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New Numbers on Euler's Totient Function with Applications

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Abstract. For any positive integer m, $\varphi(m)$ finds out how many residues of m that are co-prime to m, where φ is the Euler's totient function. In this paper, we introduce the notion of totient and hyper totient numbers. We explore the potential links of totient, super totient and hyper totient numbers. Many postulates and characterizations of these numbers have been proposed with straight forward proofs. In the end, applications of these numbers in graph labeling have also been demonstrated over a family of well known graph.

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1. Introduction

A number is called perfect if the sum of its positive divisors is twice of the number. That is, $\sigma(n) = 2n$, where $\sigma(n)$ denotes the sum of positive divisors on n [2]. A generalization of the concept of perfect number, called Zumkeller number, has been investigated and published by Zumkeller in 2003. Zumkeller generated a sequence of integers in which the positive

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divisors of every integer can be partitioned into two disjoint sets whose sums are equal [15]. The notion of Zumkeller numbers was formally investigated by Clark et al. [12]. Later on, Peng and Bhaskara Rao proved several results about Zumkeller numbers and half Zumkeller numbers in [14]. In [1], Balamurugan and et al., presented algorithms for Zumkeller labeling of bipartite and wheel graphs. In [6], Hoque generalized the concept of perfect numbers using arithmetic functions. McDaniel proved some results on non-existence of odd perfect numbers [8]. In this piece of work, we introduce and investigate new classes of numbers namely totient numbers and hyper totient numbers using Euler Totient function φ and demonstrate their applications in graph labeling over Wheel graphs. We organize our paper as follows.

First, we state some previous results without proofs and few important definitions from [2, 13] to make this paper self readable. In Section 2, we discuss super totient numbers and characterize these numbers completely. In Section 3, we introduce the notion of hyper totient numbers and prove that a Zumkeller number is either a super totient or a hyper totient number. In Section 4, we validate these numbers in graph labeling over Wheel graphs.

Theorem 1.1. [*2*] *"For $n > 1$, the sum of positive integers less than n and relatively prime to n is $\frac{n\varphi(n)}{2}$."*

Definition 1.2. [*5*] *"A perfect number is of Euclid type if it can be written in the form $2^{k-1}(2^k - 1),\ k > 1$."*

Definition 1.3. [*9*] *"A number n is said to be k-perfect if $\sigma(n) = kn$."*

Definition 1.4. [*3, 10*] *"An integer $n > 0$ is near 3-perfect if $\sigma(n) = 3n + d$, where $\sigma(n)$ is the divisor function and d is a proper divisor of n. Every 3-perfect number will be of the form $2^\alpha p_1^t p_2$, where $p_1 < p_2$ are distinct odd primes provided $\alpha \geqslant 1,\ 1 \leqslant t \leqslant 2$."*

Proposition 1.5. [*14*] *"If n is a Zumkeller number, then*
(a) $\sigma(n)$ is even.
(b) The prime factorization of n must include at least one odd prime to an odd power."

Definition 1.6. *An integer $n > 0$ is called totient, if the sum of co-prime residues of n is $2^k n,\ k \geqslant 1$. That is,*

$$\sum_{d<n,(d,n)=1} d = 2^k n,\ k \geqslant 1.$$

The numbers $5, 8, 10, 12, 15, 16, 17, 20, 24, ...$ are the examples of some totient numbers. In the following proposition, we characterize the totient numbers to view their postulates as simple consequences.

Proposition 1.7. *An integer $n > 0$ is totient if and only if $\varphi(n) = 2^{k+1}, k \geqslant 1$.*

Proof. By Theorem 1.1, n is totient number

$$\begin{aligned} &\Leftrightarrow \quad \rho(n) = \frac{n\varphi(n)}{2} = 2^k n,\ k \geqslant 1 \\ &\Leftrightarrow \quad \varphi(n) = 2^{k+1}, k \geqslant 1. \quad \square \end{aligned}$$

Consequences

1. Let n_i for $i = 1, 2, 3, ..., r$ be pairwise relatively prime totient numbers. Then $\prod_{i=1}^{r} n_i$ is a totient number.

2. If 2^k and 2^{k+1} are totient numbers, then so is their sum.

3. An odd integer n is a totient number if and only if $2n$ is a totient number.

A prime number of the form $2^{2^n} + 1, n \geqslant 1$ is called a Fermat prime [11]. Sine it is well-known that p is prime if and only if $\varphi(p) = p - 1$. Thus in case of Fermat's prime, we must get $\varphi(p) = \varphi(2^{2^n} + 1) = 2^{2^n}$, where $2^n > 1$. Thus, in the light of Proposition 1.7, it is clear that all Fermat's primes are totient numbers. This gives the following theorem.

Theorem 1.8. *Every Fermat's prime is a totient number.*

By Proposition 1.7 and Theorem 1.8, we characterize totient number by prime factorization.

Theorem 1.9. *A positive integer n is totient number if and only if $n = 2^k p_1 p_2 \cdots p_m$, $n \neq 2$, $n \neq 2^2$, $n \neq 2.3$, where each p_i, $1 \leqslant i \leqslant m$, $k \geqslant 1$, are Fermat's prime numbers.*

From Proposition 1.7 and Consequence 1, we conclude the following result.

Theorem 1.10. *The set of totient numbers is infinite.*

2. Super Totient Numbers

Recall that a positive integer n is super totient if the co-prime residues of n can be separated into disjoint sets whose sums are equal (for detail, see Definition 1.3, [7]). For example if we take $n = 5$, then the set of residues of 5 can be partitioned as, $A = \{1, 4\}$ and $B = \{2, 3\}$. Then by Definition 1.3 of [7], the integer 5 is a super totient number. Similarly, 8, 10, 12, 13, 14, 15, 16, 17, 20, $\cdots$, are all super totient numbers. In this section, we completely characterize all super totient numbers. We further show that the class of super totient numbers is a bigger class then to class of totient numbers. For the characterization of super totient numbers, the subsequent lemma of [7] is of essential importance.

