-y\|_*,\Delta\|x-y\|_*,$$

$$\Delta\|x-y\|_*,\Delta\|x-y\|_*)\int_0^1(1-s)^{\alpha-2}a_i(s)ds$$

$$=\frac{1}{\Gamma(\alpha)}\times$$

$$\sum_{i=1}^{k_0}\|\hat{a}_i\|_1\Lambda_i(\Delta\|x-y\|_*,\Delta\|x-y\|_*,\Delta\|x-y\|_*,\Delta\|x-y\|_*,\Delta\|x-y\|_*)$$

$$+\frac{\mu(1-\eta+t)-A_{\mu,\eta}}{A_{\mu,\eta}\Gamma(\alpha-1)}\times$$

$$\sum_{i=1}^{k_0}\|\hat{a}_i\|_1\Lambda_i(\Delta\|x-y\|_*,\Delta\|x-y\|_*,\Delta\|x-y\|_*,$$

$$\Delta\|x-y\|_*,\Delta\|x-y\|_*)+\frac{A_{\mu,\eta}+\mu(\eta+t-1)}{A_{\mu,\eta}\Gamma(\alpha)}\sum_{i=1}^{k_0}\|\hat{a}_i\|_1$$

$$\Lambda_i(\Delta\|x-y\|_*,\Delta\|x-y\|_*,\Delta\|x-y\|_*,\Delta\|x-y\|_*,\Delta\|x-y\|_*)$$

$$+\frac{A_{\mu,\eta}(1-\eta)+t}{A_{\mu,\eta}\Gamma(\alpha+1)}$$

$$\times\sum_{i=1}^{k_0}\|\hat{a}_i\|_1\Lambda_i(\Delta\|x-y\|_*,\Delta\|x-y\|_*,\Delta\|x-y\|_*,\Delta\|x-y\|_*,\Delta\|x-y\|_*)$$

$$= \sum_{i=1}^{k_0} \|\hat{a}_i\|_1 \Lambda_i(\Delta\|x-y\|_*, \Delta\|x-y\|_*, \Delta\|x-y\|_*, \Delta\|x-y\|_*, \Delta\|x-y\|_*)$$

$$\times(\frac{1}{\Gamma(\alpha)} + \frac{\mu(1-\eta+t) - A_{\mu,\eta}}{A_{\mu,\eta}\Gamma(\alpha-1)} + \frac{A_{\mu,\eta} + \mu(\eta+t-1)}{A_{\mu,\eta}\Gamma(\alpha)} + \frac{A_{\mu,\eta}(1-\eta)+t}{A_{\mu,\eta}\Gamma(\alpha+1)}),$$

where $\Delta = \max\{1, \frac{1}{\Gamma(2-\beta)}, m, b_1 + b_2\}$. This implies that

$$\|F_x - F_y\| \leqslant$$

$$\sum_{i=1}^{k_0} \|\hat{a}_i\|_1 \Lambda_i(\Delta\|x-y\|_*, \Delta\|x-y\|_*, \Delta\|x-y\|_*, \Delta\|x-y\|_*, \Delta\|x-y\|_*)$$

$$\times(\frac{1}{\Gamma(\alpha)} + \frac{\mu(2-\eta) - A_{\mu,\eta}}{A_{\mu,\eta}\Gamma(\alpha-1)} + \frac{A_{\mu,\eta} + \mu\eta}{A_{\mu,\eta}\Gamma(\alpha)} + \frac{A_{\mu,\eta}(1-\eta)+1}{A_{\mu,\eta}\Gamma(\alpha+1)}).$$

Let $x, y \in X$. Then, we have

$$|F'_x(t) - F'_y(t)| \leqslant$$

$$|-\frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} f(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)d\xi, \phi(x(s)))ds$$

$$+\frac{\mu}{A_{\mu,\eta}\Gamma(\alpha-1)}\times$$

$$\int_0^1 (1-s)^{\alpha-2} f(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)d\xi, \phi(x(s)))ds$$

$$+\frac{\mu}{A_{\mu,\eta}\Gamma(\alpha)}\times$$

$$\int_0^\eta (\eta-s)^{\alpha-1} f(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)d\xi, \phi(x(s)))ds$$

$$+\frac{1}{A_{\mu,\eta}\Gamma(\alpha+1)}\times$$

$$\int_0^\mu (\mu-s)^{\alpha} f(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)d\xi, \phi(x(s)))ds$$

$$+\frac{1}{\Gamma(\alpha-1)}\times$$

$$\int_0^t (t-s)^{\alpha-2} f(s, y(s), y'(s), D^\beta y(s), \int_0^s h(\xi)y(\xi)d\xi, \phi(y(s)))ds$$

$$-\frac{\mu}{A_{\mu,\eta}\Gamma(\alpha-1)}\times$$

$$\int_0^1 (1-s)^{\alpha-2} f(s, y(s), y'(s), D^\beta y(s), \int_0^s h(\xi)y(\xi)d\xi, \phi(y(s)))ds$$

$$-\frac{\mu}{A_{\mu,\eta}\Gamma(\alpha)}\times$$

$$\int_0^\eta (\eta-s)^{\alpha-1} f(s, y(s), y'(s), D^\beta y(s), \int_0^s h(\xi)y(\xi)d\xi, \phi(y(s)))ds$$

$$-\frac{1}{A_{\mu,\eta}\Gamma(\alpha+1)}\times$$

$$\int_0^\mu (\mu-s)^{\alpha} f(s, y(s), y'(s), D^\beta y(s), \int_0^s h(\xi)y(\xi)d\xi, \phi(y(s)))ds|$$

$$\leqslant \frac{1}{\Gamma(\alpha-1)}\times$$

$$\int_0^t (t-s)^{\alpha-2} |f(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)d\xi, \phi(x(s)))$$

$$-f(s, y(s), y'(s), D^\beta y(s), \int_0^s h(\xi)y(\xi)d\xi, \phi(y(s)))|ds$$

$$+\frac{\mu}{A_{\mu,\eta}\Gamma(\alpha-1)}\times$$

$$\int_0^1 (1-s)^{\alpha-2} |f(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)d\xi, \phi(x(s)))$$

$$-f(s, y(s), y'(s), D^\beta y(s), \int_0^s h(\xi)y(\xi)d\xi, \phi(y(s)))|ds$$

$$+\frac{\mu}{A_{\mu,\eta}\Gamma(\alpha)}\times$$

$$\int_0^\eta (\eta-s)^{\alpha-1} |f(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)d\xi, \phi(x(s)))$$