Lemma 2.1. [7]. *Let $n > 0$ be an integer, if $\varphi(n) \equiv 0 \ (mod\ 4)$ then n is super totient.*

Theorem 2.2. *A prime number p is super totient if and only if $p = 1+4t$ for some $t \in \mathbb{Z}^+$.*

Proof. Let be p a super totient prime number. Then by definition, $1, 2, 3, ..., p-1$ can be divided into two disjoint sets of equal sums. This sum certainly is an integer, so by Theorem 1.1, $4|p\ \varphi(p)$. But p is prime, so $\varphi(p) \equiv 0(mod\ 4)$ if and only if $p = 1 + 4t$ for some $t \in \mathbb{Z}^+$ □

Corollary 2.3. *If a prime number p is super totient number then $p^k, k \geqslant 1$ is super totient.*

Proof. Let p be a super totient prime. Then by Theorem 2.2, $p \equiv 1(mod\ 4)$. That is, $p - 1 = 4t$ for some integer t. Note that,

$$\varphi(p^k) = p^{k-1}(p-1) = p^{k-1}(4t) = 4tp^{k-1} \equiv 0(mod\ 4).$$

Thus by Lemma 2.1, p^k is a super totient. □

Observe that, if n is a totient number, then by Proposition 1.7, $\varphi(n) = 2^{k+1},\ \ k \geqslant 1$. Hence by Lemma 2.1, n must be super totient. However, the converse is not true in general, since 14 is a super totient number but $\varphi(14) = 6$ cannot be written as a power of 2. Hence, it is not a totient number. The above discussion leads to the following theorem.

Theorem 2.4. *Every totient number is super totient. The converse is not asserted and in fact, is not true in general.*

The proof of the following theorem can be obtained by means of Lemma 2.1.

Theorem 2.5.
1. *Let n_i for $i = 1, 2, 3, ..., r$ be pairwise relatively prime super totient numbers. Then $\prod_{i=1}^{r} n_i$ is a super totient number.*
2. *If $m > 0$ has at least two odd prime factors then m is a super totient number.*
3. *If $m > 0$ and n is super totient number then mn is a super totient number.*
4. *If n is super totient number, then n^2 is also super totient.*
5. *If $p|n$, where $p = 1 + 4t$ is an odd prime number, for some $t \in \mathbb{Z}^+$ then n is super totient number.*

From Lemma 2.1 and Theorem 2.5(1), we conclude the following result.

Theorem 2.6. *The set of super totient numbers is infinite.*

Theorem 2.7. *If the positive integer $n \in N$ is super totient, then $n\varphi(n)$ is a multiple of 4. Converse is true for $n \geqslant 30$, exactly if $n \geqslant 30$ and $n\varphi(n)$ is a multiple of 4, then n is super totient.*

Proof. If n is super totient, then $\{r_1, \cdots, r_{\varphi(n)}\} = A \cup B$, where A and B are disjoint sets and $\sum_{a\in A} a = \sum_{b\in B} b$. Letting s denote the common value of the above sum, we have by Theorem 1.1,

$$2s = \sum_{a\in A} a + \sum_{b\in B} b = n\varphi(n)/2,$$

so $s = n\varphi(n)/4$. Thus, $4|n\varphi(n)$. This proves the necessary condition.

For the sufficiency, assume $4|n\varphi(n)$. If $4|\varphi(n)$, then n is super totient by

Lemma 2.1. So assume that $4 \nmid \varphi(n)$. Since $n \geqslant 30, \varphi(n)$ is even, it follows that $2 \parallel \varphi(n)$ (we write $a \parallel b$, if a divides b but no higher powers of a divide b). Since $4|n\varphi(n)$ and $2 \parallel \varphi(n)$, it follows that $2|n$. In particular, $n = 2p^k$ for some positive integer k and prime p with $p \equiv 3(mod\ 4)$ (for if $p \equiv 1(mod\ 4)$, then $4|p-1 = \varphi(p)$, a contradiction against the fact that $2 \parallel \varphi(n)$).

Consider the following numbers

$$r_1, r_2, \cdots, r_{\varphi(n)/2},$$

which are all the numbers smaller than $n/2$ and coprime to n. Let $s = \varphi(n)/2$. If $p > 3$, the string of these numbers contains $1, 3, \cdots, n/2-4$, which are all smaller than $n/2$, coprime to n, and distinct since $n/2-4 > 3$, which is equivalent to $n > 14$. There are $s-3$ numbers left. Select t of them say

$$r_{i1}, \cdots, r_{it}$$

where $t = (\varphi(n)-2)/4$. The number t is a positive integer since $\varphi(n) \equiv 2(mod\ 4)$ and it is possible to choose t numbers out of $s-3$ because $s-3 \geqslant t$, an inequality equivalent to

$$\varphi(n)/2 - 3 \geqslant (\varphi(n)-2)/4,$$

which is equivalent to $\varphi(n) \geqslant 10$. To see that this is satisfied for $n \geqslant 30$, recall that $n = 2p^k$. We want to show that $p^{k-1}(p-1) \geqslant 10$. If $k = 1$, then since $n \geqslant 30$, we get that $p \geqslant 15$, so the above inequality is satisfied. If $k \geqslant 2$, then either $p = 3$, in which case $k \geqslant 3$ and so $p^{k-1}(p-1) \geqslant 9 \cdot 2 > 10$, or $p \geqslant 7$, in which case $p^{k-1}(p-1) \geqslant 7 \cdot 6 > 10$. So, the inequality $\varphi(n) \geqslant 10$ is indeed satisfied for $n \geqslant 30$ of the form $n = 2p^k$ with $p \equiv 3(mod\ 4)$.

Consider now

$$A = \{1,\ 3,\ n/2-4,\ r_{i1},\ \cdots,\ r_{it},\ n-r_{i1},\ \cdots,\ n-r_{it}\}.$$

Then A has $3+2t$ elements, the first $t+3 \leqslant s$ being distinct and smaller than $n/2$ and the last t being distinct and larger than $n/2$. The sum of elements of A is

$$(1+3+n/2-4)+\sum_{j=1}^{t}(r_{ij}+n-r_{ij}) = n/2+tn = n/2(1+2t) = n\varphi(n)/4 = s.$$

Thus, the complement B of A has sum $n\varphi(n)/2 - s = n\varphi(n)/4 = s$.

In case $p = 3$, replace the beginning elements $1, 3, n/2-4$ by $1, 7, n/2-8$ which are also coprime to $n = 2 \cdot 3^k$. We need $n/2 - 8 > 7$, so $n > 30$, which is satisfied for $n \geqslant 30$ of the form $2 \cdot 3^k$. Everything else stays the same(namely the choice of t, etc.) $\square$

It is important to note that every even numbers n must be any one of the two types namely $\varphi(n) \equiv 0(mod\ 4)$ or $\varphi(n) \equiv 2(mod\ 4)$. The numbers lying in first class have been determined as super totient numbers by means of Lemma 2.1. However, the determination of all super totient numbers from the second type is difficult and challenging. Since there are many numbers from second class which do not follow the definition of super totient numbers. For example, $\varphi(18) \equiv 2(mod\ 4)$, but 18 is not super totient, whereas 14 is a super totient number and $\varphi(14) \equiv 2(mod\ 4)$ as well. After proving the following result, we characterize the supper totient numbers completely. Thus the following theorem is of vital importance.