$$-f(s, y(s), y'(s), D^\beta y(s), \int_0^s h(\xi)y(\xi)d\xi, \phi(y(s)))|ds$$

$$+\frac{1}{A_{\mu,\eta}\Gamma(\alpha+1)}\times$$

$$\int_0^{\mu}(\mu-s)^{\alpha}|f(s,x(s),x'(s),D^{\beta}x(s),\int_0^s h(\xi)x(\xi)d\xi,\phi(x(s)))$$

$$-f(s,y(s),y'(s),D^{\beta}y(s),\int_0^s h(\xi)y(\xi)d\xi,\phi(y(s)))|ds$$

$$\leqslant\frac{1}{\Gamma(\alpha-1)}\times$$

$$\int_0^t(t-s)^{\alpha-2}\sum_{i=1}^{k_0}a_i(s)[\Lambda_i(x(s)-y(s),x'(s)-y'(s),D^{\beta}x(s)-D^{\beta}y(s),$$

$$\int_0^s h(\xi)x(\xi)d\xi-\int_0^s h(\xi)y(\xi)d\xi,\phi(x(s))-\phi(y(s)))]ds$$

$$+\frac{\mu}{A_{\mu,\eta}\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-2}\sum_{i=1}^{k_0}a_i(s)[\Lambda_i(x(s)-y(s),x'(s)-y'(s),$$

$$D^{\beta}x(s)-D^{\beta}y(s),\int_0^s h(\xi)x(\xi)d\xi-\int_0^s h(\xi)y(\xi)d\xi,\phi(x(s))-\phi(y(s)))]ds$$

$$+\frac{\mu}{A_{\mu,\eta}\Gamma(\alpha)}\int_0^{\eta}(\eta-s)^{\alpha-1}\sum_{i=1}^{k_0}a_i(s)[\Lambda_i(x(s)-y(s),x'(s)-y'(s),$$

$$D^{\beta}x(s)-D^{\beta}y(s),\int_0^s h(\xi)x(\xi)d\xi-\int_0^s h(\xi)y(\xi)d\xi,\phi(x(s))-\phi(y(s)))]ds$$

$$+\frac{1}{A_{\mu,\eta}\Gamma(\alpha+1)}\int_0^{\mu}(\mu-s)^{\alpha}\sum_{i=1}^{k_0}a_i(s)[\Lambda_i(x(s)-y(s),x'(s)-y'(s),$$

$$D^{\beta}x(s)-D^{\beta}y(s),\int_0^s h(\xi)x(\xi)d\xi-\int_0^s h(\xi)y(\xi)d\xi,\phi(x(s))-\phi(y(s)))]ds$$

$$\leqslant\frac{1}{\Gamma(\alpha-1)}\times$$

$$\sum_{i=1}^{k_0}\int_0^t(t-s)^{\alpha-2}a_i(s)[\Lambda_i(|x(s)-y(s)|,|x'(s)-y'(s)|,|D^{\beta}(x(s)-y(s))|,$$

$$|\int_0^s h(\xi)(x(\xi)-y(\xi))d\xi|, |\phi(x(s))-\phi(y(s))|)]ds$$

$$+\frac{\mu}{A_{\mu,\eta}\Gamma(\alpha-1)}\sum_{i=1}^{k_0}\int_0^1 (1-s)^{\alpha-2}a_i(s)[\Lambda_i(|x(s)-y(s)|, |x'(s)-y'(s)|,$$

$$|D^\beta(x(s)-y(s))|, |\int_0^s h(\xi)(x(\xi)-y(\xi))d\xi|, |\phi(x(s))-\phi(y(s))|)]ds$$

$$+\frac{\mu}{A_{\mu,\eta}\Gamma(\alpha)}\times$$

$$\sum_{i=1}^{k_0}\int_0^\eta (\eta-s)^{\alpha-1}a_i(s)[\Lambda_i(|x(s)-y(s)|, |x'(s)-y'(s)|, |D^\beta(x(s)-y(s))|,$$

$$|\int_0^s h(\xi)(x(\xi)-y(\xi))d\xi|, |\phi(x(s))-\phi(y(s))|)]ds$$

$$+\frac{1}{A_{\mu,\eta}\Gamma(\alpha+1)}\sum_{i=1}^{k_0}\int_0^\mu (\mu-s)^{\alpha}a_i(s)[\Lambda_i(|x(s)-y(s)|, |x'(s)-y'(s)|,$$

$$|D^\beta(x(s)-y(s))|, |\int_0^s h(\xi)(x(\xi)-y(\xi))d\xi|, |\phi(x(s))-\phi(y(s))|)]ds$$

$$\leqslant \frac{1}{\Gamma(\alpha-1)}\sum_{i=1}^{k_0}\int_0^t (t-s)^{\alpha-2}[\Lambda_i(\|x-y\|, \|x'-y'\|, \frac{\|x'-y'\|}{\Gamma(2-\beta)},$$

$$m\|x-y\|, b_1\|x-y\|+b_2\|x'-y'\|)]ds$$

$$+\frac{\mu}{A_{\mu,\eta}\Gamma(\alpha-1)}\sum_{i=1}^{k_0}\int_0^1 (1-s)^{\alpha-2}a_i(s)[\Lambda_i(\|x-y\|, \|x'-y'\|, \frac{\|x'-y'\|}{\Gamma(2-\beta)},$$

$$m\|x-y\|, b_1\|x-y\|+b_2\|x'-y'\|)]ds$$

$$+\frac{\mu}{A_{\mu,\eta}\Gamma(\alpha)}\sum_{i=1}^{k_0}\int_0^\eta (\eta-s)^{\alpha-1}a_i(s)[\Lambda_i(\|x-y\|, \|x'-y'\|, \frac{\|x'-y'\|}{\Gamma(2-\beta)},$$

$$m\|x-y\|, b_1\|x-y\|+b_2\|x'-y'\|)]ds$$

$$+\frac{1}{A_{\mu,\eta}\Gamma(\alpha+1)}\sum_{i=1}^{k_0}\int_0^\mu (\mu-s)^{\alpha}a_i(s)[\Lambda_i(\|x-y\|, \|x'-y'\|, \frac{\|x'-y'\|}{\Gamma(2-\beta)},$$

$$m\|x-y\|, b_1\|x-y\| + b_2\|x'-y'\|)]ds$$

$$\leqslant \frac{1}{\Gamma(\alpha-1)} \sum_{i=1}^{k_0} \Lambda_i(\|x-y\|_*, \|x-y\|_*, \frac{\|x-y\|_*}{\Gamma(2-\beta)},$$