Theorem 2.8. *An even integer not divisible by* 4 *of kind* $\varphi(n) \equiv 2(mod\ 4)$ *is super totient if and only if there exists residue* $1 < r_i < (n+2)/2,\ i = 1,\ 2,\ 3,\ \cdots,\ \varphi(\frac{n+2}{2})$*, such that* $(\frac{n+2}{2}, r_i) = 1$ *and* $(\frac{n+2}{2} - r_i,\ n) = 1$.

Proof. Let n be an even integer not divisible by 4 satisfying the congruence $\varphi(n) \equiv 2(mod\ 4)$. Suppose there exists a residue $r_i, 1 < r_i < (n+2)/2$, where $i = 1,\ 2,\ 3,\ \cdots,\ \varphi(\frac{n+2}{2})$, such that $(\frac{n+2}{2}, r_i) = 1 = (\frac{n+2}{2} - r_i,\ n)$. Without any loss, we take $i = 2$, and get, $(\frac{n+2}{2}, r_2) = 1 = (\frac{n+2}{2} - r_2,\ n)$. Let $k = (\varphi(n) - 2)/4 > 0$ be any integer. If $k = 1$ then $(\varphi(n) - 2)/4 = 1$ yields that $\varphi(n) = 6$. Thus in this case there are the six residues of n which can be rearranged after renaming as:

$$r_1 = 1,\ r_2 = r_2,\ (n+2)/2 - r_2 = r_3,\ n - r_3 = r_4,\ n - r_2 = r_5,\ n - 1 = r_6.$$

These can be partitioned as under:

$$A = \ \{\, r_6,\ r_2,\ r_3\}, \qquad B = \ \{\, r_1,\ r_5,\ r_4\}.$$

Then it is clear that,

$$\sum_{a \in A} a = n - 1 + \tfrac{n+2}{2} = \tfrac{3n}{2} = 1 + n - r_2 + n - (\tfrac{n+2}{2} - r_2) = \sum_{b \in B} b.$$

That is, the case $k = 1$ is true. Let $k > 1$ and after renaming, we fix six residues as, $r_1 = 1,\ r_2 = r_2,\ (n+2)/2 - r_2 = r_3,\ n - r_3 = r_4,\ n - r_2 = r_5,\ n - 1 = r_6$. The rest of the residues can be rearranged as, $r_7 < r_8 < r_9 < \cdots < r_{4k+1} < r_{4k+2}$. Then after, we can partition these co-prime residues of n in the following two disjoint sets:
$A = \{ r_6,\ r_2,\ r_3\} \cup \{r_7,\ r_8,\ \cdots,\ r_{k+5}\} \cup \{n-r_7,\ n-r_8,\ \cdots,\ n-r_{k+5}\}$,
$B = \{ r_1,\ r_5,\ r_4\} \cup \{r_{k+6}, \cdots, r_{2k+4}\} \cup \{n - r_{k+6}, \cdots, n - r_{2k+4}\}$.

This gives,

$$\begin{aligned}
\sum_{a\in A} a &= n - 1 + \frac{n+2}{2} + \sum_{i=7}^{k+5}(r_i + n - r_i)\\
&= n - 1 + \frac{n+2}{2} + (k-1)n = \frac{n(2k+1)}{2}\\
&= 1 + n - r_2 + n - (\frac{n+2}{2} - r_2) + (k-1)n\\
&= 1 + n - r_2 + n - r_3 + \sum_{j=k+6}^{2k+4}(r_j + n - r_j) = \sum_{b\in B} b.
\end{aligned}$$

Hence by definition, n is super totient.

Conversely, suppose n is a super totient number of the type $\varphi(n) \equiv 2(mod\ 4)$. Then by Theorem 1.1, the sum of co-prime residue of n is $n(2k+1)$. But then the sum of residues appearing in both the disjoint partitioned sets is $\frac{n(2k+1)}{2}$.

That is,

$$\begin{aligned}
\frac{n(2k+1)}{2} &= \sum_{a\in A} a\\
&= \sum_{i=1}^{k-1}(r_i + n - r_i) + n - 1 + s \text{ where } s = \sum_j r_j \text{ for some } j.\\
&= n(k-1) + n - 1 + s = nk - 1 + s. \qquad (1)
\end{aligned}$$

Equation (1), is balanced only if $s = \frac{n+2}{2}$. Since $4 \nmid n$, so $\frac{n+2}{4}$ must be an integer so, $s = \frac{n+2}{2}$ is even integer. On contrary we suppose that there does not exist any residue r_i, co-prime to n provided $1 <$

$r_i < (n+2)/2$, $i = 1, 2, 3, \cdots, \varphi(\frac{n+2}{2})$, satisfying the condition $(\frac{n+2}{2}, r_i) = 1$ and $(\frac{n+2}{2} - r_i, n) = 1$. Then, of course, it is impossible to find their sum as the number s. This further implies that n is not super totient, a contradiction. This completes the proof. □

It is well-known that every even perfect number is of Euclid type. That is, it can be expressed in the form $2^{k-1}(2^k - 1)$, $k > 1$. Let n be an even perfect number greater then 6. Then it can be written as $2^{k-1}(2^k - 1)$, $k > 1$. For $k = 2$, $n = 6$, which is not super totient number. Let $k > 2$ then we obtain, $\varphi(n) = \varphi(2^{k-1}(2^k - 1)) = \varphi(2^{k-1})\varphi(2^k - 1) = 4k$, for some positive integer k. Hence, n is a super totient number. This leads to the following result.

Theorem 2.9. *Every even perfect number greater then* 6 *is super totient.*

Also, in view of Definition 1.3 and 1.4, following results are easy to prove.

Theorem 2.10.
(1) Every 3*-perfect and* 4*-perfect numbers are super totient.(2) Every near* 3*-perfect is super totient.*

3. Hyper Totient Numbers

An integer $n > 0$ is termed as hyper totient, if the co-prime residues of n including n, can be divided into two separated sets of equal sum. In this section, we state and prove results regarding hyper totient numbers and show that the any Zumkeller number is either a super totient number or hyper totient number. For example the numbers 6 and 7 are not super totient since their respective set of relatively prime residues are $\{1, 5\}$ and $\{1, 2, 3, 4, 5, 6\}$. Both can never be divided into disjoint sets of equal sum. However, if we add 6 and 7 to above sets then the desired partitions are possible. These are: $\{1, 5\}$; $\{6\}$ and $\{1, 6, 7\}$;$\{2, 3, 4, 5\}$.
The following theorems give few postulates and enumerate the complete set of hyper totient numbers.