$$m\|x-y\|_*, b_1\|x-y\|_* + b_2\|x-y\|_*) \int_0^t (t-s)^{\alpha-2} a_i(s) ds$$

$$+\frac{\mu}{A_{\mu,\eta}\Gamma(\alpha-1)} \sum_{i=1}^{k_0} \Lambda_i(\|x-y\|_*, \|x-y\|_*, \frac{\|x-y\|_*}{\Gamma(2-\beta)},$$

$$m\|x-y\|_*, b_1\|x-y\|_* + b_2\|x-y\|_*) \int_0^1 (1-s)^{\alpha-2} a_i(s) ds$$

$$+\frac{\mu}{A_{\mu,\eta}\Gamma(\alpha)} \sum_{i=1}^{k_0} \Lambda_i(\|x-y\|_*, \|x-y\|_*, \frac{\|x-y\|_*}{\Gamma(2-\beta)},$$

$$m\|x-y\|_*, b_1\|x-y\|_* + b_2\|x-y\|_*) \int_0^\eta (\eta-s)^{\alpha-1} a_i(s) ds$$

$$+\frac{1}{A_{\mu,\eta}\Gamma(\alpha+1)} \sum_{i=1}^{k_0} \Lambda_i(\|x-y\|_*, \|x-y\|_*, \frac{\|x-y\|_*}{\Gamma(2-\beta)},$$

$$m\|x-y\|_*, b_1\|x-y\|_* + b_2\|x-y\|_*) \int_0^\mu (\mu-s)^{\alpha} a_i(s) ds$$

$$\leqslant \frac{1}{\Gamma(\alpha-1)} \sum_{i=1}^{k_0} \Lambda_i(\Delta\|x-y\|_*, \Delta\|x-y\|_*, \Delta\|x-y\|_*,$$

$$\Delta\|x-y\|_*, \Delta\|x-y\|_*) \int_0^1 (1-s)^{\alpha-2} a_i(s) ds$$

$$+\frac{\mu}{A_{\mu,\eta}\Gamma(\alpha-1)} \sum_{i=1}^{k_0} \Lambda_i(\Delta\|x-y\|_*, \Delta\|x-y\|_*, \Delta\|x-y\|_*,$$

$$\Delta\|x-y\|_*, \Delta\|x-y\|_*) \int_0^1 (1-s)^{\alpha-2} a_i(s) ds$$

$$+\frac{\mu}{A_{\mu,\eta}\Gamma(\alpha)} \sum_{i=1}^{k_0} \Lambda_i(\Delta\|x-y\|_*, \Delta\|x-y\|_*, \Delta\|x-y\|_*,$$

$$\Delta\|x-y\|_*, \Delta\|x-y\|_*) \int_0^1 (1-s)^{\alpha-2} a_i(s) ds$$

$$+\frac{1}{A_{\mu,\eta}\Gamma(\alpha+1)} \sum_{i=1}^{k_0} \Lambda_i(\Delta\|x-y\|_*, \Delta\|x-y\|_*, \Delta\|x-y\|_*,$$

$$\Delta\|x-y\|_*, \Delta\|x-y\|_*) \int_0^1 (1-s)^{\alpha-2} a_i(s) ds$$

$$= \frac{1}{\Gamma(\alpha-1)} \times$$

$$\sum_{i=1}^{k_0} \|\hat{a}_i\|_1 \Lambda_i(\Delta\|x-y\|_*, \Delta\|x-y\|_*, \Delta\|x-y\|_*, \Delta\|x-y\|_*, \Delta\|x-y\|_*)$$

$$+\frac{\mu}{A_{\mu,\eta}\Gamma(\alpha-1)} \sum_{i=1}^{k_0} \|\hat{a}_i\|_1 \Lambda_i(\Delta\|x-y\|_*, \Delta\|x-y\|_*, \Delta\|x-y\|_*,$$

$$\Delta\|x-y\|_*, \Delta\|x-y\|_*) + \frac{\mu}{A_{\mu,\eta}\Gamma(\alpha)} \sum_{i=1}^{k_0} \|\hat{a}_i\|_1$$

$$\Lambda_i(\Delta\|x-y\|_*, \Delta\|x-y\|_*, \Delta\|x-y\|_*, \Delta\|x-y\|_*, \Delta\|x-y\|_*) + \frac{1}{A_{\mu,\eta}\Gamma(\alpha+1)}$$

$$\times \sum_{i=1}^{k_0} \|\hat{a}_i\|_1 \Lambda_i(\Delta\|x-y\|_*, \Delta\|x-y\|_*, \Delta\|x-y\|_*, \Delta\|x-y\|_*, \Delta\|x-y\|_*)$$

$$= \sum_{i=1}^{k_0} \|\hat{a}_i\|_1 \Lambda_i(\Delta\|x-y\|_*, \Delta\|x-y\|_*, \Delta\|x-y\|_*, \Delta\|x-y\|_*, \Delta\|x-y\|_*)$$

$$\times (\frac{1}{\Gamma(\alpha-1)} + \frac{\mu}{A_{\mu,\eta}\Gamma(\alpha-1)} + \frac{\mu}{A_{\mu,\eta}\Gamma(\alpha)} + \frac{1}{A_{\mu,\eta}\Gamma(\alpha+1)})$$

Hence,

$$\|F'_x - F'_y\| \leqslant$$

$$\sum_{i=1}^{k_0} \|\hat{a}_i\|_1 \Lambda_i(\Delta\|x-y\|_*, \Delta\|x-y\|_*, \Delta\|x-y\|_*, \Delta\|x-y\|_*, \Delta\|x-y\|_*)$$

$$\times(\frac{1}{\Gamma(\alpha-1)}+\frac{\mu}{A_{\mu,\eta}\Gamma(\alpha-1)}+\frac{\mu}{A_{\mu,\eta}\Gamma(\alpha)}+\frac{1}{A_{\mu,\eta}\Gamma(\alpha+1)})$$

and so

$$\|F_x-F_y\|_* \leqslant$$

$$\sum_{i=1}^{k_0}\|\hat{a}_i\|_1\Lambda_i(\Delta\|x-y\|_*,\Delta\|x-y\|_*,\Delta\|x-y\|_*,\Delta\|x-y\|_*,\Delta\|x-y\|_*)$$

$$\times\max\{\frac{1}{\Gamma(\alpha)}+\frac{\mu(2-\eta)-A_{\mu,\eta}}{A_{\mu,\eta}\Gamma(\alpha-1)}+\frac{A_{\mu,\eta}+\mu\eta}{A_{\mu,\eta}\Gamma(\alpha)}+\frac{A_{\mu,\eta}(1-\eta)+1}{A_{\mu,\eta}\Gamma(\alpha+1)},$$

$$\frac{1}{\Gamma(\alpha-1)}+\frac{\mu}{A_{\mu,\eta}\Gamma(\alpha-1)}+\frac{\mu}{A_{\mu,\eta}\Gamma(\alpha)}+\frac{1}{A_{\mu,\eta}\Gamma(\alpha+1)}\}.$$