Theorem 3.1. *A positive integer* n *is hyper totient if* $4|(\varphi(n)+2)$. *However, the converse is not asserted and, in fact, is false in general.*

Proof. We note that, $(r_i, n) = 1$ if and only if $(n - r_i, n) = 1$. Let $k = (\varphi(n) + 2)/4$. If $k = 1$, then $1 = (\varphi(n) + 2)/4$ gives that $\varphi(n) = 2$.

This means that 1 and $n-1$ are the only co-prime residues of n, in this case, we find the desired partition as, $A = \{1,\ n-1\}$ and $B = \ \{n\}$. This clearly shows that n is hyper totient. Next we let $k > 1$. A partition of the set of co-prime residues of n namely $1 = r_1 < r_2 < r_3 < \cdots < r_{\varphi(n)} < n$ and including n, is given as,
$A = \{\ r_1,\ \ r_2,\ \ r_3 \cdots,\ r_k\} \cup \{n - r_1,\ n - r_2,\ n - r_3 \cdots,\ n - r_k\}$,
$B = \{r_{k+1},\ r_{k+2}, \cdots,\ r_{2k-1}\} \cup \{n - r_{k+1},\ n - r_{k+2}, \cdots,\ n - r_{2k-1}\} \cup \{n\}$.
Then it is clear that,

$$\sum_{a\in A} a = \sum_{i=1}^{k}(r_i + n - r_i) = nk = \sum_{j=k+1}^{2k-1}(r_j + n - r_j) + n = \sum_{b\in B} b.$$

Conversely, if we take $n = 8$ then it is hyper totient since its set of co-prime residues can be partitioned as $\{1, 3, 8\}$ and $\{5, 7\}$. While, $\frac{\varphi(8)+2}{4}$ is not integer. □

The following results can be proved using Theorem 3.1.

Corollary 3.2. *A prime number p is hyper totient if and only if $p = 3 + 4t$ for some $t \in \mathbb{Z}^+$.*

Proof. Let p be any hyper totient prime number. Then by definition, the numbers $1, 2, 3, ..., p-1$ and p can be divided into two disjoint sets of equal sum. This sum certainly is an integer, so by Theorem 1.1, $4|p(\varphi(p) + 2)$. But p is prime, thus $4|\varphi(p) + 2$. Note that $\varphi(p) + 2 \equiv 0(mod\ 4)$ if and only if $p = 3 + 4t$ for some $t \in \mathbb{Z}^+$. Consequently, any prime number p is hyper totient if and only if $p = 3 + 4t$ for some $t \in \mathbb{Z}^+$. □

Corollary 3.3. *If a prime p is a hyper totient then $p^k, k \geqslant 1$ is also hyper totient.*

The proof is similar to Corollary 2.3.

Theorem 3.4. *Every integer divisible by 4 is hyper totient.*

Proof. Let n be any integer divisible by 4. In view of Theorem 3.1, all integers of the type $\varphi(n) \equiv 2(mod\ 4)$ divisible 4 are hyper totient. Now we discuss the other case. That is, all integers of kind $\varphi(n) \equiv 0(mod\ 4)$ and divisible by 4. So we, let $k = \varphi(n)/4$. It is easy to verify that $(n,\ \frac{n}{2} - 1) = 1$ and $(n,\ \frac{n}{2} + 1) = 1$. If $k = 1$, there are four co-prime residues of n. These can be rearranged as, $r_1 = 1,\ r_2 = \frac{n}{2} - 1,\ r_3 = \frac{n}{2} + 1,\ r_4 = n - 1$. Then, $A = \{r_1,\ r_2,\ n\}$ and $B = \{\ r_3,\ r_4\}$, is the desired partition. Hence n is hyper totient as,

$$\sum_{a\in A} a = 1 + \tfrac{n}{2} - 1 + n = \tfrac{3n}{2} = \tfrac{n}{2} + 1 + n - 1 = \sum_{b\in B} b.$$

Let $k > 1$. We fix the above four co-prime residues of n and the remaining co-prime residues are given as, $r_5 < r_6 < r_7 < \cdots < r_{4k}$. Then we can find the desired partition as,
$A = \{\ r_1,\ r_2,\ n\} \cup \{r_5,\ r_6,\ \cdots,\ r_{k+3}\} \cup \{n - r_5,\ n - r_6,\ \cdots,\ n - r_{k+3}\}$,
$B = \{\ r_3, r_4\} \cup \{r_{k+4},\ \cdots, r_{2k+2}\} \cup \{n - r_{k+4}, \cdots, n - r_{2k+2}\}$.
But then,

$$\begin{aligned}\sum_{a\in A} a &= 1 + \frac{n}{2} - 1 + n + \sum_{i=5}^{k+3}(r_i + n - r_i)\\ &= \frac{3n}{2} + (k-1)n = \frac{n(2k+1)}{2}\\ &= \frac{n}{2} + 1 + n - 1 + (k-1)n\\ &= \frac{3n}{2} + \sum_{j=k+4}^{2k+2}(r_j + n - r_j) + n = \sum_{b\in B} b. \quad \square\end{aligned}$$

Again we note that the number lying in the type $\varphi(n) \equiv 2(mod\ 4)$ have been determined as hyper totient numbers by means of Theorem 3.1. However, the determination of all hyper totient numbers from the type $\varphi(n) \equiv 0(mod\ 4)$ is much difficult and challenging. Since there are many numbers from second class which do not follow the definition of hyper totient numbers. For example, $\varphi(30) \equiv 0(mod\ 4)$, but 30 is not hyper totient, whereas 26 is a hyper totient number and $\varphi(26) \equiv 0(mod\ 4)$ as well. After proving the following result, we characterize the hyper totient numbers completely. Thus the following theorem is of vital importance.