If

$$L_{\alpha,\mu,\eta}=\max\{\frac{1}{\Gamma(\alpha)}+\frac{\mu(2-\eta)-A_{\mu,\eta}}{A_{\mu,\eta}\Gamma(\alpha-1)}+\frac{A_{\mu,\eta}+\mu\eta}{A_{\mu,\eta}\Gamma(\alpha)}+\frac{A_{\mu,\eta}(1-\eta)+1}{A_{\mu,\eta}\Gamma(\alpha+1)},$$
$$\frac{1}{\Gamma(\alpha-1)}+\frac{\mu}{A_{\mu,\eta}\Gamma(\alpha-1)}+\frac{\mu}{A_{\mu,\eta}\Gamma(\alpha)}+\frac{1}{A_{\mu,\eta}\Gamma(\alpha+1)}\},$$

then

$$\|F_x-F_y\|_* \leqslant L_{\alpha,\mu,\eta}\sum_{i=1}^{k_0}\|\hat{a}_i\|_1\Lambda_i(\Delta\|x-y\|_*,\Delta\|x-y\|_*,\Delta\|x-y\|_*,\Delta\|x-y\|_*,\Delta\|x-y\|_*). \tag{2}$$

Let $0<\epsilon\leqslant 1$ be given. Since $\lim_{x\to 0^+}\frac{\Lambda_i(z,z,z,z,z)}{z^{\gamma_i}}=q_i$ for $1\leqslant i\leqslant k_0$, there exists $\delta_i:=\delta_i(\epsilon)$ such that $z\in(0,\delta_i]$ implies $|\frac{\Lambda_i(z,z,z,z,z)}{z^{\gamma_i}}-q_i|<\epsilon$ and so $\frac{\Lambda_i(z,z,z,z,z)}{z^{\gamma_i}}<\epsilon+q_i$. This consequents $0\leqslant\Lambda_i(z,z,z,z,z)<(\epsilon+q_i)z^{\gamma_i}$. Put $\delta:=\min\{\delta_1,...,\delta_{k_0},\epsilon\}$. In this case, $z\in(0,\delta]$ implies

$$0\leqslant\Lambda_i(z,z,z,z,z)<(\epsilon+q_i)z^{\gamma_i} \tag{3}$$

for all $1\leqslant i\leqslant k_0$. By using (3), we get

$$\Lambda_i(\Delta\|x-y\|_*,...,\Delta\|x-y\|_*)\leqslant(\epsilon+q_i)(\Delta\|x-y\|_*)^{\gamma_i}\leqslant(\epsilon+q_i)\Delta^{\gamma_i}\epsilon^{\gamma_i}. \tag{4}$$

Now using (2) and (4), we obtain

$$\|F_x - F_y\|_* \leqslant L_{\alpha,\mu,\eta} \sum_{i=1}^{k_0} \|\hat{a_i}\|_1 (\epsilon + q_i) \Delta^{\gamma_i} \epsilon^{\gamma_i}.$$

Put $\gamma := \min\{\gamma_1, ..., \gamma_{k_0}\}$. Since $\epsilon \geqslant 1$, $\epsilon^{\gamma} \geqslant \epsilon^{\gamma_i}$ for all $1 \leqslant i \leqslant k_0$ and so

$$\|F_x - F_y\|_* \leqslant \epsilon^{\gamma} L_{\alpha,\mu,\eta} \sum_{i=1}^{k_0} \|\hat{a_i}\|_1 (\epsilon + q_i) \Delta^{\gamma_i}.$$

This shows that F is continuous. Since $L_{\alpha,\mu,\eta} \sum_{i=1}^{k_0} \|\hat{a_i}\|_1 q_i \Delta^{\gamma_i} < 1$, there is $\epsilon_1 > 0$ such that $L_{\alpha,\mu,\eta} \sum_{i=1}^{k_0} \|\hat{a_i}\|_1 (q_i + \epsilon_1) \Delta^{\gamma_i} < 1$. Put $\theta := \lim_{z \to 0^+} \frac{K(z,z,z,z,z)}{z} \in [0, B)$. Then, we have $\theta := \lim_{z \to 0^+} \frac{K(\Delta z, ..., \Delta z)}{\Delta z}$ and so for each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that $z \in (0, \delta(\epsilon)]$ implies $0 \leqslant \frac{K(\Delta z, ..., \Delta z)}{\Delta z} - \theta < \epsilon$. Hence, $0 \leqslant K(\Delta z, ..., \Delta z) < (\theta + \epsilon)\Delta z$ and

$$0 \leqslant K(\Delta\delta(\epsilon), ..., \Delta\delta(\epsilon)) < (\theta + \epsilon)\Delta\delta(\epsilon).$$

Since $\theta \in [0, B)$, $\frac{\theta}{B} < 1$. Choose $\epsilon_0 > 0$ such that $\frac{\theta + \epsilon_0}{B} < 1$. Let $r_0 := \min\{\delta(\epsilon_0), \delta(\epsilon_1)\}$. Then, $r \leqslant r_0$ implies $0 \leqslant K(\Delta r, ..., \Delta r) < (\theta + \epsilon_0)\Delta r$. Since $\lim_{z \to 0^+} \frac{\Lambda_i(z,z,z,z,z)}{z^{\gamma_i}} = q_i$, there is $r_1 > 0$ such that $z \in (0, r_1]$ implies

$$\Lambda_i(\Delta z, ..., \Delta z) < (q_i + \epsilon_0)(\Delta z)^{\gamma_i} \tag{5}$$

for $i = 1, ..., k_0$. Put $r = \min\{r_0, \frac{r_1}{2}, \frac{1}{2}\}$ and $C = \{x \in X : \|x\|_* \leqslant r\}$. Define $\alpha : X^2 \to \mathbb{R}$ by $\alpha(x, y) = 1$ whenever $x, y \in C$ and $\alpha(x, y) = 0$ otherwise. Let $x, y \in X$ be given. If $\alpha(x, y) \geqslant 1$, then $x, y \in C$ and so for every $t \in [0, 1]$ we have

$$|F_x(t)| \leqslant \int_0^t |G(t,s)| f(s, x(s), x'(s), D^{\beta}x(s), \int_0^s h(\xi)x(\xi)d\xi, \phi(x(s)))ds$$

$$\leqslant \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, x(s), x'(s), D^{\beta}x(s), \int_0^s h(\xi)x(\xi)d\xi, \phi(x(s)))| ds$$

$$+ \frac{\mu(1 - \eta + t) - A_{\mu,\eta}}{A_{\mu,\eta}\Gamma(\alpha - 1)} \times$$

$$\int_0^1 (1-s)^{\alpha-2}|f(s,x(s),x'(s),D^\beta x(s),\int_0^s h(\xi)x(\xi)d\xi,\phi(x(s)))|ds$$
$$+\frac{A_{\mu,\eta}+\mu(\eta+t-1)}{A_{\mu,\eta}\Gamma(\alpha)}\times$$
$$\int_0^\eta (\eta-s)^{\alpha-1}|f(s,x(s),x'(s),D^\beta x(s),\int_0^s h(\xi)x(\xi)d\xi,\phi(x(s)))|ds$$
$$+\frac{A_{\mu,\eta}(1-\eta)+t}{A_{\mu,\eta}\Gamma(\alpha+1)}\times$$
$$\int_0^\mu (\mu-s)^{\alpha}|f(s,x(s),x'(s),D^\beta x(s),\int_0^s h(\xi)x(\xi)d\xi,\phi(x(s)))|ds$$
$$\leqslant \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g(s)K(x(s),x'(s),D^\beta x(s),\int_0^s h(\xi)x(\xi)d\xi,\phi(x(s)))ds$$
$$+\frac{\mu(1-\eta+t)-A_{\mu,\eta}}{A_{\mu,\eta}\Gamma(\alpha-1)}\times$$
$$\int_0^1 (1-s)^{\alpha-2}g(s)K(x(s),x'(s),D^\beta x(s),\int_0^s h(\xi)x(\xi)d\xi,\phi(x(s)))ds$$
$$+\frac{A_{\mu,\eta}+\mu(\eta+t-1)}{A_{\mu,\eta}\Gamma(\alpha)}\times$$
$$\int_0^\eta (\eta-s)^{\alpha-1}g(s)K(x(s),x'(s),D^\beta x(s),\int_0^s h(\xi)x(\xi)d\xi,\phi(x(s)))ds$$
$$+\frac{A_{\mu,\eta}(1-\eta)+t}{A_{\mu,\eta}\Gamma(\alpha+1)}\times$$
$$\int_0^\mu (\mu-s)^{\alpha}g(s)K(x(s),x'(s),D^\beta x(s),\int_0^s h(\xi)x(\xi)d\xi,\phi(x(s)))ds$$
$$\leqslant \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g(s)K(|x(s)|,|x'(s)|,|D^\beta x(s)|,\int_0^s |h(\xi)||x(\xi)|d\xi,|\phi(x(s))|)ds$$
$$+\frac{\mu(1-\eta+t)-A_{\mu,\eta}}{A_{\mu,\eta}\Gamma(\alpha-1)}\times$$
$$\int_0^1 (1-s)^{\alpha-2}g(s)K(|x(s)|,|x'(s)|,|D^\beta x(s)|,\int_0^s |h(\xi)||x(\xi)|d\xi,|\phi(x(s))|)ds$$

$$+\frac{A_{\mu,\eta}+\mu(\eta+t-1)}{A_{\mu,\eta}\Gamma(\alpha)}\times$$

$$\int_0^\eta(\eta-s)^{\alpha-1}g(s)K(|x(s)|,|x'(s)|,|D^\beta x(s)|,\int_0^s|h(\xi)||x(\xi)|d\xi,|\phi(x(s))|)ds$$

$$+\frac{A_{\mu,\eta}(1-\eta)+t}{A_{\mu,\eta}\Gamma(\alpha+1)}\times$$

$$\int_0^\mu(\mu-s)^\alpha g(s)K(|x(s)|,|x'(s)|,|D^\beta x(s)|,\int_0^s|h(\xi)||x(\xi)|d\xi,|\phi(x(s))|)ds$$

$$\leqslant\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}g(s)K(\|x\|,\|x'\|,\frac{\|x'\|}{\Gamma(2-\beta)},m\|x\|,b_1\|x\|+b_2\|x'\|)ds$$

$$+\frac{\mu(1-\eta+t)-A_{\mu,\eta}}{A_{\mu,\eta}\Gamma(\alpha-1)}\times$$

$$\int_0^1(1-s)^{\alpha-2}g(s)K(\|x\|,\|x'\|,\frac{\|x'\|}{\Gamma(2-\beta)},m\|x\|,b_1\|x\|+b_2\|x'\|)ds$$

$$+\frac{A_{\mu,\eta}+\mu(\eta+t-1)}{A_{\mu,\eta}\Gamma(\alpha)}\times$$

$$\int_0^\eta(\eta-s)^{\alpha-1}g(s)K(\|x\|,\|x'\|,\frac{\|x'\|}{\Gamma(2-\beta)},m\|x\|,b_1\|x\|+b_2\|x'\|)ds$$

$$+\frac{A_{\mu,\eta}(1-\eta)+t}{A_{\mu,\eta}\Gamma(\alpha+1)}\times$$

$$\int_0^\mu(\mu-s)^\alpha g(s)K(\|x\|,\|x'\|,\frac{\|x'\|}{\Gamma(2-\beta)},m\|x\|,b_1\|x\|+b_2\|x'\|)ds$$

$$\leqslant\frac{1}{\Gamma(\alpha)}K(\|x\|,\|x'\|,\frac{\|x'\|}{\Gamma(2-\beta)},m\|x\|,b_1\|x\|+b_2\|x'\|)\int_0^1(1-s)^{\alpha-2}g(s)ds$$

$$+\frac{\mu(1-\eta+t)-A_{\mu,\eta}}{A_{\mu,\eta}\Gamma(\alpha-1)}K(\|x\|,\|x'\|,\frac{\|x'\|}{\Gamma(2-\beta)},m\|x\|,b_1\|x\|+b_2\|x'\|)$$

$$\times\int_0^1(1-s)^{\alpha-2}g(s)ds$$

$$+\frac{A_{\mu,\eta}+\mu(\eta+t-1)}{A_{\mu,\eta}\Gamma(\alpha)}K(\|x\|,\|x'\|,\frac{\|x'\|}{\Gamma(2-\beta)},m\|x\|,b_1\|x\|+b_2\|x'\|)$$

$$\times \int_0^1 (1-s)^{\alpha-2} g(s)ds$$

$$+\frac{A_{\mu,\eta}(1-\eta)+t}{A_{\mu,\eta}\Gamma(\alpha+1)}\times$$

$$K(\|x\|, \|x'\|, \frac{\|x'\|}{\Gamma(2-\beta)}, m\|x\|, b_1\|x\|+b_2\|x'\|) \times \int_0^1 (1-s)^{\alpha-2} g(s)ds$$