Theorem 3.5. *An even integer not divisible by* 4 *of kind* $\varphi(n) \equiv 0(mod\ 4)$ *is hyper totient if and only if there exists residue* $1 < r_i < (n+2)/2,\ i = 1,\ 2,\ 3,\ \cdots,\ \varphi(\frac{n+2}{2})$, *such that* $(\frac{n+2}{2}, r_i) = 1$ *and* $(\frac{n+2}{2} - r_i,\ n) = 1.$

Proof. Let n be an even integer not divisible by 4 satisfying the congruence $\varphi(n) \equiv 0(mod\ 4)$. Suppose there exists a residue $r_i, 1 < r_i < (n+2)/4$, where $i = 1,\ 2,\ 3,\ \cdots,\ \varphi(\frac{n+2}{4})$, such that $(\frac{n+2}{2}, r_i) = 1 = (\frac{n+2}{2} - r_i,\ n)$. Without any loss, we take $i = 2$ and get, $(\frac{n+2}{2}, r_2) = 1 = (\frac{n+2}{2} - r_2,\ n)$. Let $k = \varphi(n)/4 > 1$ be any integer. The six residues of n which can be rearranged after renaming as:

$$r_1 = 1,\ r_2 = r_2,\ (n+2)/2 - r_2 = r_3,\ n - r_3 = r_4,\ n - r_2 = r_5,\ n - 1 = r_6.$$

The rest of the residues can be rearranged as, $r_7 < r_8 < r_9 < \cdots < r_{4k}$. we can partition the set of co-prime residues and including n in the following two disjoint sets:

$A = \ \{r_6,\ r_2,\ r_3\} \cup \{r_7,\ r_8,\ \cdots,\ r_{k+5}\} \cup \{n - r_7,\ n - r_8,\ \cdots,\ n - r_{k+5}\}$,
$B = \ \{r_1,\ r_5,\ r_4\} \cup \{r_{k+6},\ \cdots,\ r_{2k+3}\} \cup \{n - r_{k+6},\ \cdots,\ n - r_{2k+3}\} \cup \{n\}$.

Then it is clear that,

$$\begin{aligned}\sum_{a \in A} a &= n - 1 + \frac{n+2}{2} + \sum_{i=7}^{k+5}(r_i + n - r_i)\\ &= n - 1 + \frac{n+2}{2} + (k-1)n\\ &= \frac{n(2k+1)}{2}\end{aligned}$$

$$\begin{aligned}&= 1 + n - r_2 + n - (\frac{n+2}{2} - r_2) + (k-2)n + n\\ &= 1 + n - r_2 + n - r_3 + \sum_{j=k+6}^{2k+3}(r_j + n - r_j) + n = \sum_{b \in B} b.\end{aligned}$$

Hence, n is a hyper totient number.

Conversely, suppose n is a hyper totient number of the type $\varphi(n) \equiv 0(mod\ 4)$. Then by Theorem 1.1, the sum of co-prime residues of n including n is $n(2k+1)$. But then the sum of residues appearing in both the disjoint partitioned sets is $\frac{n(2k+1)}{2}$.

That is,

$$\begin{aligned}
\frac{n(2k+1)}{2} &= \sum_{a\in A} a \\
&= \sum_{i=1}^{k-1}(r_i + n - r_i) + n - 1 + s \quad \text{where} \quad s = \sum_{j} r_j \text{ for some } \; j. \\
&= n(k-1) + n - 1 + s \\
&= nk - 1 + s. \qquad (2)
\end{aligned}$$

Equation (2) is balanced only if $s = \frac{n+2}{2}$. Since $4 \nmid n$, so $\frac{n+2}{4}$ must be an integer so, $s = \frac{n+2}{2}$ is even integer. On contrary we suppose that there does not exist any residue r_i, co-prime to n provided $1 < r_i < (n+2)/2$, $i = 1,\ 2,\ 3,\ \cdots\ ,\ \varphi(\frac{n+2}{2})$, satisfying the condition $(\frac{n+2}{2}, r_i) = 1$ and $(\frac{n+2}{2} - r_i,\ n) = 1$. Then of course, it is impossible to find their sum as the number s. This further implies that n is not hyper totient, a contradiction. This completes the proof. □

Theorem 3.6. *If the positive integer $n \in N$ is hyper totient, then $n(\varphi(n)+2)$ is a multiple of 4. Conversely it is true for $n \geqslant 32$, exactly if $n \geqslant 32$ and $n(\varphi(n)+2)$ is a multiple of 4, then n is hyper totient.*

Proof. If n is hyper totient, then $\{r_1, \cdots, r_{\varphi(n)},\ n\} = A \cup B$, where A and B are disjoint and $\sum_{a\in A} a = \sum_{b\in B} b$. Letting s denote the common value of the above sum, we have by Theorem 1.1,

$$2s = \sum_{a\in A} a + \sum_{b\in B} b = n(\varphi(n)+2)/2,$$

so $s = n(\varphi(n)+2)/4$. Thus, $4|n(\varphi(n)+2)$. This proves the necessary condition.

For the sufficiency, assume $4|n(\varphi(n)+2)$. If $4|(\varphi(n)+2)$, then n is super totient by Theorem 3.1. So assume that $4 \nmid (\varphi(n)+2)$. Since $n \geqslant 32, \varphi(n)$ is even. Since $4 \nmid (\varphi(n)+2)$, it follows that $2 \parallel \varphi(n)$. Since $4 \nmid (\varphi(n)+2)$ but $4|n(\varphi(n)+2)$, it follows that $2|n$. In particular, $n = 2p^k$ for some positive integer k and prime p with $p \equiv 1(mod\, 4)$. Consider the following numbers

$$r_1, r_2, \cdots, r_{\varphi(n)/2},$$

which are all the numbers smaller than $n/2$ and coprime to n. Let $s = (\varphi(n)+2)/2$. If $p \geqslant 5$, the string of these numbers contains $1, 3, \cdots, n/2-4$, which are all smaller than $n/2$, coprime to n, and distinct since $n/2-4 > 5$, which is equivalent to $n > 18$, which is satisfied for us. There are $s-3$ numbers left. Select t of them say

$$r_{i1}, \cdots, r_{it},$$

where $t = \varphi(n)/4$. The number t is a positive integer since $\varphi(n) \equiv 0(mod\ 4)$ and $n \geqslant 32$, it is possible to choose t numbers out of $s-3$ because $s-3 \geqslant t$, an inequality equivalent to

$$(\varphi(n)+2)/2 - 3 \geqslant \varphi(n)/4,$$

which is equivalent to $\varphi(n) \geqslant 8$. To see that this is satisfied for $n \geqslant 32$, recall that $n = 2p^k$. We want to show that $p^{k-1}(p-1) \geqslant 8$. If $k = 1$, then since $n \geqslant 32$, we get that $p \geqslant 17$, so the above inequality is satisfied. If $k \geqslant 2$, then either $p = 5$, in which case $k \geqslant 3$ and so $p^{k-1}(p-1) \geqslant 25 \cdot 4 > 10$, or $p \geqslant 13$, in which case $p^{k-1}(p-1) \geqslant 13 \cdot 12 > 10$. So, the inequality $\varphi(n) \geqslant 8$ is indeed satisfied for $n \geqslant 32$ of the form $n = 2p^k$ with $p \equiv 1(mod\ 4)$.

Consider now

$$A = \{1,\ 3,\ n/2-4,\ r_{i1},\ \cdots,\ r_{it},\ n-r_{i1},\ \cdots,\ n-r_{it}\}.$$

Then A has $3+2t$ elements, the first $t+3 \leqslant s$ being distinct and smaller than $n/2$ and the last t being distinct and larger than $n/2$. The sum of elements of A is

$$(1+3+n/2-4)+\sum_{j=1}^{t}(r_{ij}+n-r_{ij}) = n/2(1+2\varphi(n)/4) = n(\varphi(n)+2)/4 = s.$$

Thus, the complement B of A has sum $n(\varphi(n)+2)/2 - s = n(\varphi(n)+2)/4 = s$.

Now if $n = 2p_1^{k_1}p_2^{k_2}\cdots p_m^{k_m}$, then clearly n is even and not divisible by 4 and of the kind $\varphi(n) \equiv 0(mod\ 4)$, so by Theorem 3.5, n is hyper totient. If $n = 2^k p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$, $k > 1$ then n is divisible by four, so by Theorem 3.4, n is hyper totient. □

Remark 3.7. *A hyper totient number my not be super totient. For example* 4 *is hyper totient but not a super totient number.*

Finally, we prove that the class of Zumkeller numbers is either a sub class of super totient number or of hyper totient numbers. The assertion can be entertained in the following theorem.

Theorem 3.8. *Every zumkeller number is either a super totient or a hyper totient number.*

Proof. Let n be a zumkeller number then $n \neq 2$, so by Proposition 1.5 the canonical representation of n must contains an odd prime number with odd exponent. This odd prime number must be of the form $p \equiv 1(mod\ 4)$ or $p \equiv 3(mod\ 4)$. If $p \equiv 1(mod\ 4)$ then by Theorem 2.2, n is super totient number. However, if $p \equiv 3(mod\ 4)$, then by Corollary 3.2, n is hyper totient number. □

Table 1: Totient, Super Totient and Hyper Totient Numbers.

Totient Numbers	Super Totient Numbers	Hyper Totient Numbers
5, 8, 10, 12, 15, 16, 17, 20, 24, 30, 32, 34, 40, 48, 51, 60, 64, 68, 80, 85, 96.	5, 8, 10, 12, 13, 14, 15, 16, 17, 20, 21, 22, 24, 25, 26, 28, 29, 30, 32, 33, 34, 35, 36, 37, 38, 40, 41, 42, 44, 45, 46, 48, 50, 51, 52, 54, 55, 56, 57, 58,60, 61, 62, 63, 64, 65, 66, 68, 69, 70, 72, 73, 74, 75, 76, 77,78, 80, 82, 84, 85, 86, 87, 88,89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100.	3, 4, 6, 7, 8, 9, 11, 12, 14, 16, 18, 19, 20, 22, 23, 24, 26, 27, 28, 31, 32, 34, 36, 38, 40, 42, 43, 44, 46, 47, 48, 49,50, 52, 54, 55, 56, 58, 59, 60, 62, 64, 66,67, 68, 70, 71, 72, 74,76, 78, 79, 80, 81, 82, 83,84, 86, 88, 90, 92, 94, 96,98, 100.

Conjecture 3.9. The set of hyper totient numbers is infinite.

In Table 1, we list first hundred numbers of each of the classes for totient numbers, super totient numbers and hyper totient numbers.

4. Applications of Super Totient Numbers

In previous sections, we introduced and investigated new numbers by means of Euler's totient function. It would be more interesting and of great worth if these numbers could be employed in some well known mathematics. In our previous work, a super totient labeling over many graphs such as Wheel graphs, Bipartite graphs, Friendship graphs and Cyclic graphs has been considered via multiplication [7]. While in this paper, we demonstrate super toitient labeling in a different way only for Wheel graphs. The rest of the new defined labeling over other classes can be validated in a similar technique.

The definition of super totient labeling can also be defined with the operation of addition together with a constant multiple. Keeping in view that the super totient labeling was lacking over addition in a previous work (see, [7]). By incorporating this new definition, we would be able to develop a new labeling which also agrees as super totient labeling. The validity of this new labeling has been shown in Fig.1. For instance, there is no edge between vertices 3 and 4 as $1 \times 3 + 4 = 7$, is not a super totient number for $k = 1$. While in [7], there was an edge between 3 and 4 since $3 \times 4 = 12$, is a super totient number.

Definition 4.1. *Let V be the set of vertices and E be the set of edges of given graph G. A one-one function $h : V \to \mathbb{N}$ is call as super totient labeling of the graph G, if the induced function $h^* : E \to \mathbb{N}$ given by $h^*(xy) = kh(x) + h(y)$ allocates a super totient number, $\forall\ xy \in E$, where $x, y \in V$ and k is any positive integer but not fixed.*

Definition 4.2. *We call a graph as super totient graph if it satisfies a super totient labeling.*

Example 4.3. Let $V = \{3, 4, 5, 6, 7, 8\}$ be the vertex set of graph G, then by the induced function $h^*(xy) = h(x) + h(y)$, we obtain a super totient graph in Fig.1.

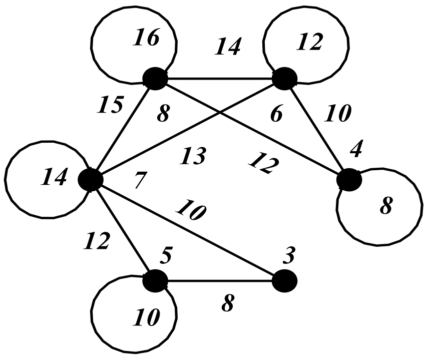

Figure 1. Super totient graph

Definition 4.4. [4] *A graph G with $n+1$ vertices is called a wheel graph if the vertices of cyclic graph $v_1, v_2, v_3, ..., v_n$ are adjacent with central vertex v_0.*

Theorem 4.5. *The Wheel graph W_n is super totient graph i.e, W_n states a super totient labeling.*