$$\leqslant K(\Delta\|x\|_*, \Delta\|x\|_*, \Delta\|x\|_*, \Delta\|x\|_*, \Delta\|x\|_*)\|\hat{g}\|_1[\frac{1}{\Gamma(\alpha)}+$$

$$\frac{\mu(1-\eta+t)-A_{\mu,\eta}}{A_{\mu,\eta}\Gamma(\alpha-1)}+\frac{A_{\mu,\eta}+\mu(\eta+t-1)}{A_{\mu,\eta}\Gamma(\alpha)}+\frac{A_{\mu,\eta}(1-\eta)+t}{A_{\mu,\eta}\Gamma(\alpha+1)}]$$

$$\leqslant K(\Delta r, \Delta r, \Delta r, \Delta r, \Delta r)\|\hat{g}\|_1[\frac{1}{\Gamma(\alpha)}+\frac{\mu(1-\eta+t)-A_{\mu,\eta}}{A_{\mu,\eta}\Gamma(\alpha-1)}$$

$$+\frac{A_{\mu,\eta}+\mu(\eta+t-1)}{A_{\mu,\eta}\Gamma(\alpha)}+\frac{A_{\mu,\eta}(1-\eta)+t}{A_{\mu,\eta}\Gamma(\alpha+1)}]$$

$$\leqslant \Delta r(\theta+\epsilon)\|\hat{g}\|_1[\frac{1}{\Gamma(\alpha)}+\frac{\mu(1-\eta+t)-A_{\mu,\eta}}{A_{\mu,\eta}\Gamma(\alpha-1)}+\frac{A_{\mu,\eta}+\mu(\eta+t-1)}{A_{\mu,\eta}\Gamma(\alpha)}$$

$$+\frac{A_{\mu,\eta}(1-\eta)+t}{A_{\mu,\eta}\Gamma(\alpha+1)}]$$

$$= r(\Delta(\theta+\epsilon)\|\hat{g}\|_1[\frac{1}{\Gamma(\alpha)}+\frac{\mu(1-\eta+t)-A_{\mu,\eta}}{A_{\mu,\eta}\Gamma(\alpha-1)}+\frac{A_{\mu,\eta}+\mu(\eta+t-1)}{A_{\mu,\eta}\Gamma(\alpha)}$$

$$+\frac{A_{\mu,\eta}(1-\eta)+t}{A_{\mu,\eta}\Gamma(\alpha+1)}]).$$

and so

$$\|F_x\| \leqslant$$

$$r(\Delta(\theta+\epsilon)\|\hat{g}\|_1[\frac{1}{\Gamma(\alpha)}+\frac{\mu(2-\eta)-A_{\mu,\eta}}{A_{\mu,\eta}\Gamma(\alpha-1)}+\frac{A_{\mu,\eta}+\mu\eta}{A_{\mu,\eta}\Gamma(\alpha)}+\frac{A_{\mu,\eta}(1-\eta)+1}{A_{\mu,\eta}\Gamma(\alpha+1)}]) \leqslant r.$$

Also, we have

$$|F'_x(t)| \leqslant$$

$$|-\frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2} f(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)d\xi, \phi(x(s)))ds$$

$$+\frac{\mu}{A_{\mu,\eta}\Gamma(\alpha-1)}\times$$

$$\int_0^1 (1-s)^{\alpha-2} f(s,x(s),x'(s),D^\beta x(s),\int_0^s h(\xi)x(\xi)d\xi,\phi(x(s)))ds$$

$$+\frac{\mu}{A_{\mu,\eta}\Gamma(\alpha)}\times$$

$$\int_0^\eta (\eta-s)^{\alpha-1} f(s,x(s),x'(s),D^\beta x(s),\int_0^s h(\xi)x(\xi)d\xi,\phi(x(s)))ds$$

$$+\frac{1}{A_{\mu,\eta}\Gamma(\alpha+1)}\times$$

$$\int_0^\mu (\mu-s)^{\alpha} f(s,x(s),x'(s),D^\beta x(s),\int_0^s h(\xi)x(\xi)d\xi,\phi(x(s)))ds|$$

$$\leqslant \frac{1}{\Gamma(\alpha-1)}\times$$

$$\int_0^t (t-s)^{\alpha-2} |f(s,x(s),x'(s),D^\beta x(s),\int_0^s h(\xi)x(\xi)d\xi,\phi(x(s)))|ds$$

$$+\frac{\mu}{A_{\mu,\eta}\Gamma(\alpha-1)}\times$$

$$\int_0^1 (1-s)^{\alpha-2} |f(s,x(s),x'(s),D^\beta x(s),\int_0^s h(\xi)x(\xi)d\xi,\phi(x(s)))|ds$$

$$+\frac{\mu}{A_{\mu,\eta}\Gamma(\alpha)}\times$$

$$\int_0^\eta (\eta-s)^{\alpha-1} |f(s,x(s),x'(s),D^\beta x(s),\int_0^s h(\xi)x(\xi)d\xi,\phi(x(s)))|ds$$

$$+\frac{1}{A_{\mu,\eta}\Gamma(\alpha+1)}\times$$

$$\int_0^\mu (\mu-s)^{\alpha} |f(s,x(s),x'(s),D^\beta x(s),\int_0^s h(\xi)x(\xi)d\xi,\phi(x(s)))|ds$$

$$\leqslant \frac{1}{\Gamma(\alpha-1)}\times$$

$$\int_0^t (t-s)^{\alpha-2} g(s) K(x(s),x'(s),D^\beta x(s),\int_0^s h(\xi)x(\xi)d\xi,\phi(x(s)))ds$$

$$+\frac{\mu}{A_{\mu,\eta}\Gamma(\alpha-1)}\times$$

$$\int_0^1 (1-s)^{\alpha-2}g(s)K(x(s),x'(s),D^\beta x(s),\int_0^s h(\xi)x(\xi)d\xi,\phi(x(s)))ds$$

$$+\frac{\mu}{A_{\mu,\eta}\Gamma(\alpha)}\times$$

$$\int_0^\eta (\eta-s)^{\alpha-1}g(s)K(x(s),x'(s),D^\beta x(s),\int_0^s h(\xi)x(\xi)d\xi,\phi(x(s)))ds$$

$$+\frac{1}{A_{\mu,\eta}\Gamma(\alpha+1)}\times$$

$$\int_0^\mu (\mu-s)^{\alpha}g(s)K(x(s),x'(s),D^\beta x(s),\int_0^s h(\xi)x(\xi)d\xi,\phi(x(s)))ds$$

$$\leqslant \frac{1}{\Gamma(\alpha-1)}\times$$

$$\int_0^t (t-s)^{\alpha-2}g(s)K(\|x\|,\|x'\|,\frac{\|x'\|}{\Gamma(2-\beta)},m\|x\|,b_1\|x\|+b_2\|x'\|)ds$$

$$+\frac{\mu}{A_{\mu,\eta}\Gamma(\alpha-1)}\times$$

$$\int_0^1 (1-s)^{\alpha-2}g(s)K(\|x\|,\|x'\|,\frac{\|x'\|}{\Gamma(2-\beta)},m\|x\|,b_1\|x\|+b_2\|x'\|)ds$$

$$+\frac{\mu}{A_{\mu,\eta}\Gamma(\alpha)}\times$$

$$\int_0^\eta (\eta-s)^{\alpha-1}g(s)K(\|x\|,\|x'\|,\frac{\|x'\|}{\Gamma(2-\beta)},m\|x\|,b_1\|x\|+b_2\|x'\|)ds$$

$$+\frac{1}{A_{\mu,\eta}\Gamma(\alpha+1)}\times$$

$$\int_0^\mu (\mu-s)^{\alpha}g(s)K(\|x\|,\|x'\|,\frac{\|x'\|}{\Gamma(2-\beta)},m\|x\|,b_1\|x\|+b_2\|x'\|)ds$$

$$\leqslant \frac{1}{\Gamma(\alpha-1)}\times$$

$$K(\|x\|,\|x'\|,\frac{\|x'\|}{\Gamma(2-\beta)},m\|x\|,b_1\|x\|+b_2\|x'\|)\int_0^1 (1-s)^{\alpha-2}g(s)ds$$

$$+\frac{\mu}{A_{\mu,\eta}\Gamma(\alpha-1)}\times$$

$$K(\|x\|,\|x'\|,\frac{\|x'\|}{\Gamma(2-\beta)},m\|x\|,b_1\|x\|+b_2\|x'\|)\int_0^1(1-s)^{\alpha-2}g(s)ds$$

$$+\frac{\mu}{A_{\mu,\eta}\Gamma(\alpha)}\times$$

$$K(\|x\|,\|x'\|,\frac{\|x'\|}{\Gamma(2-\beta)},m\|x\|,b_1\|x\|+b_2\|x'\|)\int_0^1(1-s)^{\alpha-2}g(s)ds$$

$$+\frac{1}{A_{\mu,\eta}\Gamma(\alpha+1)}\times$$

$$K(\|x\|,\|x'\|,\frac{\|x'\|}{\Gamma(2-\beta)},m\|x\|,b_1\|x\|+b_2\|x'\|)\int_0^1(1-s)^{\alpha-2}g(s)ds$$

$$\leqslant K(\Delta\|x\|_*,...,\Delta\|x\|_*)\|\hat{g}\|_1\times$$

$$[\frac{1}{\Gamma(\alpha-1)}+\frac{\mu}{A_{\mu,\eta}\Gamma(\alpha-1)}+\frac{\mu}{A_{\mu,\eta}\Gamma(\alpha)}+\frac{1}{A_{\mu,\eta}\Gamma(\alpha+1)}]$$

$$\leqslant K(\Delta r,...,\Delta r)\|\hat{g}\|_1[\frac{1}{\Gamma(\alpha-1)}+\frac{\mu}{A_{\mu,\eta}\Gamma(\alpha-1)}+\frac{\mu}{A_{\mu,\eta}\Gamma(\alpha)}+\frac{1}{A_{\mu,\eta}\Gamma(\alpha+1)}]$$

$$\leqslant (\Delta r)(\theta+\epsilon_0)\|\hat{g}\|_1[\frac{1}{\Gamma(\alpha-1)}+\frac{\mu}{A_{\mu,\eta}\Gamma(\alpha-1)}+\frac{\mu}{A_{\mu,\eta}\Gamma(\alpha)}+\frac{1}{A_{\mu,\eta}\Gamma(\alpha+1)}]$$

and so

$$\|F'_x\|\leqslant$$

$$(\Delta r)(\theta+\epsilon_0)\|\hat{g}\|_1[\frac{1}{\Gamma(\alpha-1)}+\frac{\mu}{A_{\mu,\eta}\Gamma(\alpha-1)}+\frac{\mu}{A_{\mu,\eta}\Gamma(\alpha)}+\frac{1}{A_{\mu,\eta}\Gamma(\alpha+1)}]\leqslant r.$$

Hence, $\|F_x\|_*\leqslant r$ and so $F_x\in C$. Using a similar proof, we can show that $F_y\in C$. This implies $\alpha(F_x,F_y)\geqslant 1$ and so F is α- admissible. It is obvious that $C\neq\emptyset$. Choose $x_0\in C$. Hence, $F_{x_0}\in C$ and so $\alpha(x_0,F_{x_0})\geqslant 1$. Let $x,y\in C$. Then, $\|x-y\|_*\leqslant\|x\|_*+\|y\|_*\leqslant 2r\leqslant r_1$. Also using (2), we have

$$\|F_x-F_y\|_*\leqslant L_{\alpha,\mu,\eta}\sum_{i=1}^{k_0}\|\hat{a}_i\|_1\Lambda_i(\Delta\|x-y\|_*,...,\Delta\|x-y\|_*).$$

Now using (5), we conclude that

$$\|F_x - F_y\|_* \leqslant L_{\alpha,\mu,\eta} \sum_{i=1}^{k_0} \|\hat{a}_i\|_1 (q_i + \epsilon_1)(\Delta \|x-y\|_*)^{\gamma_i}$$

$$\leqslant L_{\alpha,\mu,\eta} \sum_{i=1}^{k_0} \|\hat{a}_i\|_1 (q_i + \epsilon_1)\Delta^{\gamma_i} \|x-y\|_*^{\gamma_i}$$

$$\leqslant L_{\alpha,\mu,\eta} [\sum_{i=1}^{k_0} \|\hat{a}_i\|_1 (q_i + \epsilon_1)\Delta^{\gamma_i}] \|x-y\|_*^{\gamma},$$

where $\gamma = \min\{\gamma_1, ..., \gamma_{k_0}\}$. Since $\|x-y\|_* \leqslant 1$, $\|x-y\|_*^{\gamma_i} \leqslant \|x-y\|_*^{\gamma}$ for all $1 \leqslant i \leqslant k_0$. Put $\tau := L_{\alpha,\mu,\eta} \sum_{i=1}^{k_0} \|\hat{a}_i\|_1 q_i \Delta^{\gamma_i}$. Note that, $0 \leqslant \tau < 1$. Define the map $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\psi(t) = \tau t^{\gamma}$ whenever $t \in [0,1)$ and $\psi(t) = \tau t$ whenever $t \in [1, \infty)$. Then, ψ is nondecreasing and