Proof. Let v_0 be the central vertex of a given wheel graph and $v_1, v_2, v_3, \cdots, v_n$. Be the remaining vertices of wheel graph. Define the edge set E of W_n by: $E = \{e_i^{'} = v_0 v_i,\ i = 1, 2, ..., n\} \cup \{e_i = v_i v_{i+1},\ i = 1, 2, ..., n-1\} \cup \{v_n v_1 = e_n\}$. We deliberate the cases $n = 2t$ and $n = 2t+1$ for some t. Let $n = 2t,\ t \in \mathbb{Z}^+$ and p, q be distinct odd primes. We establish an one-one function $h : V \to \mathbb{N}$ as:

$$h(v_i) = \begin{cases} 1, & & \text{if } i = 0 \\ p^{\frac{i+1}{2}}, & p \equiv 1(mod\ 4) & \text{if } i \text{ is odd} \\ q^{\frac{i}{2}}, & 3 \neq q \equiv 3(mod\ 4) & \text{if } i \text{ is even} \end{cases}$$

Now, we define an induced function h^* on h:

$$\begin{aligned} h^*(e_i) &= h^*(v_i v_{i+1}) = h(v_i) + h(v_{i+1}), i = 1 + 2t, t \in \mathbb{Z}^+, && (3) \\ h^*(e_i) &= h^*(v_i v_{i+1}) = h(v_{i+1}) + h(v_i), i = 2t,\ t \in \mathbb{O}^+, && (4) \\ h^*(e_i) &= h^*(v_i v_{i+1}) = 3h(v_{i+1}) + h(v_i), i = 4t,\ 4t-1,\ t \in \mathbb{Z}^+, && (5) \\ h^*(e_i^{'}) &= h^*(v_0 v_i) = h(v_0) + h(v_i),\ t \in \mathbb{O}^+, && (6) \\ h^*(e_i^{'}) &= h^*(v_0 v_i) = 3h(v_0) + h(v_i), i = 4t,\ t \in \mathbb{Z}^+, && (7) \\ h^*(e_i^{'}) &= h^*(v_0 v_i) = 3h(v_0) + h(v_i), i = 1 + 2t, t \in \mathbb{Z}^+. && (8) \end{aligned}$$

Where, $\mathbb{O}^+$ is the set of odd positive integers and $\mathbb{Z}^+$ is the set of positive integers. Applying definition of h, we obtain,

$$\begin{aligned} h^*(e_i) &= h^*(v_iv_{i+1}) = h(v_i) + h(v_{i+1}) \\ &= p^{\frac{i+1}{2}} + q^{\frac{i+1}{2}}, i = 1 + 2t, t \in \mathbb{Z}^+, \quad (9) \\ h^*(e_i) &= h^*(v_iv_{i+1}) = h(v_{i+1}) + h(v_i) \\ &= q^{\frac{i}{2}} + p^{\frac{i+2}{2}}, i = 2t,\ t \in \mathbb{O}^+, \quad (10) \\ h^*(e_i) &= h^*(v_iv_{i+1}) = 3h(v_{i+1}) + h(v_i) \\ &= 3p^{\frac{i+2}{2}} + q^{\frac{i}{2}}, i = 4t,\ 4t-1,\ t \in \mathbb{Z}^+, \quad (11) \end{aligned}$$

$$\begin{aligned} h^*(e_i^{'}) &= h^*(v_0v_i) = h(v_0) + h(v_i) = 1 + q^{\frac{i}{2}}, i = 2t,\ t \in \mathbb{O}^+, \quad (12) \\ h^*(e_i^{'}) &= h^*(v_0v_i) = 3h(v_0) + h(v_i) = 3 + q^{\frac{i}{2}}, i = 4t,\ t \in \mathbb{Z}^+, \quad (13) \\ h^*(e_i^{'}) &= h^*(v_0v_i) = 3h(v_0) + h(v_i) \\ &= 3 + p^{\frac{i+1}{2}}, i = 1 + 2t, t \in \mathbb{Z}^+. \quad (14) \end{aligned}$$

Since, $p \equiv 1(mod\ 4)$ and $3 \neq q \equiv 3(mod\ 4)$, thus $p^{\frac{i+1}{2}} + q^{\frac{i+1}{2}}$, $q^{\frac{i}{2}} + p^{\frac{i+2}{2}}, 3p^{\frac{i}{2}} + q^{\frac{i+2}{2}}$, $1 + q^{\frac{i}{2}}$, $3 + q^{\frac{i}{2}}$ and $3 + p^{\frac{i+1}{2}}$ all are multiple of 4 and greater then 4, so by Lemma 2.1, equation (9)$-$(14) are super totient number.

Now if $n = 2t + 1$, $t \in \mathbb{Z}^+$ then, we take distinct odd primes p and q. We define a one-to-one function h by:

$$h(v_i) = \begin{cases} 1, & & \text{if } i = 0 \\ p^{\frac{i+1}{2}}, & p \equiv 1(mod\ 4) & \text{if } i \text{ is odd} \\ q^{\frac{i}{2}}, & 3 \neq q \equiv 3(mod\ 4) & \text{if } i \text{ is even} \\ 3, & & \text{if } i = n \end{cases}$$

Also we define an induced function h^* to h as follows,

Equations (15)$-$(20) follow the previous case, so we only to prove that the equations (21)$-$(24) assign super totient numbers.

$$\begin{aligned}
h^*(e_i) &= h^*(v_iv_{i+1}) = h(v_i) + h(v_{i+1}), i = 1 + 2t, t \in \mathbb{Z}^+, && (15)\\
h^*(e_i) &= h^*(v_iv_{i+1}) = h(v_{i+1}) + h(v_i), i = 2t,\ t \in \mathbb{O}^+, && (16)\\
h^*(e_i) &= h^*(v_iv_{i+1}) = 3h(v_{i+1}) + h(v_i) \\
&\qquad i = 4t \text{ or } i+1 = 4t,\ t \in \mathbb{Z}^+, && (17)\\
h^*(e_i^{'}) &= h^*(v_0v_i) = h(v_0) + h(v_i),\ t \in \mathbb{O}^+, && (18)\\
h^*(e_i^{'}) &= h^*(v_0v_i) = 3h(v_0) + h(v_i), i = 4t,\ t \in \mathbb{Z}^+, && (19)\\
h^*(e_i^{'}) &= h^*(v_0v_i) = 3h(v_0) + h(v_i), i = 1 + 2t, t \in \mathbb{Z}^+, && (20)\\
h^*(e_n^{'}) &= h^*(v_0v_n) = 5h(v_0) + h(v_n), && (21)\\
h^*(e_{n-1}) &= h^*(v_{n-1}v_n) = h(v_{n-1}) + 3h(v_n) \\
&\qquad n-1 = 2t,\ t \in \mathbb{O}^+, && (22)\\
h^*(e_{n-1}) &= h^*(v_{n-1}v_n) = h(v_{n-1}) + h(v_n) \\
&\qquad n-1 = 4t,\ t \in \mathbb{Z}^+, && (23)\\
h^*(e_n) &= h^*(v_nv_1) = h(v_n) + h(v_1). && (24)
\end{aligned}$$

$$\begin{aligned}
h^*(e_n^{'}) &= h^*(v_0v_n) = 5h(v_0) + h(v_n) = 5 + 3 = 8, && (25)\\
h^*(e_{n-1}) &= h^*(v_{n-1}v_n) = h(v_{n-1}) + 3h(v_n) = q^{\frac{n-1}{2}} + 9, && (26)\\
h^*(e_{n-1}) &= h^*(v_{n-1}v_n) = h(v_{n-1}) + h(v_n) = q^{\frac{n-1}{2}} + 3, && (27)\\
h^*(e_n) &= h^*(v_nv_1) = h(v_n) + h(v_1) = 3 + p. && (28)
\end{aligned}$$