$$\sum_{i=1}^{\infty} \psi^i(t) = \tau t^{\gamma} + \tau^2 t^{2\gamma} + ... \leqslant \sum_{i=1}^{\infty} \tau^i t^{\gamma} = \frac{\tau}{1-\tau} t^{\gamma} < \infty$$

for $t \in [0,1)$.
Also, we have $\sum_{i=1}^{\infty} \psi^i(t) = \frac{\tau}{1-\tau} t < \infty$ for $t \in [1, \infty)$. Thus, $\sum_{i=1}^{\infty} \psi^i(t)$ is a convergent series for all $t \geqslant 0$ and so $\psi \in \Psi$. Also, we have $\alpha(x,y)\|F_x, F_y\|_* \leqslant \psi(\|x-y\|_*)$. If $x \notin C$ or $y \notin C$, then the above inequality holds obviously. This shows that $\alpha(x,y)d(F_x, F_y) \leqslant \psi(d(x,y))$ for all $x, y \in X$. Now using Lemma 1.1, F has a fixed point that is the solution for problem (1). $\square$

Here, we provide an example to illustrate our main result.

Example 2.3.: Consider the pointwise defined problem

$$D^{\frac{5}{2}} x(t) = \frac{1}{100p(t)} (|x(t)| + |x'(t)| + |D^{\frac{1}{2}} x(t)| + |\int_0^t \frac{x(\xi)}{\sqrt{\xi}} d\xi| + |sin(x(t))|) \quad (*)$$

with boundary conditions $\int_0^{\frac{1}{3}} x(\xi) d\xi = 0$, $x'(1) = x(\frac{1}{4})$ and $x''(0) = 0$, where $p(t) = 0$ whenever $t \in [0,1] \cap \mathbb{Q}$ and $p(t) = 1 - t$ whenever

$t \in [0,1] \cap \mathbb{Q}^c$. Put $\alpha = \frac{1}{2}$, $\beta = \frac{1}{2}$, $\mu = \frac{1}{3}$, $\eta = \frac{1}{4}$, $k_0 = 1$, $\gamma_1 = 1$, $a_1(t) = g(t) = \frac{1}{p(t)}$, $b_1 = 1$, $b_2 = 0$, $h(\xi) = \frac{x(\xi)}{\sqrt{\xi}}$, $\phi(x) = sin(x)$ and

$$K(x_1, ..., x_5) = \Lambda_1(x_1, ..., x_5) = \frac{1}{100}(|x_1| + ... + |x_5|).$$

Then, we have

$$|\phi(x) - \phi(y)| = |sin(x) - sin(y)| \leqslant |x - y| = b_1|x - y|,$$

$$|f(t, x_1, ...x_5) - f_1(t, y_1, ..., y_5)| \leqslant a_1(t)[|x_1 - y_1| + ... + |x_5 - y_5|],$$

$q_1 = \lim_{z\to 0^+} \frac{\Lambda_1(z,z,z,z,z)}{z^{\gamma_1}} = \lim_{z\to 0^+} \frac{5|z|}{100z} = 0.05$, $a_1, g \in L^1$, $m := \|h\|_1 = 2$, $\|\hat{g}\|_{[0,1]} = \|\hat{a_1}\|_{[0,1]} = \int_0^1 \frac{1}{p(s)}(1-s)^{\alpha-2} ds = \int_0^1 \frac{(1-s)^{\frac{1}{2}}}{1-s} ds = \int_0^1 \frac{1}{\sqrt{1-s}} ds = 2$, $|f(t, x_1, ..., x_5)| \leqslant g(t)K(x_1, ..., x_5)$, K, Λ_1 are nonnegative and nondecreasing respect to $x_1, ..., x_5$, $A_{\mu,\eta} = \mu(1-\eta) + \frac{\mu^2}{2} = \frac{11}{36}$,

$$\Delta = \max\{1, \frac{1}{\Gamma(2-\beta)}, m, b_1 + b_2\} = \max\{1, \frac{1}{\Gamma(\frac{3}{2})}, 2, 1\} = 2,$$

$$\begin{aligned} L_{\alpha,\mu,\eta} &= \max\{\frac{1}{\Gamma(\alpha)} + \frac{\mu(2-\eta) - A_{\mu,\eta}}{A_{\mu,\eta}\Gamma(\alpha-1)} + \frac{A_{\mu,\eta} + \mu\eta}{A_{\mu,\eta}\Gamma(\alpha)} + \frac{A_{\mu,\eta}(1-\eta)+1}{A_{\mu,\eta}\Gamma(\alpha+1)}, \\ &\quad \frac{1}{\Gamma(\alpha-1)} + \frac{\mu}{A_{\mu,\eta}\Gamma(\alpha-1)} + \frac{\mu}{A_{\mu,\eta}\Gamma(\alpha)} + \frac{1}{A_{\mu,\eta}\Gamma(\alpha+1)}\} \\ &= \max\{\frac{1}{\Gamma(\frac{5}{2})} + \frac{\frac{1}{3}(2-\frac{1}{4}) - \frac{11}{36}}{\frac{11}{36}\Gamma(\frac{3}{2})} + \frac{\frac{11}{36} + \frac{1}{3}.\frac{1}{4}}{\frac{11}{36}\Gamma(\frac{5}{2})} + \frac{\frac{11}{36}(1-\frac{1}{4})+1}{\frac{11}{36}\Gamma(\frac{7}{2})}, \\ &\quad \frac{1}{\Gamma(\frac{3}{2})} + \frac{\frac{1}{3}}{\frac{11}{36}\Gamma(\frac{3}{2})} + \frac{\frac{1}{3}}{\frac{11}{36}\Gamma(\frac{5}{2})} + \frac{1}{\frac{11}{36}\Gamma(\frac{7}{2})}\} \\ &\quad \geqslant \max\{3.944, 3.957\} = 3.957, \end{aligned}$$

$$B = (\Delta\|\hat{g}\|_1 L_{\alpha,\mu,\eta})^{-1} \geqslant (4 \times 3.957)^{-1} \geqslant 0.0695,$$

$$\lim_{z\to 0^+} \frac{K(z,z,z,z,z)}{z} = 0.05 \in [0, B)$$

and

$$\sum_{i=1}^{k_0} \|\hat{a}_i\|_{[0,1]} q_i \Delta^{\gamma_i} = 2 \times 0.05 \times 2^1 = 0.2 < \frac{1}{3.957} = \frac{1}{L_{\alpha,\mu,\eta}}.$$

Now using Theorem 2.2, the problem $(*)$ has a solution.

Competing interests

The authors declare that they have no competing interests.

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contributions

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