Since, $p \equiv 1(mod\ 4)$ and $3 \neq q \equiv 3(mod\ 4)$, thus $q^{\frac{n-1}{2}} + 9,\ q^{\frac{n-1}{2}} + 3$ and $3 + p$ are multiple 4 and greater then 4, so by Lemma 2.1, equation (25)−(28) assign super totient number. In both cases wheel graph admits a super totient labeling. □

Example 4.6. For $x_0 = 1,\ p = 5$ and $q = 7$, the super totient wheel graph W_7 and for $x_0, p = 5,\ q = 7$ and $x_7 = 3$ the super totient wheel graph W_8 is described in Fig.2.

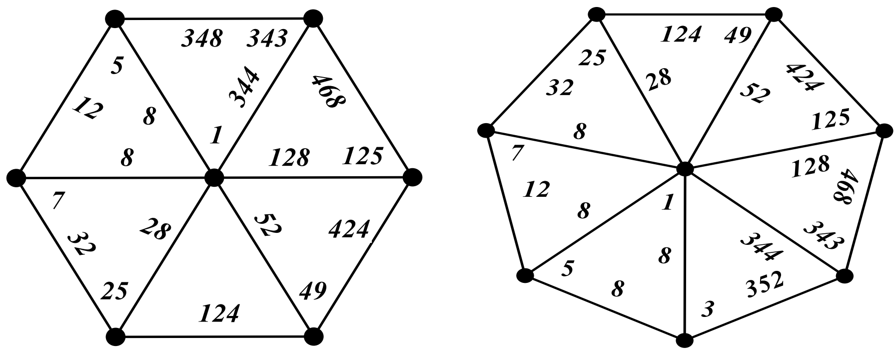

Figure 2. Super totient wheel graphs

Algorithm

This algorithm gives the set of wheel graph W_n in such a way that edges of wheel graph states a super totient number.

Step (a).

W_n, a wheel graph over n vertices;

V : Set of vertices of W_n;

E : Set of edges of W_n and $E = \{e_i^{'} = v_0v_i,\ i = 1, 2, ..., n\} \cup \{e_i = v_iv_{i+1},\ i = 1, 2, ..., n-1\} \cup \{v_nv_1 = e_n\}$;

h : h is an injective function on V;

$p,\ q$: $p,\ q$ are distinct odd prime;

Set $h(v_0) = 1$;

Step (b). do

{ **if** $n = 2t,\ t \in \mathbb{Z}^+$ **then**

{

for $i = 1, 3, ..., n-1$ **do**

{

$h(v_i) = p^{\frac{i+1}{2}}$

$h(v_{i+1}) = q^{\frac{i+1}{2}}$

}

}

else

{

for $i = 1, 3, ..., n-2$ **do**

{
$h(v_i) = p^{\frac{i+1}{2}}$
$h(v_{i+1}) = q^{\frac{i+1}{2}}$
}
}
if $i = n$ **then** $h(v_n) = 3$
}
{
for $1 \leqslant i \leqslant n$ **do**
for $1 \leqslant j \leqslant n$ **do**
Step (c). If n is even **then**
$h^*(e_i) = h^*(v_i v_{i+1}) = h(v_i) + h(v_{i+1}), i = 1 + 2t, t \in \mathbb{Z}^+$
$h^*(e_i) = h^*(v_i v_{i+1}) = h(v_{i+1}) + h(v_i), i = 2t,\ t \in \mathbb{O}^+$
$h^*(e_i) = h^*(v_i v_{i+1}) = 3h(v_{i+1}) + h(v_i), i = 4t$ or $i+1 = 4t, ,\ t \in \mathbb{Z}^+$
$h^*(e_i^{'}) = h^*(v_0 v_i) = h(v_0) + h(v_i),\ t \in \mathbb{O}^+$
$h^*(e_i^{'}) = h^*(v_0 v_i) = 3h(v_0) + h(v_i), i = 4t,\ t \in \mathbb{Z}^+$
$h^*(e_i^{'}) = h^*(v_0 v_i) = 3h(v_0) + h(v_i), i = 1 + 2t, t \in \mathbb{Z}^+$
}
else
{
$h^*(e_i) = h^*(v_i v_{i+1}) = h(v_i) + h(v_{i+1}), i = 1 + 2t, t \in \mathbb{Z}^+$
$h^*(e_i) = h^*(v_i v_{i+1}) = h(v_{i+1}) + h(v_i), i = 2t,\ t \in \mathbb{O}^+$
$h^*(e_i) = h^*(v_i v_{i+1}) = 3h(v_{i+1}) + h(v_i), i = 4t$ or $i+1 = 4t, ,\ t \in \mathbb{Z}^+$
$h^*(e_i^{'}) = h^*(v_0 v_i) = h(v_0) + h(v_i),\ t \in \mathbb{O}^+$
$h^*(e_i^{'}) = h^*(v_0 v_i) = 3h(v_0) + h(v_i), i = 4t,\ t \in \mathbb{Z}^+$
$h^*(e_i^{'}) = h^*(v_0 v_i) = 3h(v_0) + h(v_i), i = 1 + 2t, t \in \mathbb{Z}^+)$
$h^*(e_n^{'}) = h^*(v_0 v_n) = 5h(v_0) + h(v_n)$
$h^*(e_{n-1}) = h^*(v_{n-1} v_n) = h(v_{n-1}) + 3h(v_n),\ n - 1 = 2t,\ t \in \mathbb{O}^+$
$h^*(e_{n-1}) = h^*(v_{n-1} v_n) = h(v_{n-1}) + h(v_n),\ n - 1 = 4t,\ t \in \mathbb{Z}^+$
$h^*(e_n) = h^*(v_n v_1) = h(v_n) + h(v_1)$
}
Step (d). Output (super totient wheel graph W_n).
}

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